Exercises, Chapter 10 Atiyah-MacDonald (AM)

Exercise 1. *Let I be an ideal in a ring A, and let $\hat{A}$ denote the I-adic completion. That is*

$$\hat{A} = \varprojlim_n A/I^n.$$

Let M be an A-module. Show that two different I-stable filtrations of I, give the same I-adic topology on M. In other words, if (M_n) and (M'_n) are I-stable filtrations, we have an equality of $\hat{A}$-modules

$$\varprojlim_n M/M_n = \varprojlim_n M/M'_n.$$

Exercise 2. *Let A be a ring and I an ideal. For any A-module M we let $\hat{M}$ denote the I-adic completion of M, that is $\hat{M} = \varprojlim_n M/I^n M$. Show the following*

(1) *If $F \to M$ is a map of A-modules, we get an induced map of $\hat{A}$-modules $\hat{F} \to \hat{M}$.*
(2) *If F is free A-module, then $\hat{F} = F \otimes_A \hat{A}$. Do you need that F is of finite rank?*
(3) *If $F \to M$ is a surjection, with F free of finite rank, then the induced map $\hat{F} \to \hat{M}$ is surjective. Do you need that F is finite rank, and do you need that F is free?*

Exercise 3 (AM, 10.9, p. 115)**.** *Let A be a local ring with maximal ideal $\mathfrak{m}$. For any polynomial $f(x) \in A[x]$ we let $\overline{f}(x) \in A/\mathfrak{m}[x]$ denote its reduction modulo $\mathfrak{m}$. Assume that A is $\mathfrak{m}$-adically complete. Prove Hensel's Lemma: Assume that $f(x)$ is monic of degree n, and that $\overline{f}(x) = \overline{g}(x)\overline{h}(x)$, where $\overline{g}(x)$ and $\overline{h}(x)$ are monic and coprime. Then we can find monic polynomials $g(x)$ and $h(x)$ whose reductions modulo $\mathfrak{m}$ are $\overline{g}(x)$ and $\overline{h}(x)$, and where $f(x) = g(x)h(x)$.*