WEIL RESTRICTION AND THE QUOT SCHEME

ROY MIKAEL SKJELNES

Abstract. We introduce a concept that we call module restriction, which generalizes the classical Weil restriction. After having established some fundamental properties, as existence and étaleness of the module restrictions, we apply our results to show that the Quot functor $\underline{\mathrm{Quot}}^n_{F_X/S}$ of Grothendieck is representable by an algebraic space, for any quasi-coherent sheaf F_X on any separated algebraic space X/S.

Introduction

The main novelty in this article is the introduction of the *module restriction*, which is a generalization of the classical Weil restriction. Our main motivation for introducing the module restriction is given by our application to the Quot functor of Grothendieck.

If F_X is a quasi-coherent sheaf on a scheme $X \longrightarrow S$, then the Quot functor $\underline{\mathrm{Quot}}_{F_X/S}$ parametrizes quotients of F_X that are flat and with proper support over the base. For projective schemes $X \longrightarrow S$ the Quot functor is represented by a scheme given as a disjoint union of projective schemes [Gro95]. When $X \longrightarrow S$ is locally of finite type and separated, Artin showed that the Quot functor is representable by an algebraic space ([Art69] and erratum in [Art74]).

When the fixed sheaf $F_X = \mathscr{O}_X$ is the structure sheaf of X the Quot functor is referred to as the Hilbert functor $\underline{\mathrm{Hilb}}_{X/S}$.

Grothendieck who both introduced the Quot functor and showed representability for projective $X \longrightarrow S$, also pointed out the connection between the Hilbert scheme and the Weil restriction [Gro95, 4. Variantes]. If $f \colon Y \longrightarrow X$ is a morphism with X separated over the base, there is an open subset $\Omega_{Y \to X}$ of $\underline{\mathrm{Hilb}}_{Y/S}$ from where the push-forward map f_* is defined. The fibers of $f_* \colon \Omega_{Y \to X} \longrightarrow \underline{\mathrm{Hilb}}_{X/S}$ are identified with the Weil restrictions.

However, even though the Weil restriction appears naturally in connection with Hilbert schemes, there does not appear to exist any description of the more general situation with the Quot scheme replacing the Hilbert scheme. The purpose of this article is to give such a description with the Quot functor $\underline{\mathrm{Quot}}^n_{F_X/S}$ parametrizing quotients of

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F_X that are flat, *finite* and of relative rank n. In order to do so we will need to introduce a generalization of the Weil restriction.

Generalizations of the Weil restriction exist ([Ols06] and [Ver72]), but those generalizations leap off in different directions than what is needed for the present discussion. The generalization we undertake here is in the direction from ideals to modules.

We fix a homomorphism of A-algebras $B \longrightarrow R$, and a B-module M. The module restriction $\mathscr{M}od^M_{B\to R}$ parametrizes, as a functor from A-algebras to sets, R-module structures extending the fixed B-module structure on M.

When M is finitely generated and projective as an A-module, we show that the module restriction $\mathscr{M}od^M_{B\to R}$ is representable by an A-algebra. We show representability by constructing the representing object in the free algebra situation, and using Fitting ideals in the general situation.

Furthermore, when the B-module M is a quotient module of B, then we obtain that the module restriction $\mathscr{M}od^M_{B\to R}$ coincides with the Weil restriction.

Our investigations about the module restriction can be summarized as follows.

Theorem 1. *Let $X \longrightarrow S$ be a separated morphism of schemes (or algebraic spaces) and let $\mathscr{C}oh^n_{X/S}$ denote the stack of quasi-coherent sheaves on X that are flat, of finite support, and of relative rank n over the base S. For any affine morphism $f\colon Y \longrightarrow X$ the push-forward map*

$$(\star) \quad f_*\colon \mathscr{C}oh^n_{Y/S} \longrightarrow \mathscr{C}oh^n_{X/S}$$

is schematically representable. Moreover, if $f\colon Y \longrightarrow X$ is étale, then the push-forward map f_* is also étale.

The fibers of the push-forward map $(\star)$ are the module restrictions parameterizing sheaves on $\mathscr{F}$ on Y that are flat, finite, and of relative rank n over the base, such that the push-forward $f_*\mathscr{F}$ is isomorphic to a fixed $\mathscr{E}$ on X.

When $f\colon Y \longrightarrow X$ is étale the push-forward map $(\star)$ is also étale, but the analogous statement for smooth maps does not hold. This is in contrast with the situation with the Weil restriction where smoothness of $f\colon Y \longrightarrow X$ implies smoothness of the Weil restriction [BLR90].

Our main application and motivation for introducing the module restriction follows from the above result. Let F_Y denote the pull-back of the quasi-coherent sheaf F_X along $f\colon Y \longrightarrow X$. There is an open subfunctor $\Omega^F_{Y\to X}$ of $\underline{\mathrm{Quot}}^n_{F_Y/S}$ on where the push-forward map f_* is defined. In fact, we have that $\Omega^F_{Y\to X}$ is given by the fiber product of $(\star)$ and the natural forgetful map from $\underline{\mathrm{Quot}}^n_{F_X/S}$ to $\mathscr{C}oh^n_{X/S}$. Then using the fact that $\underline{\mathrm{Quot}}^n_{F_Y/S}$ is representable by a scheme for *affine* schemes $Y \longrightarrow S$, together with the Theorem 1 above, proves the following.

Theorem 2. *Let* $X \longrightarrow S$ *be a separated map of algebraic spaces, and* F_X *a quasi-coherent sheaf on* X*. Then the Quot functor* $\underline{\mathrm{Quot}}^n_{F_X/S}$ *is representable by an algebraic space.*

In particular this generalizes the result about the representability of the Hilbert functor $\underline{\mathrm{Hilb}}^n_{X/S}$ described in [ES04]. See also the generalization to Hilbert stacks in [Ryd].

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1. Weil restriction and Fitting ideals

In this section we use Fitting ideals to construct algebras that parameterize algebra homomorphisms between two fixed algebras. Both the problem and its solution is well-known. The material in this section is included, not only to make the article self-contained, but also because we could not find references for the results in the needed generality.

1.1. **Notation.** If V is an A-module, we let $A[V]$ denote the symmetric quotient algebra of the full tensor algebra $T_A(V) = \bigoplus_{n\geq 0} V^{\otimes^n}$. The dual module $\mathrm{Hom}_A(V, A)$ we write as $V^\star$.

1.2. **Parametrizing algebra maps.** Let $g\colon A \longrightarrow B$ be a homomorphism of commutative, unital rings. Let $f\colon B \longrightarrow R$ and $\mu\colon B \longrightarrow E$ be homomorphism of rings, where R and E are not necessarily commutative. Thus we fix the following data

(1.2.1)
$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "c"] & B \arrow[r, "f"] \arrow[d, "\mu"] & R \\ & E & \end{tikzcd}$$

where $c = \mu \circ g$. We consider the functor $\underline{\mathrm{Hom}}_B(R, E)$, from the category of commutative A-algebras to sets, that for any A-algebra A' assigns the set of B-algebra homomorphisms

$$\mathrm{Hom}_{B\text{-alg}}(R, E \bigotimes_A A').$$

1.3. We occasionally write $\underline{\mathrm{Hom}}_A(R, E)$ having implicitely made the assumption that $A = B$.

1.4. Let A' be an A-algebra, and let $\xi\colon R \longrightarrow E \bigotimes_A A'$ be a B-algebra homomorphism. The morphism ξ will factorize as a $B \bigotimes_A A'$-algebra homomorphism $R \bigotimes_A A' \longrightarrow E \bigotimes_A A'$, which is the identity on A'. Therefore the set set of A'-valued points of $\underline{\mathrm{Hom}}_B(R, E)$ is the set

$$\mathrm{Hom}_{B \bigotimes_A A'\text{-alg}}(R \bigotimes_A A', E \bigotimes_A A').$$

1.5. We are primarily interested in commutative A-algebras R, but it will be important to allow E to be non-commutative. The reason for allowing R to also be non-commutative is mainly that some of our proofs are simplified by doing so.

