Exercises, Chapter 4 Atiyah-MacDonald (AM)

Exercise 1. *A linear form l in the polynomial ring $k[x, y]$, over a field k, is an element of the form $l = ax + by + c$, with scalars a, b and c. If $(a, b) \neq (0, 0)$, then the closed set $V(l)$ is a line in $\mathrm{Spec}(k[x, y])$.*

1. *Determine an expression in the coefficients of the linear form that distinguishes parallel lines.*
2. *Let l_1 and l_2 be two lines that are not parallel. Show that we have an isomorphism of rings $k[x, y] = k[l_1, l_2]$.*
3. *Let l_1, l_2 and l_3 be three lines, of which none are parallel. Compute the vector space dimension of $k[x, y]/(l_1l_2, l_1l_3, l_2l_3)$.*

Exercise 2 (Eisenbud=E, 1.19, p. 53)**.** *In $k[x, y, z, w]$, the polynomial ring over a field k, let $I = (xz - y^2, xw - yz, yw - z^2)$. The quotient algebra $R = k[x, y, z, w]/I$ is an $S = k[x, w]$-module. Show that R is a free, of finite rank, S-module. However, not even finitely generated as an $k[x, y]$-module.*

Exercise 3. *Show that $A = k[t]_t$ is isomorphic to $k[x, y]/(xy - 1)$. Why is the spectrum $\mathrm{Spec}(A)$ denoted G_m or k^*?*

Exercise 4. *Show that $\mathbb{C}[x, y]/(y-x^2)$ (the parabola) is not isomorphic to the ring $\mathbb{C}[x, y]/(xy - 1)$ (the hyperbola). But the ring $\mathbb{C}[x, y]/(x^2 + y^2 - 1)$ (the circle) is isomorphic to the hyperbola.*

Exercise 5. *Let V be a free $\mathbb{Z}$-module of rank 2. For any ring A, let $V_A = V \otimes_{\mathbb{Z}} A$, which is also free of rank 2 as an A-module.*

1. *Construct a ring R and a R-linear map $\xi\colon V_R \to V_R$ with the following universal property. For any ring A, and any A-linear map $f\colon V_A \to V_A$ there exists a unique ring homomorphism $\varphi : R \to A$ such that $\xi \otimes 1 = f$ as endomorphisms on $V_R \otimes_R A = V_A$.*
2. *Construct a ring R' and a R'-linear map $\xi'\colon V_{R'} \to V_{R'}$ with the following universal property. For any ring A, and any A-linear invertible map $f\colon V_A \to V_A$ there exists a unique ring homomorphism $\varphi\colon R' \to A$ such that $\xi' \otimes 1 = f$ as endomorphisms on $V_{R'} \otimes_{R'} A = V_A$.*