Autumn 2011 O. Sisask

Extremal Combinatorics examples sheet 2

There are also exercises in the notes; some of these are included below and some are not, but you should attempt all exercises to ensure a thorough understanding of the course material. The examples sheets are unassessed, but you are welcome to hand in your attempts for feedback.

1. The purpose of this exercise is to guide you through a proof of a slightly weakened version of Turán's theorem, starting with another proof of Mantel's theorem. So, complete the steps:

- Given a graph G on vertex set $\{1, 2, \ldots, n\}$ with at least one edge, define the polynomial
$$f_G(x_1, \ldots, x_n) = \sum_{ij \in E(G)} x_i x_j.$$
This is sometimes known as the *Lagrangian* of G.
- Let S denote the region $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geqslant 0 \text{ and } \sum_{i=1}^n x_i = 1\}$. Explain why the supremum $\sup_{\mathbf{x} \in S} f_G(\mathbf{x})$ is actually a maximum. Let us call this maximum λ.
- Show that there is a tuple $\mathbf{a} \in S$ such that $f_G(\mathbf{a}) = \lambda$ and if $ij \notin E(G)$ then at least one of a_i and a_j is 0. (*Hint: consider a variational argument.*)
- Supposing G is triangle-free, conclude that $\lambda = a_i a_j$ for some edge ij, and hence that $\lambda \leqslant 1/4$.
- Finally evaluate $f_G\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$ and use the fact that it is at most λ to give a bound for $e(G)$.

Now extend this argument to show that $\mathrm{ex}(n, K_{r+1}) \leqslant \left(1 - \frac{1}{r}\right) n^2/2$. (*Exercise for the enthusiast: go further and modify the argument to yield* $\mathrm{ex}(n, K_{r+1}) = t_r(n)$.)

2. Let A be the graph on 4 vertices comprising 2 triangles sharing an edge. Let B be the graph on 5 vertices comprising 2 triangles sharing a vertex.
   a) Show by induction on n that if G is a graph on $n \geqslant 4$ vertices with $e(G) \geqslant \lfloor \frac{n^2}{4} \rfloor + 1$ then G contains a copy of A as a subgraph.
   b) Show by induction on n that if G is a graph on $n \geqslant 5$ vertices with $e(G) \geqslant \lfloor \frac{n^2}{4} \rfloor + 2$ then G contains a copy of B as a subgraph.
   c) Determine $\mathrm{ex}(n, A)$ and $\mathrm{ex}(n, B)$.

3. By modifying the proof of our upper bound for $\mathrm{ex}(n, C_4)$, show that
$$\mathrm{ex}(n, K_{r,r}) \leqslant C n^{2-1/r}$$
for some constant C and all $r \geqslant 2$. (*Hint: you can use the inequality* $\mathbb{E}_{x \in X} |f(x)|^r \geqslant (\mathbb{E}_{x \in X} |f(x)|)^r$ *instead of the square-mean inequality.*) Deduce that $\pi(H) = 0$ for any bipartite graph H.

4. For a bipartite graph H, define $z(n, H)$ to be the maximum number of edges in an H-free *bipartite* graph with n vertices in each part. By modifying the proof of our upper bound for $\mathrm{ex}(n, C_4)$, show that
$$z(n, K_{2,2}) \leqslant \tfrac{1}{2} n \left(1 + \sqrt{4n - 3}\right)$$
for all $n \geqslant 1$. By considering the construction giving lower bounds for $\mathrm{ex}(n, C_4)$, show that when $n = q^2 + q + 1$ for a prime power q the above inequality is an equality. Determining $z(n, K_{r,r})$ and related numbers is known as the *problem of Zarankiewicz*.

Please let me know if you have any comments or corrections.

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