Lemma 1.6. *Let $c\colon A \longrightarrow E$ be a homomorphism of rings, where E is not necessarily commutative. Assume that E is finitely generated and projective as an A-module. For any A-module V we have that the A-algebra $A[V \bigotimes_A E^\star]$ represents $\underline{\mathrm{Hom}}_A(R, E)$, with $R = T_A(V)$.*

Proof. An A-algebra homomorphism $u_{A'}\colon T_A(V) \longrightarrow E \bigotimes_A A'$ is determined by an A'-linear map $u_1\colon V \bigotimes_A A' \longrightarrow E \bigotimes_A A'$. Since E is finitely generated and projective as an A-module, the A'-linear map u_1 is equivalent with an A'-linear map $\varphi_1\colon V \bigotimes_A A' \bigotimes_{A'} (E \bigotimes_A A')^\star \longrightarrow A'$. Moreover, the canonical map

$$\mathrm{Hom}_A(E, A) \bigotimes_A A' \longrightarrow \mathrm{Hom}_{A'}(E \bigotimes_A A', A')$$

is an isomorphism ([Bou98, 4.3. Proposition 7]). Therefore φ_1 corresponds to an A-algebra homomorphism $\varphi\colon A[V \bigotimes_A E^\star] \longrightarrow A'$. See e.g. [Die62]. □

Lemma 1.7. *Let $g\colon A \longrightarrow B$ be a homomorphism of commutative rings. Let $h\colon A \longrightarrow D$ be homomorphism of rings, and let $\mu\colon B \longrightarrow E$ be an A-algebra homomorphism. Let $R = B \otimes_A D$, where the B-algebra structure is the canonical one. Then we have that $\underline{\mathrm{Hom}}_B(R, E)$ is naturally identified with $\underline{\mathrm{Hom}}_A(D, E)$.*

Proof. Let A' be an A-algebra. It follows readily that a B-algebra homomorphism $B \bigotimes_A D \longrightarrow E \bigotimes_A A'$ is the same as an A-algebra homomorphism $D \longrightarrow E \bigotimes_A A'$. Thus $\underline{\mathrm{Hom}}_B(R, E)(A') = \underline{\mathrm{Hom}}_A(D, E)(A')$. □

Lemma 1.8. *Let E be a projective A-module of rank n. Let $E \longrightarrow Q$ be a quotient module, and let $F_{n-1}(Q) \subseteq A$ denote the $(n-1)$'st Fitting ideal of Q. Then the A-module map $E \longrightarrow Q$ is an isomorphism if and only if $F_{n-1}(Q) = 0$ is the zero ideal.*

Proof. The statement can be checked locally on A, hence we may assume that E is free of finite rank n. The result then follows from the definition of the Fitting ideal. □

1.9. Rank of projective modules. The rank of a projective module E is constant on the connected components of $\mathrm{Spec}(A)$. Therefore, if $E \longrightarrow Q$ is a quotient module of a finitely generated and projective A-module E, we will denote by

$$F_{\mathrm{rk}E-1}(Q) \subseteq A$$

the Fitting ideal we obtain by assigning on each connected component of $\mathrm{Spec}(A)$ the Fitting ideal $F_{n-1}(Q)$, where n is the rank of E on that particular component.

1.10. Extension of scalars to the universal map. Let $g\colon A \longrightarrow B$ and let $\mu\colon B \longrightarrow E$ be homomorphism of rings, with E not necessarily commutative. Assume that E is projective and finitely generated as an A-module, and let V be an A-module. We have (Lemma 1.6) that

$$C_V^E = A[V \bigotimes_A E^\star]$$

represents A-algebra homomorphisms from $T_A(V)$ to the A-algebra E. In particular the identity morphism on C_V^E corresponds to an A-algebra homomorphism

$$u\colon T_A(V) \longrightarrow E \bigotimes_A C_V^E\ . \tag{1.10.1}$$

We have that E is both an A and a B-algebra. And consequently we get an induced algebra homomorphism

$$u_B\colon B \bigotimes_A T_A(V) \longrightarrow E \bigotimes_A C_V^E\ , \tag{1.10.2}$$

by extension of scalars to the universal map u of 1.10.1

Definition 1.11. Assume the A-algebra E is finitely generated and projective A-module, and let V be an A-module. For any given ideal $I \subseteq B \bigotimes_A T_A(V)$, we have the C_V^E-module generated by the image $u_B(I) \subseteq E \bigotimes_A C_V^E$. We define

$$F(I^B) = F_{\mathrm{rk}E-1}(E \bigotimes_A C_V^E / u_B(I) C_V^E)$$

as the $(\mathrm{rk}E-1)$'st Fitting ideal of the C_V^E-module given as the quotient module $E \bigotimes_A C_V^E / u_B(I) C_V^E$ of the projective module $E \bigotimes_A C_V^E$.

Proposition 1.12. *Let $g\colon A \longrightarrow B$ be a homomorphism of commutative rings. Let $f\colon B \longrightarrow R$ be homomorphism of rings, and let $\mu\colon B \longrightarrow E$ be an A-algebra homomorphism, where E is not necessarily commutative. Assume that E is finitely generated and projective as an A-module. Then the functor $\underline{\mathrm{Hom}}_B(R, E)$ is representable. In fact, write $R = B \bigotimes_A T_A(V)/I$ with V an A-module, and some two-sided ideal I in $B \bigotimes_A T_A(V)$. Then we have that the functor $\underline{\mathrm{Hom}}_B(R, E)$ is represented by the A-algebra*

$$A[V \bigotimes_A E^\star]/F(I^B)\ .$$

Proof. Consider first the situation with R the full tensor algebra over B. We can then write $R = B \bigotimes_A T_A(V)$, for some free A-module V. In that case we have by Lemma 1.7 that $\underline{\mathrm{Hom}}_B(R, E)$ equals $\underline{\mathrm{Hom}}_A(T_A(V), E)$. By Lemma 1.6 we get that $\underline{\mathrm{Hom}}_B(R, E)$ is represented by $A[V \bigotimes_A E^\star]$. Consider now the situation with general R, and write R as the quotient of the full tensor algebra $B \bigotimes_A T_A(V)$ modulo some two-sided ideal $I \subseteq B \bigotimes_A T_A(V)$. We are then interested in describing B-algebra homomorphisms $\sigma\colon B \bigotimes T_A(V) \longrightarrow E \bigotimes_A A'$ that factors via R, for any A-algebra A'. The condition that $\sigma(I) = 0$ is equivalent with

$$E \bigotimes_A A' \longrightarrow Q = E \bigotimes_A A'/\sigma(I) \tag{1.12.1}$$

being an isomorphism. As observed in Lemma 1.8, we have that $\sigma(I) = 0$ if and only if the Fitting ideal $F_{\mathrm{rk}-1}(Q)$ is the zero ideal in A'. The B-algebra homomorphism $\sigma\colon B \bigotimes_A T_A(V) \longrightarrow E \bigotimes_A A'$ corresponds to an A-algebra homomorphism $\varphi\colon C_V^E = A[V \bigotimes_A E^\star] \longrightarrow A'$. The correspondance is given as $\sigma = (1 \otimes \varphi) \circ u_B$, where u_B is the map 1.10.2. Therefore we have that the map of C_V^E-modules

$$E \bigotimes_A C_V^E \longrightarrow E \bigotimes_A C_V^E / u_B(I) C_V^E$$

tensored with $- \bigotimes_{C_V^E} A'$ specializes to the map of A'-modules in 1.12.1. And, for Fitting ideals we have the property that

$$C_V^E / F(I^B) \bigotimes_{C_V^E} A' = A'/F_{\mathrm{rk}E-1}(Q)\ ,$$

from where the result follows. □

Corollary 1.13. *Let $g\colon A \longrightarrow B$ be a homomorphism of commutative rings, where B is finitely generated and projective as an A-module. Let $f\colon B \longrightarrow R$ be homomorphisms of rings. Write $R = B \bigotimes_A T_A(V)/I$ as a quotient of the full tensor algebra, where V is some A-module. Then the Weil restriction $\mathfrak{R}_{R/B}$ is representable by the A-algebra*

$$A[V \bigotimes_A B^\star]/F(I^B)\ .$$

Proof. We have that the Weil restriction $\mathfrak{R}_{R/B}$ (see e.g. [BLR90]) by definition equals $\underline{\mathrm{Hom}}_B(R, B)$. □

Remark 1.14. The defining properties of the full tensor algebra $T_A(V)$ as well as the symmetric quotient $A[V]$ are well-known, and can be found in e.g. [Die62]. The situation with the Weil restriction as in Corollary 1.13, that is to parametrize B-algebra homomorphisms $R \longrightarrow B$ can be found in e.g. the proof of [BLR90, Theorem 7.4]. There, however the Fitting ideal $F(I^B)$ is not mentioned.

