Exercises, Chapter 1 Atiyah-MacDonald (AM)

Exercise 1 (AM, 1.14, p.12). *In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.*

Exercise 2 (AM, 1.15, p.12). *Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that the collection of sets of the form $V(E)$ satisfy the axioms for the closed sets in a topological space. The topological space X is called the prime spectrum of A, and is written* $\operatorname{Spec}(A)$.

Exercise 3. *Draw pictures of the prime spectrum of the rings*

$$\mathbb{Z}, \quad \mathbb{C}[X], \quad \mathbb{C}[X,Y], \quad \mathbb{C}[X,Y]/(XY) \quad \text{and} \quad \mathbb{C}[X^2, XY, Y^2]$$

and the maps arising from $\mathbb{C}[X] \to \mathbb{C}[X,T]$ taking $X \mapsto X$, and the map taking $X \mapsto X - T$. Finally, a nice picture of the natural map $\mathbb{C}[X] \to \mathbb{C}[X,T]/(X^2+TX+T^2)$, and the map $\mathbb{C}[X,Y] \to \mathbb{C}[T]$ taking $X \mapsto T$ and $Y \mapsto T$.

Exercise 4 (AM, 1.17, p.12). *For each $f \in A$, let X_f denote the complement of $V(f)$ in $X = \operatorname{Spec}(A)$. The sets are open. Show that they form a basis for the Zariski topology on X. Show that X_f is always quasi-compact.*

Exercise 5 (AM, 1.21+, p.13). *Let $\phi\colon A \to B$ be a ring homomorphism. Let $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$. We have a natural induced mapping $\phi^*\colon Y \to X$. Show that*

(1) *If $f \in A$ then $\phi^{*-1}(X_f) = Y_{\phi(f)}$, and hence that ϕ^* is continuous.*
(2) *If I is an ideal of A, then $\phi^{*-1}(V(I)) = V(\phi(I)B)$.*
(3) *If J is an ideal of B, then $\overline{\phi^*(V(J))} = V(\phi^{-1}(J))$.*
(4) *Given an example where $\phi^*(V(J))$ not equals $V(\phi^{-1}(J))$.*
(5) *If ϕ is surjective, then ϕ^* is a homeomorphism of Y onto the closed subset $V(\operatorname{Ker}\phi)$.*
(6) *If ϕ is injective, then $\phi^*(Y)$ is dense in X.*

Exercise 6 (AM, 1.22, p.13). *Let $A = \prod_{i=1}^n A_i$ be the direct product of rings A_i. Show that $\operatorname{Spec}(A)$ is the disjoint union open (and closed) subspaces $\operatorname{Spec}(A_i)$.*

Exercise 7. *Let A be a ring with only a finite number of prime ideals, that all are maximal. Show that A is the direct of local rings A_i.*

Exercise 8. *Let P_1, P_2 and P_3 be three different points in $\mathbb{C}^2$, and where the points are not aligned. Let $l_i \in \mathbb{C}[X,Y]$ be a linear form such that the maximal elements in $V(l_i)$ is the unique line passing through* *two of the points P_1, P_2, P_3, but not P_i ($i = 1, 2, 3$). Show that the ring $\mathbb{C}[X, Y]/(l_1 l_2, l_2 l_3, l_3 l_1)$ is isomorphic to three copies of $\mathbb{C}$.*