Extremal Combinatorics examples sheet 1

There are also exercises in the notes; some of these are included below and some are not, but you should attempt all exercises to ensure a thorough understanding of the course material. The examples sheets are unassessed, but you are welcome to hand in your attempts the week after they are handed out for feedback.

1. Determine all the extremal graphs for Mantel's theorem, i.e. all triangle-free graphs on n vertices with $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.

2. In our first proof of Mantel's theorem we showed that
$$\sum_{x \in V} d(x)^2 \leqslant n\, e$$
for any triangle-free graph on n vertices with e edges. Show that if one replaces the right-hand side with $2(n-1)e$ then the inequality is valid for any graph; one thus 'saves' roughly a factor of 2 for triangle-free graphs. When is this new inequality an equality?

3. By taking the number of triangles t in G into account, generalise the above inequality to one that is valid for any graph and has the right-hand side growing with t.

4. Prove, by using your answer to the previous question or otherwise, that if a graph G on n vertices has e edges then the number of triangles t in G satisfies
$$t \geqslant \frac{e}{3n}\left(4e - n^2\right).$$
(Thus, if G has $\delta\binom{n}{2}$ edges then the number of triangles is roughly at least $\delta(2\delta - 1)\binom{n}{3}$.) Give some examples of families of graphs for which equality holds.

5. By considering the deletion of any two endvertices of an edge in a triangle-free graph, give an alternative inductive proof of Mantel's theorem. (No calculations involving the floor function should be needed via this route!)

6. Describe $T_r(n)$ for $n \leqslant r$.

7. Prove that if $n = ar + b$ with $0 \leqslant b < r$, then $t_r(n) = \left(\frac{r-1}{r}\right)\frac{n^2}{2} - \frac{b(r-b)}{2r}$.

8. What might be the largest graph on n vertices without a 5-cycle C_5?

Please let me know if you have any comments or corrections.

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