2. Weil and Module restrictions

In this section we will introduce the module restriction, which is the main novelty of the article. At the end we relate the module restriction to the Weil restriction.

2.1. Module structure. Recall that an A-module structure on an Abelian group M is to have a ring homomorphism $\rho\colon A \longrightarrow \mathrm{End}_{\mathbf{Z}}(M)$. The image of ρ will factorize via the subring of A-linear endomorphisms $\mathrm{End}_A(M)$. If M is an A-module, then $\mathrm{End}_A(M)$ is an A-algebra via the canonical map $\mathrm{can}\colon A \longrightarrow \mathrm{End}_A(M)$.

2.2. Extension of module structures. Let $g\colon A \longrightarrow B$ be a homomorphism of rings. If M is an A-module, then a B-module structure on the set M, extending the fixed A-module structure, is a B-module structure on M that is compatible with the A-module structure. That is, a B-module structure on M extending the A-module structure is a ring homomorphism $\mu\colon B \longrightarrow \mathrm{End}_A(M)$ making the commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\mu"] & \mathrm{End}_A(M) . \\
A \arrow[u, "g"] \arrow[ur, "\mathrm{can}"] &
\end{tikzcd}$$

In other words, a B-module structure on M extending the A-module structure, is an A-algebra homomorphism $\mu\colon B \longrightarrow \mathrm{End}_A(M)$.

Definition 2.3. Let $g\colon A \longrightarrow B$ and $f\colon B \longrightarrow R$ be algebras and homomorphisms, and let M be a B-module. We define the functor $\mathscr{M}od^M_{B\to R}$, from the category of A-algebras to sets, by assigning to each A-algebra A' the set

$$\mathscr{M}od^M_{B\to R}(A') = \left\{ \begin{array}{l} R \bigotimes_A A'\text{-module structures on } M \bigotimes_A A'\text{, extending} \\ \text{the fixed } B \bigotimes_A A'\text{-module structure on } M \bigotimes_A A'. \end{array} \right\}$$

We call this functor the module restriction.

Theorem 2.4. *Let $g\colon A \longrightarrow B$ be a homomorphism of commutative rings. Let $f\colon B \longrightarrow R$ be homomorphism of rings, and let M be a B-module. Assume that M, considered as an A-module, is projective and finitely generated. Then the functor $\mathscr{M}od^M_{B\to R}$ is naturally identified with $\underline{\mathrm{Hom}}_B(R, E)$, where $E = \mathrm{End}_A(M)$, and in particular the functor $\mathscr{M}od^M_{B\to R}$ is representable.*

Proof. Since M is a B-module, we have that M is also an A-module. In particular we have that the B-module structure on M extends the A-module structure. Thus, the B-module structure on M is given by an A-algebra homomorphism $\mu\colon B \longrightarrow \mathrm{End}_A(M)$. Let A' be an A-algebra, and let ξ be an A'-valued point of the module restriction

$\mathscr{M}od^M_{B\to R}$. Then we have that the A'-valued point ξ is a B-algebra homomorphism making the following commutative diagram

$$\begin{tikzcd}
R \bigotimes_A A' \arrow[r, "\xi"] & \mathrm{End}_{A'}(M \bigotimes_A A') \\
B \bigotimes_A A' \arrow[u, "f\otimes \mathrm{id}"] \arrow[r, "\mu\otimes\mathrm{id}"] & \mathrm{End}_A(M) \bigotimes_A A', \arrow[u, "\nu"']
\end{tikzcd} \tag{2.4.1}$$

where ν is the canonical map. Since M is finitely generated and projective, the map ν is an isomorphism ([Bou98, 4.3. Proposition 7]). In other words, we have a natural identification of functors

$$\mathscr{M}od^M_{B\to R} = \underline{\mathrm{Hom}}_B(R, E),$$

with $E = \mathrm{End}_A(M)$. As M is projective and finitely generated, so is E, and the statement about representability follows from Proposition 1.12. $\square$

2.5. Weil restriction revisited. In the rest of the present section we will show that the functor $\mathscr{M}od^M_{B\to R}$ specializes to the Weil restriction for particular choices of the module M.

Let $s\colon A \longrightarrow A'$ be homomorphism of rings. Consider $M = A'$ as an A-module. The evaluation map $\mathrm{ev}_1\colon \mathrm{End}_A(M) \longrightarrow M$ evaluating endomorphism at the identity element of $A' = M$, is an A-algebra homomorphism.

Proposition 2.6. *Let $g\colon A \longrightarrow B$ and $s\colon A \longrightarrow A'$ be homomorphism of commutative rings. Consider $A' = M$ with the induced A-module structure.*

1. *For each A-algebra homomorphism $\mu\colon B \longrightarrow \mathrm{End}_A(M)$ the composition $\mathrm{ev}_1 \circ \mu\colon B \longrightarrow A'$ is an A-algebra homomorphism.*
2. *If $s\colon A \longrightarrow A'$ is surjective, then (1) gives a bijection between the set of B-module structures on M, extending the A-module structure, and the set of A-algebra homomorphisms $B \longrightarrow A'$.*

Proof. The composition of A-algebra homomorphisms is again an A-algebra homomorphism, so the first assertion is clear.

To prove Assertion (2) we note that a B-module structure extending the A-module structure on M is to give an A-algebra homomorphism $\mu\colon B \longrightarrow \mathrm{End}_A(M)$. The composition $\mathrm{ev}_1 \circ \mu\colon B \longrightarrow A'$ is an A-algebra homomorphism. If $f\colon B \longrightarrow A'$ is an A-algebra homomorphism, then the induced B-module structure on A' is such that the multiplication map $\mathrm{ev}_1 \circ \mu = f$. Thus we need only to prove that a B-module structure on M, extending the A-module structure, is determined by the multiplication map μ_1. This is clear since $s\colon A \longrightarrow A'$ is assumed surjective. $\square$

Corollary 2.7. *Let $g\colon A \longrightarrow B$ and $f\colon B \longrightarrow R$ be homomorphisms of commutative rings. Let $M = B/J$ for some ideal $J \subseteq B$, and assume* *that $M = B/J$ is finitely generated and projective as an A-module. Then we have a natural identification*

$$\mathscr{M}od^M_{B\to R} = \underline{\mathrm{Hom}}_{B/J}(R/J, B/J),$$

where $R/J = R \bigotimes_B B/J$. In other words, the module restriction generalizes the notion of Weil restriction.

Proof. By setting $R' = R \bigotimes_A A'$, $B' = B \bigotimes_A A'$, etc. we may assume that $A = A'$. From Proposition (2.6) we have that an R-module structure on B/J, extending the B-module structure, is precisely the same as an B-algebra homomorphism $R \longrightarrow B/J$. By the usual properties of tensor product, a B-algebra homomorphism $R \longrightarrow B/J$ is the same as a B/J-algebra homomorphism $R \bigotimes_B B/J \longrightarrow B/J$. Thus we may assume $B = B/J$, and $R = R \bigotimes_B B/J$. We then have that $\mathscr{M}od^M_{B\to R}$ parametrizes B-algebra homomorphisms $R \longrightarrow B$, thus being equal to $\underline{\mathrm{Hom}}_B(R, B)$. The last statement is given by Corollary 1.13. □

3. Trace map and Fitting ideals

In this section we will describe the Fitting ideals we have be considering, using the trace map.

3.1. Trace map. We will use the notation from the previous sections. Let $u_B \colon B \bigotimes_A T_A(V) \longrightarrow E \bigotimes_A C^E_V$ be the map 1.10.2, with $C^E_V = A[V \bigotimes_A E^\star]$, where V is an A-module, and where E is an A-algebra that is finitely generated and projective as an A-module. Let

$$u^*_B \colon B \bigotimes_A T_A(V) \bigotimes_A E^\star \longrightarrow E \bigotimes_A C^E_V \bigotimes_A E^\star \longrightarrow C^E_V \tag{3.1.1}$$

denote the induced composite map.

