Exercise V

November 9, 2007

Let R be commutativ and with 1.

Exercise 1

Let $S \subseteq R$ be a multiplicatively closed subset with 1. For any R-module M we define $S^{-1}M$ as the set of pairs (m, s) with $m \in M$ and $s \in S$, modulo the equivalence relation $(m, s) \cong (n, t)$ if there exist $u \in S$ such that $u(tm - sn) = 0$ in M. We have an induced R-linear map $M \longrightarrow S^{-1}M$. Show that if $f : M \longrightarrow N$ is an R-linear map, we get a commutative diagram of R-modules

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ S^{-1}M & \longrightarrow & S^{-1}N. \end{array} \tag{1}$$

Show that if $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is a short exact sequence of R-modules, then the induced sequence

$$0 \longrightarrow S^{-1}M' \longrightarrow S^{-1}M \longrightarrow S^{-1}M'' \longrightarrow 0$$

is short exact.

Show finally that we have a natural map $M \otimes_R S^{-1}R \longrightarrow S^{-1}M$ which is an isomorphism. Conclude that $S^{-1}R$ is a flat R-module.

Exercise 2

For any R-module M we let $S(M)$ denote the symmetric quotient of the tensor algebra. Show that $S(M \oplus N)$ is isomorphic to $S(M) \otimes_R S(N)$. We have that, for any module M, the symmetric algebra $S(M)$ is naturally graded. Use the isomorphism deduced above,to describe the induced grading on $S(M) \otimes_R S(N)$.