Proposition 3.2. *Let $g \colon A \longrightarrow B$ be a homomorphism of commutative rings, and let V be an A-module. Let $\mu \colon B \longrightarrow E$ be an A-algebra homomorphism, where E is not necessarily commutative. Assume that E is finitely generated an projective as an A-module. For any two-sided ideal $I \subseteq B \bigotimes_A T_A(V)$, let $I^E \subseteq A[V \bigotimes_A E^\star]$ denote the the ideal generated by $u^*_B(I \bigotimes_A E^\star)$, where u^*_B is the map 3.1.1. Then we have that I^E is the Fitting ideal $F(I^B)$ defined in 1.11.*

Proof. Both the Fitting ideal and the ideal I^E commute with base change, and we may therefore assume that E is free as an A-module. Let $e_1, \ldots, e_n$ be a basis of E. If $(f_\alpha)_{\alpha\in\mathscr{A}}$ generates the ideal $I \subseteq B \bigotimes_A T_A(V)$, then we have the map of free C^E_V-modules

$$\bigoplus\nolimits_{\alpha\in\mathscr{A}} C^E_V \longrightarrow E \bigotimes_A C^E_V \tag{3.2.1}$$

determined by sending 1_α on α'th component to $u_B(f_\alpha)$. The cokernel of the map 3.2.1 is $E \bigotimes_A C^E_V / u_B(I) C^E_V$, and consequently the $(n-1)$-minors of the map generate the Fitting ideal $F(I^B)$. For any element $f \in B \bigotimes_A T_A(V)$, the element $u_B(f)$ is in the free C_V^E-module $E \bigotimes_A C_V^E$. Therefore we can write

$$u_B(f) = \sum_{ik=1}^{n} e_k \otimes f_k^E,$$

with $f_k^E \in C_V^E$, with $k = 1, \ldots, n$. Then we obtain that the $(n-1)$-minors of the map 3.2.1 are $\{(f_\alpha)_1^E, \ldots, (f_\alpha)_n^E\}_{\alpha \in \mathscr{A}}$. In order to relate the Fitting ideal $F(I^B)$ to the other ideal I^E, let $e_1^\star, \ldots, e_n^\star$ denote the dual basis of $E^\star$. For each $k = 1, \ldots, n$ we have that

$$u_B^*(f \otimes e_k^\star) = u_B(f) \otimes e_k^\star = (\sum_{i=1}^{n} e_i \otimes f_i^E) \otimes e_k^\star = f_k^E.$$

Thus $\{(f_\alpha)_1^E, \ldots, (f_\alpha)_n^E\}_{\alpha \in \mathscr{A}}$ also generate $u_B^*(I \bigotimes_A E^\star)$, and we have the equality $I^E = F(I^B)$. □

Remark 3.3. It follows by the characterization of the Fitting ideal $F(I^B)$ given in Proposition 1.12, and the above result, that an A-algebra homomorphism $\varphi \colon C_V^E \longrightarrow A'$ vanishes on I^E if and only if the composite map

$$B \bigotimes_A T_A(V) \xrightarrow{u_B} E \bigotimes_A C_V^E \xrightarrow{\mathrm{id} \otimes \varphi} E \bigotimes_A A'$$

vanishes on I. This fact can also be checked directly, and consequently one could establish the result in this article without using the theory of Fitting ideals.

Example 3.4. We will in this example explicitely describe the correspondance between $T_A(V)$ and $E \bigotimes_A C_V^E$, where $C_V^E = A[V \bigotimes_A E^\star]$. Assume that the A-algebra E is free as an A-module with basis $e_1, \ldots, e_n$, and let V be a free A-module with basis $\{t_i\}_{i \in \mathscr{I}}$. For any monomial $f = t_{i_1} \otimes \cdots \otimes t_{i_p}$ in $T_A(V)$, we consider the element f^E in $E \bigotimes_A A[V \bigotimes_A E^\star]$ given as

$$f^E = \big(\sum_{k=1}^{n} e_k \otimes (t_{i_1} \otimes e_k^\star)\big) \cdots \big(\sum_{k=1}^{n} e_k \otimes (t_{i_p} \otimes e_k^\star)\big), \tag{3.4.1}$$

where $e_1^\star, \ldots, e_n^\star$ is the dual basis of $E^\star$. The element $f^E = u(f)$, where u is the universal map 1.10.1. Describing the correspondance given by u for monomials will suffice to descrbie the correspondance for arbitrary elements. We have a unique decomposition

$$f^E = \sum_{k=1}^{n} e_k \otimes f_k^E,$$

with $f_k^E \in A[V \bigotimes_A E^\star]$, for $k = 1, \dots, n$. If we expand the defining expression 3.4.1 of f^E we get that

$$f^E = \sum_{\substack{1 \leq k_i \leq n \\ i=1,\dots,p}} e_{k_1} \cdots e_{k_p} \otimes (t_{i_1} \otimes e^\star_{k_1}) \cdots (t_{i_p} \otimes e^\star_{k_p}).$$

Each monomial expression $e_{k_1} \cdots e_{k_p}$ in the free A-module E, can be written $\sum_{j=1}^n m^j(\underline{k}) e_j$ for some $m^j(\underline{k}) \in A$, with $j = 1, \dots, n$, and each ordered tuple $\underline{k} = k_1, \dots, k_p$. Therefore we get that

$$f^E = \sum_{j=1}^n e_j \otimes \big(\sum_{\substack{1 \leq k_i \leq n \\ i=1,\dots,p}} (t_{i_1} \otimes e^\star_{k_1}) \cdots (t_{i_p} \otimes e^\star_{k_p}) \cdot m^j(\underline{k}) \big).$$

In particular we have that

$$f_j^E = \sum_{\substack{1 \leq k_i \leq n \\ i=1,\dots,p}} (t_{i_1} \otimes e^\star_{k_1}) \cdots (t_{i_p} \otimes e^\star_{k_p}) \cdot m^j(\underline{k}),$$

for each $j = 1, \dots, n$.

4. Étaleness and non-commutative test rings

In this section we will show that if $B \longrightarrow R$ is étale, then the A-algebra $\underline{\mathrm{Hom}}_B(R, E)$ is étale. It is notable that the corresponding statement for smoothness does not hold. From now on all rings, unless otherwise stated, are commutative.

Proposition 4.1. *Let $g\colon A \longrightarrow B$ and $f\colon B \longrightarrow R$ be homomorphism of rings, and let $\mu\colon B \longrightarrow E$ be an A-algebra homomorphism where E is not necessarily commutative. If $f\colon B \longrightarrow R$ is of finite presentation, then also the functor $\underline{\mathrm{Hom}}_B(R, E)$ is of finite presentation.*

Proof. Let $\{A_\alpha\}_{\alpha \in \mathscr{A}}$ be a directed system of A-algebras (and A-algebra homomorphisms). We need to show that the natural map of sets

$$\varinjlim \underline{\mathrm{Hom}}_B(R, E)(A_\alpha) \longrightarrow \underline{\mathrm{Hom}}_B(R, E)(\varinjlim A_\alpha) \tag{4.1.1}$$

is a bijection. Injectivity is clear. We will show surjectivity. Let $A' = \lim_\to A_\alpha$, and let $\xi \in \underline{\mathrm{Hom}}_B(R, E)(A')$. Then we have the following commutative diagram of B-algebras

$$\begin{tikzcd}
R \arrow[r] & R \bigotimes_A A' \arrow[dr, "\xi"] & \\
B \arrow[u, "f"] \arrow[r] & B \bigotimes_A A' \arrow[u, "f\otimes 1"] \arrow[r, "\mu\otimes 1"] & E \bigotimes_A A'.
\end{tikzcd}$$

We let $S = \mathrm{im}(\xi) \subseteq E \bigotimes_A A'$, which is a commutative B-algebra. For each α we let

$$c_\alpha\colon E \bigotimes_A A_\alpha \longrightarrow E \bigotimes_A A' = \varinjlim (E \bigotimes_A A_\alpha)$$

denote the canonical map. Since R is finitely generated as a B-algebra, so is the B-algebra S. Let $s_1, \dots, s_m$ generate S. Then there exists an index $\beta' \in \mathscr{A}$ such that for $\alpha \geq \beta'$ the B-algebra $E \bigotimes_A A_\alpha$ contains elements $x_1^\alpha, \dots, x_m^\alpha$ such that their images $c_\alpha(x_i^\alpha) = s_i$, for $i = 1, \dots, m$. Let S_α be the B-subalgebra $S_\alpha \subseteq E \bigotimes_A A_\alpha$ generated by $x_1^\alpha, \dots, x_m^\alpha$. Furthermore, for each x_α and y_α in S_α the image of the commutator $x_\alpha y_\alpha - y_\alpha x_\alpha$ is zero in S, since S is commutative. Therefore there exists an index $\beta \in \mathscr{A}$ such that for any $\alpha \geq \beta$ the B-algebra S_α contains preimages $x_1^\alpha, \dots, x_m^\alpha$ of the generators of S, and these preimages commute. Hence $\{S_\alpha\}_{\alpha \geq \beta}$ is a directed system of commutative B-algebras. Moreover, we have that $S = \lim_{\rightarrow} S_\alpha$. As $B \longrightarrow R$ is of finite presentation, we have ([Gro66, 8.14.2.2]) a bijection of sets

$$\lim_{\rightarrow}(\mathrm{Hom}_{B\text{-alg}}(R, S_\alpha)) = \mathrm{Hom}_{B\text{-alg}}(R, S).$$

Hence the B-algebra homomorphism $\xi \colon R \longrightarrow S \subseteq E \bigotimes_A A'$ corresponds to $\xi = \{\xi_\alpha\}$, where ξ_α is a B-algebra homomorphism that fits into the commutative diagram

$$\begin{tikzcd}
R \bigotimes_A A_\alpha \arrow[rd, "\xi_\alpha"] & \\
B \bigotimes_A A_\alpha \arrow[u, "f\otimes 1"] \arrow[r, "\mu\otimes 1"] & E \bigotimes_A A_\alpha .
\end{tikzcd}$$

Thus $\xi_\alpha \in \underline{\mathrm{Hom}}_B(R, E)(A_\alpha)$, and we have that 4.1.1 is surjective. $\square$

Remark 4.2. It follows from the construction of the representing object, given in Proposition 1.12, that if $f \colon B \longrightarrow R$ is of finite type, then $\underline{\mathrm{Hom}}_B(R, E)$ is of finite type.

Proposition 4.3. *Let $g \colon A \longrightarrow B$ and $f \colon B \longrightarrow R$ he homomorphism of rings, and let $\mu \colon B \longrightarrow E$ be an A-algebra homomorphism where E is not necessarily commutative. If $f \colon B \longrightarrow R$ is étale, then the functor $\underline{\mathrm{Hom}}_B(R, E)$ satisfies the formal lifting criteria for étaleness. In particular, if E is finitely generated and projective as an A-module, then the A-algebra $\underline{\mathrm{Hom}}_B(R, R)$ is étale.*

Proof. We show that $\underline{\mathrm{Hom}}_B(R, E)$ satisfies the formal lifting property for étaleness. Having established that together with Proposition 4.1 proves étaleness. Therefore, let A' be an A-algebra, and let $N \subseteq A'$ be a nilpotent ideal. An A'/N-valued point of $\underline{\mathrm{Hom}}_B(R, E)$ is given by a B-algebra homomorphism $\xi \colon R \longrightarrow E \bigotimes_A A'/N$ making the commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\xi"] & E \bigotimes_A A'/N \\
B \arrow[u, "f"] \arrow[r, "\mu\otimes 1"] & E \bigotimes_A A' . \arrow[u]
\end{tikzcd}$$

We need to show that there exists a unique lift $\tilde{\xi}\colon R \longrightarrow E \bigotimes_A A'$ of the homomorphism ξ. The twist is that E is not necessarily commutative.

Assume first that B is a local ring, and let B^{hs} denote its strict Henselisation. We have that $f \otimes 1 = f^{hs}\colon B^{hs} \longrightarrow R \bigotimes_B B^{hs}$ is étale. By [Gro67, Proposition 18.5.11] it follows that $R \bigotimes_B B^{hs}$ is finite over B^{hs}, and therefore that $R \bigotimes_B B^{hs}$ is a finite product $\prod_{i=1}^p B^{hs}$. Then there clearly exists a lifting $\tilde{\xi^{hs}}\colon R \bigotimes_B B^{hs} \longrightarrow E \bigotimes_A A' \bigotimes_B B^{hs}$ of the B^{hs}-algebra homomorphism

$$\xi^{hs} = \xi \otimes 1\colon R \bigotimes_B B^{hs} \longrightarrow E \bigotimes_A A'/N \bigotimes_B B^{hs} .$$

The lifting $\tilde{\xi}^{hs}$ necessarily equals the fixed map $B^{hs} \longrightarrow E \bigotimes_A A' \bigotimes_B B^{hs}$ on each of the components of $R \bigotimes_A B^{hs}$, thus the lifting is unique. It is moreover clear that this lifting will satisfy the "cocycle" condition for descent when passing to the situation over $B^{hs} \bigotimes_B B^{hs}$. Therefore, as the map $B \longrightarrow B^{hs}$ is faithfully flat, we get by descent (see e.g. [BLR90, Chapter 6.1]) the existence of a unique lifting $\tilde{\xi}\colon R \longrightarrow E \bigotimes_A A'$ of the homomorphism ξ. This shows étaleness when B is a local ring.

In the general situation we get the lifting property for each localization B_p, for any prime p in $\operatorname{Spec}(B)$. Thus there is a B_p-algebra homomorphism $\tilde{\xi}_p\colon R \bigotimes_B B_p \longrightarrow E \bigotimes_A A' \bigotimes_B B_p$ lifting the induced map $\xi \otimes 1\colon R \bigotimes_B B_p \longrightarrow E \bigotimes_A A'/N \bigotimes_B B_p$. Since R is of finite type over B, the B_p-algebra homomorphism $\tilde{\xi}_p$ lifts to a B_b-algebra homomorphism $\tilde{\xi}_b\colon R \bigotimes_B B_b \longrightarrow E \bigotimes_A A' \bigotimes_B B_b$, for some b not in $p \subset B$. As the liftings $\tilde{\xi}_p$ were unique, the different $\tilde{\xi}_b$ will coincide on intersections, and therefore give a unique lifting $\tilde{\xi}\colon R \longrightarrow E \bigotimes_A A'$ of the A'/N-valued point ξ of $\underline{\operatorname{Hom}}_B(R, E)$. □

Example 4.4. We give here an example showing that if $B \longrightarrow R$ is smooth, then $\underline{\operatorname{Hom}}_B(R, E)$ is not always smooth. Consider first the matrices

$$x = \begin{bmatrix} 0 & \epsilon \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} a_1 + a_2\epsilon & b_1 + b_2\epsilon \\ c_1 + c_2\epsilon & d_1 + d_2\epsilon \end{bmatrix},$$

where the entries of the matrices are in some ring, where ϵ is a non-zero element such that $\epsilon^2 = 0$. Since

$$xy = \begin{bmatrix} a_1\epsilon & b_1\epsilon \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad yx = \begin{bmatrix} 0 & b_1\epsilon \\ 0 & c_1\epsilon \end{bmatrix},$$

these matrices do not in general commute. However, when we set $\epsilon = 0$, the matrix x becomes the zero matrix and the reduced matrices clearly commute. Therefore we have the following. Let $A = k[\epsilon]/(\epsilon^2)$, over a field k. Let $B = A[X]$ the polynomial ring in one variable X over A, and let $M = A \oplus A$. The matrix x gives a B-module structure on M by sending the variable X to the matrix x. Thus we have an A-algebra homomorphism $\mu\colon B \longrightarrow \mathrm{End}_A(A \oplus A) = E$. We let $R = A[X,Y]$ be the polynomial ring in two variables over A. We let furthermore $A' = A$, and the nilpotent ideal $N = (\epsilon) \subseteq A$. We then have the following commutative diagram

$$\begin{tikzcd}
A \arrow[r] & B = A[X] \arrow[r, "f"] \arrow[d, "\mu"] & R = A[X,Y] \arrow[d, "\xi"] \\
 & \mathrm{End}_A(A \oplus A) \arrow[r] & \mathrm{End}_k(k \oplus k) = E \otimes_A k,
\end{tikzcd}$$

where $\xi\colon R \longrightarrow \mathrm{End}_k(k \oplus k)$ is determined by sending X to 0 and sending Y to the endomorphism given by the matrix $\overline{y} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Any lifting of $\overline{y}$ to an element in $\mathrm{End}_A(A \oplus A)$ is of the form

$$y = \begin{bmatrix} 1 + a_2\epsilon & b_2\epsilon \\ c_2\epsilon & d_2\epsilon \end{bmatrix}$$

with elements a_2, b_2, c_2, d_2 in k. From the considerations above we have that no such lifting will commute with the matrix x. Therefore there exist no B-algebra homomorphism $\tilde{\xi}\colon R \longrightarrow \mathrm{End}_A(A \oplus A)$ that extends ξ. Thus, even if $f\colon B \longrightarrow R$ is smooth, the A-algebra $\underline{\mathrm{Hom}}_B(R, E)$, and $\mathscr{M}od^M_{B \to R}$, are not necessarily smooth.

4.5. A result about glueing. For any element x in a ring R we let $R_x = R[T]/(Tx - 1)$ denote the localization of R at x.

Lemma 4.6. *Let E be an A-algebra which is finitely generated and projective as an A-module, and let $A \longrightarrow B \longrightarrow R$ be homomorphism of commutative rings. For any element x in R the scheme* $\mathrm{Spec}(\underline{\mathrm{Hom}}_B(R_x, E))$ *is an open subscheme of* $\mathrm{Spec}(\underline{\mathrm{Hom}}_B(R, E))$. *Moreover, if y is another element of R, then we have*

$$\mathrm{Spec}(\underline{\mathrm{Hom}}_B(R_{xy}, E)) = \mathrm{Spec}(\underline{\mathrm{Hom}}_B(R_x, E)) \cap \mathrm{Spec}(\underline{\mathrm{Hom}}_B(R_y, E)).$$

Proof. Let C denote the A-algebra that represents $\underline{\mathrm{Hom}}_B(R, E)$, and let $\xi\colon R \longrightarrow E \bigotimes_A C$ denote the universal map. Let x be an element of R. Then the universal map $\xi\colon R \longrightarrow E \bigotimes_A C$ factorizes via $R \longrightarrow R_x$ if and only if $\xi(x)$ is a unit in $E \bigotimes_A C$. The element $\xi(x)$ in the finitely generated and projective C-module $E \bigotimes_A C$ is a unit if and only if $d(x) = \det(e \mapsto e \cdot \xi(x))$ in C is invertible. It follows that $C_{d(x)}$ represents $\underline{\mathrm{Hom}}_B(R_x, E)$. □

5. Representability of the Quot functor

In this section we will put our result about module restricitions in a geometric context. Thereafter we will apply our results to show representability of the Quot functor.

5.1. Relative rank and finite support. Let $\mathscr{E}$ be quasi-coherent sheaf of modules on an algebraic space X. The support of $\mathscr{E}$ is the closed subspace $\mathrm{Supp}(\mathscr{E})$ of X determined by the annihilator $\mathrm{ann}(\mathscr{E})$.

Definition 5.2. Let $g\colon X \longrightarrow S$ be a morphism of algebraic spaces, and $\mathscr{E}$ a quasi-coherent sheaf on X. We say that $\mathscr{E}$ is *finite, flat of relative rank* n over S, if $\mathscr{E}$ is flat over S, and $\mathrm{Supp}(\mathscr{E})$ is finite over S, and the locally free $\mathscr{O}_S$-module $g_*\mathscr{E}$ has constant rank n.

5.3. If the quasi-coherent $\mathscr{O}_X$-module $\mathscr{E}$ is finite, flat of rank n over the base S, then it follows that $\mathscr{E}$ is coherent $\mathscr{O}_X$-module. In particular the underlying set $|\mathrm{Supp}(\mathscr{E})|$ of the support, is precisely the set of point where the sheaf $\mathscr{E}$ is non-zero.

5.4. We denote by $\mathscr{C}oh^n_{X/S}$ the stack [LMB00] of quasi-coherent sheaves on X, that are finite, flat and of relative rank n over S. A morphism between two objects $\mathscr{E}$ and $\mathscr{F}$ in $\mathscr{C}oh_{X/S}(T)$, where $T \longrightarrow S$ is a scheme over S, is an $\mathscr{O}_{X\times_S T}$-module isomorphism $\varphi\colon \mathscr{E} \longrightarrow \mathscr{F}$. If $T \longrightarrow S$ is a morphism then we have an isomorphism of stacks

$$\mathscr{C}oh^n_{X\times_S T/T} \simeq \mathscr{C}oh^n_{X/S} \times_S T. \tag{5.4.1}$$

Lemma 5.5. *Let $X \longrightarrow S$ be a separated map of algebraic spaces, and let $f\colon Y \longrightarrow X$ be a morphism of S-spaces. Then the push-forward gives a map $f_*\colon \mathscr{C}oh^n_{Y/S} \longrightarrow \mathscr{C}oh^n_{X/S}$.*

Proof. Let $T \longrightarrow S$ be a morphism with T a scheme, and let $\mathscr{E}$ be an element of $\mathscr{C}oh^n_{Y\times_S T/T}$. Since the support $Z = \mathrm{Supp}(\mathscr{E})$ by assumption is finite over T, and the map $f\colon Y \longrightarrow X$ is separated, we have that the composition $f_T\colon Z \subseteq Y \times_S T \longrightarrow X \times_S T$ is finite [Gro61, Proposition 6.15]. In particular we have that $f_{T*}\mathscr{E} = \mathscr{E}_X$ is quasi-coherent. The support of $\mathscr{E}_X$ is the image of the support of $\mathscr{E}$. Thus the support $\mathscr{E}_X$ is proper and quasi-finite over T, hence finite. If $g_T\colon X \times_S T \longrightarrow T$ is the projection map, we have by definition that $g_T \circ f_T$ is the projection map from $Y \times_S T$. It follows that $f_{T*}\mathscr{E}$ is finite, flat of relative rank n over T. □

5.6. Since the support of an element $\mathscr{E}$ in $\mathscr{C}oh^n_{X/S}(T)$ is finite, and in particular proper over the base T, it follows that

$$\mathscr{C}oh^n_{U/S} \subseteq \mathscr{C}oh^n_{X/S}$$

is an open substack for an open subspace $U \subseteq X$ (see e.g. [Gro95]).

Theorem 5.7. *Let $f\colon Y \longrightarrow X$ and $g\colon X \longrightarrow S$ be morphisms of affine schemes X, Y and S. Then we have that the push-forward map $f_*\colon \mathscr{C}oh^n_{Y/S} \longrightarrow \mathscr{C}oh^n_{X/S}$ is schematically representable.*

Proof. We want to see that for arbitrary scheme T, the fiber product

$$T \times_{\mathscr{C}oh^n_{X/S}} \mathscr{C}oh^n_{Y/S} \tag{5.7.1}$$

is representable by a module restriction. By Lemma 4.6 the module restrictions are Zariski sheaves, and we may assume that T is affine. We may, by 5.4.1, assume that $T = S$. Let $\mathscr{E}$ in $\mathscr{C}oh^n_{X/S}(S)$ be the element corresponding to a given map $S \longrightarrow \mathscr{C}oh^n_{X/S}$. Let $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, and let M be the B-module corresponding to the sheaf $\mathscr{E}$ on X. Then M is projective and finitely generated as an A-module, and we have by Theorem 2.4 the A-algebra $\mathscr{M}od^M_{B\to R}$ where $Y = \operatorname{Spec}(R)$. We have a natural map

$$\alpha\colon \operatorname{Spec}(\mathscr{M}od^M_{B\to R}) \longrightarrow S \times_{\mathscr{C}oh^n_{X/S}} \mathscr{C}oh^n_{Y/S}$$

given as follows. Let $u'\colon \operatorname{Spec}(A') \longrightarrow \operatorname{Spec}(\mathscr{M}od^M_{B\to R})$ be a morphism of affine schemes over S. By the defining properties of the module restriction $\mathscr{M}od^M_{B\to R}$, the morphism u' corresponds to a $R' = R \bigotimes_A A'$-module structure on $M' = M \bigotimes_A A'$, extending the fixed $B' = B \bigotimes_A A'$-module structure. Let $\xi\colon R' \longrightarrow \operatorname{End}_{B'}(M')$ be the B-algebra homomorphism corresponding to the R'-module structure on M'. And let $\mathscr{F}_{Y'}$ denote the corresponding quasi-coherent sheaf on $Y' = Y \times_S S'$. Then $\alpha(u') = (s', \mathscr{F}_{Y'}, \mathrm{id})$, where $s'\colon S' \longrightarrow S$ is the structure map. The map α is a monomorphism, and we need to see that it also is essentially surjective.

Let $(s', \mathscr{F}, \psi)$ be a $S' = \operatorname{Spec}(A')$-valued point of the fiber product 5.7.1. Let N be the $R' = R \bigotimes_A A'$-module corresponding to the sheaf $\mathscr{F}$ on $Y \times_S S'$. Then ψ corresponds to a $B' = B \bigotimes_A A'$-module isomorphism $\psi\colon M' = M \bigotimes_A A' \longrightarrow N$. We get an induced B'-algebra isomorphism

$$\tilde{\psi}\colon \operatorname{End}_{B'}(M') \longrightarrow \operatorname{End}_{B'}(N) \; .$$

Finally, let $\xi'\colon R' \longrightarrow \operatorname{End}_{B'}(N)$ be the B'-algebra homomorphism corresponding to the R'-module structure on N. The composition of ξ' with $\tilde{\psi}^{-1}$ gives a B'-algebra homomorphism $\xi\colon R' \longrightarrow \operatorname{End}_{B'}(M')$. By the defining properties of the module restriction $\mathscr{M}od^M_{B\to R}$ there exists a unique $u'\colon S' \longrightarrow \operatorname{Spec}(\mathscr{M}od^M_{B\to R})$ corresponding to ξ. Thus $\alpha(u')$ is isomorphic to $(s', \mathscr{F}, \psi)$, and we have shown that α is essentially surjective. □

Corollary 5.8. *Let $f\colon X \longrightarrow S$ be a separated map of an algebraic space X over a scheme S, and let $f\colon Y \longrightarrow X$ be an affine morphism of S-spaces. Then the push-forward map*

$$f_*\colon \mathscr{C}oh^n_{Y/S} \longrightarrow \mathscr{C}oh^n_{X/S}$$

is schematically representable. Moreover, if $f\colon Y \longrightarrow X$ is étale, then the representable push-forward map f_* is étale.

Proof. By 5.4.1 it suffices to show the result for affine base scheme S. By Lemma 4.6 it suffices to show representability for fibers on T-valued points of $\mathscr{C}oh^n_{X/S}$, with affine T. Then the result follows from the theorem. The result about étaleness follows from Proposition 4.3. $\square$

5.9. The Quot stack. Fix a quasi-coherent sheaf F_X on an algebraic space $X \longrightarrow S$. For any S-scheme T we let F_{X_T} denote the pull-back of F_X along the first projection $p_X \colon X \times_S T \longrightarrow X$. The T-valued points of the quot stack $\mathfrak{Quot}^n_{F_X/S}$ are all $\mathscr{O}_{X\times_S T}$-module morphisms $q \colon F_{X_T} \longrightarrow \mathscr{E}$, from F_{X_T} to a quasi-coherent sheaf $\mathscr{E}$ on $X \times_S T$, where $\mathscr{E}$ is flat, finite of relative rank n over T. A morphism between two objects $q \colon F_{X_T} \longrightarrow \mathscr{E}$ and $q' \colon F_{X_T} \longrightarrow \mathscr{E}'$ is an $\mathscr{O}_{X\times_S T}$-module isomorphism $\varphi \colon \mathscr{E} \longrightarrow \mathscr{E}'$ such that $q' = \varphi \circ q$.

Remark 5.10. Note that the maps $q \colon F_{X_T} \longrightarrow \mathscr{E}$ are not assumed to be surjective, and in particular the T-valued points of the quot stack $\mathfrak{Quot}^n_{F_X/S}$ are not quotients of F_X. The definition of the quot stack is motivated by the definition of the Hilbert stack in [Ryd], and in [Ols06].

5.11. Identification of pull-backs. We will in the sequel of this article return to a particular situation that we describe below. Let $T \longrightarrow S$ be a morphism. Then we have the following Cartesian diagram

$$\begin{tikzcd} Y \times_S T \arrow[r, "f_T"] \arrow[d, "p_Y"] & X \times_S T \arrow[d, "p_X"] \\ Y \arrow[r, "f"] & X. \end{tikzcd} \tag{5.11.1}$$

For any sheaf F_X on X there is a canonical identification between the two sheaves $f_T^* F_{X_T}$ and $p_Y^* f^* F_X$. We will denote both these two sheaves with F_{Y_T}.

Lemma 5.12. *Let $X \longrightarrow S$ be a separated map of algebraic spaces, and $f \colon Y \longrightarrow X$ a map of S-spaces. Let F_X be a quasi-coherent sheaf on X, and let $F_Y = f^* F_X$ denote its pull-back to Y. There is a natural induced map*

$$f_* \colon \mathfrak{Quot}^n_{F_Y/S} \longrightarrow \mathfrak{Quot}^n_{F_X/S}\ .$$

Proof. Let $q \colon F_{Y_T} \longrightarrow \mathscr{E}$ be a T-valued point of $\mathfrak{Quot}^n_{F_Y/S}$. The canonical map $F_{X_T} \longrightarrow f_{T*} f_T^* F_{X_T}$, where we use the notation of 5.11.1, combined with the identification $f_T^* F_{X_T} = F_{Y_T}$, gives the composition

$$F_{X_T} \longrightarrow f_{T*} f_T^* F_{X_T} = f_{T*} F_{Y_T} \longrightarrow f_T^* \mathscr{E}\ .$$

By Lemma 5.5 the above sequence is a T-valued point of $\mathfrak{Quot}^n_{F_X/S}$. $\square$

5.13. We have a natural map

$$c\colon \mathfrak{Quot}^n_{F_X/S} \longrightarrow \mathscr{C}oh^n_{X/S}$$

that takes a T-valued point of the quotient stack $q\colon F_{X_T} \longrightarrow \mathscr{E}$ to the sheaf $\mathscr{E}$.

Lemma 5.14. *Let $X \longrightarrow S$ be a separated map of algebraic spaces, and let $f\colon Y \longrightarrow X$ be a morphism of S-spaces. For any quasi-coherent sheaf F_X on X, we have the Cartesian diagram*

$$\begin{tikzcd}
\mathfrak{Quot}^n_{F_X/S} \arrow[r, "c"] & \mathscr{C}oh^n_{X/S} \\
\mathfrak{Quot}^n_{F_Y/S} \arrow[r, "c"] \arrow[u, "f_*"] & \mathscr{C}oh^n_{Y/S}. \arrow[u, "f_*"']
\end{tikzcd}$$

Proof. By Lemma 5.12 we have have the needed map. The proof is then a formal consequence of adjunction. □

5.15. The Quot functor. Let $X \longrightarrow S$ be a morphism of algebraic spaces, and let F_X be a quasi-coherent sheaf on X. The Quot functor $\underline{\mathrm{Quot}}^n_{F_X/S}$ defined by Grothendieck ([Gro95], [Art69]) is the functor that to each S-scheme $T \longrightarrow S$ assigns the set of surjective $\mathscr{O}_{X_T}$-module maps $q\colon F_{X_T} \longrightarrow \mathscr{E}$, where $\mathscr{E}$ is finite, flat and of relative rank n. Two surjective maps $q\colon F_{X_T} \longrightarrow \mathscr{E}$ and $q'\colon F_{X_T} \longrightarrow \mathscr{E}'$ are considered as equal if their kernels coincide as subsheaves of F_{X_T}.

5.16. Let $q\colon F_{X_T} \longrightarrow \mathscr{E}$ be a T-valued point of $\underline{\mathrm{Quot}}^n_{F_X/S}$. We define

$$\iota(\mathscr{E}) = \mathscr{O}_{X\times_S T}/\mathrm{ker} q.$$

This determines a map $\iota\colon \underline{\mathrm{Quot}}^n_{F_X/S} \longrightarrow \mathfrak{Quot}^n_{F_X,S}$.

5.17. Open subfunctor. Let $f\colon Y \longrightarrow X$ be a morphism of S-spaces, with $X \longrightarrow S$ separated, and let F_X be a quasi-coherent sheaf on X. We define $\Omega^F_{Y\to X}$ as the subfunctor of $\underline{\mathrm{Quot}}^n_{F_Y/S}$ whose T-valued points are surjective $\mathscr{O}_{Y_T}$-module maps $q\colon F_{Y_T} \longrightarrow \mathscr{E}$, where $\mathscr{E}$ is finite, flat of rank n over T, such that the induced map of $\mathscr{O}_{X_T}$-modules

$$f_*(q)\colon F_{X_T} \longrightarrow f_{T*}\mathscr{E}$$

is surjective, where f_* is the map of Lemma 5.12.

Lemma 5.18. *Let $X \longrightarrow S$ be a separated map of algebraic spaces, and let $f\colon Y \longrightarrow X$ be a map of S-spaces. For any quasi-coherent sheaf F_X on X we have that $\Omega^F_{Y\to X}$ is an open subfunctor of $\underline{\mathrm{Quot}}^n_{F_Y/S}$.*

Moreover, we have the following Cartesian diagram

$$\begin{tikzcd}
\underline{\mathrm{Quot}}^n_{F_X/S} \arrow[r, "\iota"] & \mathfrak{Quot}^n_{F_X/S} \\
\Omega^F_{Y\to X} \arrow[u] \arrow[r, "\iota"] & \mathfrak{Quot}^n_{F_Y/S}. \arrow[u, "f_*"]
\end{tikzcd}$$

Proof. Let $q\colon F_{Y_T} \longrightarrow \mathscr{E}$ be a T-valued point of $\Omega^F_{Y\to X}$. By Lemma 5.5 the sheaf $f_{T*}\mathscr{E}$ is quasi-coherent on $X \times_S T$. Furthermore, since $f_{T*}\mathscr{E}$ is finite over the base it follows that surjectivity of the map $f_*(q)\colon F_{X_T} \longrightarrow f_{T*}\mathscr{E}$ is an open condition on the base, proving the first claim.

To prove the second claim, we start by noting that there is a natural map $\alpha\colon \Omega^F_{Y\to X} \longrightarrow P$, where P is the fiber product in question. If $q\colon F_{Y_T} \longrightarrow \mathscr{E}$ is a T-valued point of $\Omega^F_{Y\to X}$, then by assumption $f_*(q)\colon F_{X_T} \longrightarrow f_{T*}\mathscr{E}$ is surjective. So $\alpha(q) = (f_*(q), i(q), \mathrm{id})$. The map α is full and faithful.

Let (q_X, s, ψ) be a T-valued point of the fiber product P, where $q_X\colon F_{X_T} \longrightarrow \mathscr{E}$ is surjective, $s\colon F_{Y_T} \longrightarrow \mathscr{F}$ is a map of $\mathscr{O}_{Y\times_S T}$-modules, $\mathscr{E}$ and $\mathscr{F}$ are finite, flat of relative rank n over the base, and where ψ is an isomorphism of $\mathscr{O}_{X\times_S T}$-modules, making the commutative diagram

$$\begin{tikzcd}
f_{T*}f_T^*F_{X_T} = f_{T*}F_{Y_T} \arrow[r] & f_{T*}\mathscr{F} \\
F_{X_T} \arrow[u] \arrow[r, "q_X"] & \mathscr{E}. \arrow[u, "\psi"]
\end{tikzcd}$$

Consider now the pull-back $f_T^*(q_X)$ composed with the adjoint of ψ,

$$q_Y\colon F_{Y_T} = f_T^*F_{X_T} \longrightarrow f_T^*\mathscr{E} \longrightarrow \mathscr{F}.$$

We claim that the $\mathscr{O}_{Y\times_S T}$-module map q_Y is surjective. To see this we may restrict ourselves to the support $\mathrm{Supp}(\mathscr{F}) \subseteq Y \times_S T$ of $\mathscr{F}$. The restriction of $f\colon Y \times_S T \longrightarrow X \times_S T$ to the support of $\mathscr{F}$ is finite ([Gro61, Proposition 6.15]), and in particular affine. It is then clear that since $\psi\colon \mathscr{E} \longrightarrow f_{T*}\mathscr{F}$ is an isomorphism, and then in particular surjective, the adjoint map $f^*\mathscr{E} \longrightarrow \mathscr{F}$ is also surjective. Since q_X is surjective by assumption, so is its pull-back $f_T^*(q)$. Thus, the map q_Y is the composition of two surjective maps, and therefore surjective. So q_Y is consequently a T-valued point of $\underline{\mathrm{Quot}}^n_{F_Y/S}$. Moreover, since ψ is an isomorphism it follows that q_Y is a T-valued point of $\Omega^F_{Y\to X}$. We then have that $f_*(q_Y)$ is isomorphic to (q_X, s, ψ), and thus that α is essentially surjective. □

Theorem 5.19. *Let $X \longrightarrow S$ be a separated morphism of algebraic spaces, and let F_X be a quasi-coherent sheaf on X. For each integer n, the functor $\underline{\mathrm{Quot}}^n_{F_X/S}$ is representable by an algebraic space.*

Proof. As the Quot functor commutes with base change, we may reduce to the case with the base S being an affine scheme $S = \mathrm{Spec}(A)$. Moreover, any T-valued point $q\colon F_{X_T} \longrightarrow \mathscr{E}$ of $\underline{\mathrm{Quot}}^n_{X_F/S}$ is such that the support $\mathrm{Supp}(\mathscr{E})$ is finite over the base. Hence the support of the quotient $\mathscr{E}$ is contained in an open quasi-compact $U \subseteq X$. Therefore we have that

$$\lim_{\substack{U\subseteq X \\ \text{open, q-compact}}} \underline{\mathrm{Quot}}^n_{F_U/S} = \underline{\mathrm{Quot}}^n_{F_X/S}.$$

Hence it suffices to show the theorem for $X \longrightarrow S$ being quasi-compact. With $X \longrightarrow S$ quasi-compact we can find an affine scheme $Y \longrightarrow S$ with an étale, affine and surjective map $f\colon Y \longrightarrow X$. With affine schemes $Y \longrightarrow S$ we have that $\underline{\mathrm{Quot}}^n_{F_Y/S}$ is represented by a scheme ([GLS07]). By Lemma 5.18 the subfunctor $\Omega^F_{Y\to X}$ is open in $\underline{\mathrm{Quot}}^n_{F_Y/S}$, hence a scheme. Combining the Cartesian diagrams in 5.18 and 5.14, together with Theorem 5.8, give that the induced map

$$\Omega^F_{Y\to X} \longrightarrow \underline{\mathrm{Quot}}^n_{F_X/S} \tag{5.19.1}$$

is representable and étale.

We then want to see that the map (5.19.1) is surjective. Let k be a field and $F_{X_k} \longrightarrow \mathscr{E}$ be a $\mathrm{Spec}(k)$-valued point of $\underline{\mathrm{Quot}}^n_{F_X/S}$ where we use the notation $X_k = X \times_S \mathrm{Spec}(k)$. We want to show that there exists a separable field extension $k \longrightarrow L$ such that the corresponding $\mathrm{Spec}(L)$-valued point lifts to $\Omega^F_{Y\to X}$.

The reduced support $Z = |\,\mathrm{Supp}(\mathscr{E})\,|$ is a disjoint union of a finite set of points, given by finite field extensions $k \longrightarrow k_i$ with $i = 1, \dots, m$. Then $f^{-1}(\mathrm{Spec}(k_i))$ is also a finite union of points $\sqcup_{j_i=1}^{m_i} \mathrm{Spec}(L_{j_i})$, with $k_i \longrightarrow L_{j_i}$ a finite separable field extension for $j_i = 1, \dots, m_i$. There exists a finite separable field extension $k \longrightarrow L$ such that the induced map $k_i \otimes_k L \longrightarrow L_{j_i} \otimes_k L$ splits, for all $i = 1, \dots, m$ and all $j_i = 1, \dots, m_i$. Then

$$f^{-1}(Z) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L) \longrightarrow Z \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$$

has a section, and we have that the corresponding $\mathrm{Spec}(L)$-valued point of $\underline{\mathrm{Quot}}^n_{F_X/S}$ lifts to $\Omega^F_{Y\to X}$. We then have proven surjectivity, and consequently [RG71] that $\underline{\mathrm{Quot}}^n_{F_X/S}$ is an algebraic space. □

Remark 5.20. With $F_X = \mathscr{O}_X$ the structure sheaf on X, the Quot functor $\underline{\mathrm{Quot}}^n_{F_X/S}$ is the Hilbert functor $\underline{\mathrm{Hilb}}^n_{X/S}$. The situation with the Hilbert scheme was considered in [ES04].

Remark 5.21. The separated assumption of $X \longrightarrow S$ is a necessary condition for representability [LS08]. On the other hand there exist examples of separated schemes $X \longrightarrow S$ for which the Quot functor is not represented by a scheme [Knu71], but only an algebraic space. Thus when considering representability, the setting with separated algebraic spaces $X \longrightarrow S$ is the natural one.

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DEPARTMENT OF MATHEMATICS, KTH, STOCKHOLM, SWEDEN
E-mail address: `skjelnes@kth.se`