# RECOVERING THE GOOD COMPONENT OF THE HILBERT SCHEME

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Abstract. We give an explicit construction, for a flat map $X \to S$ of algebraic spaces, of an ideal in the $n$'th symmetric product of $X$ over $S$. Blowing up this ideal is then shown to be isomorphic to the schematic closure in the Hilbert scheme of length $n$ subschemes of the locus of $n$ distinct points. This generalises Haiman's corresponding result ([14]) for the affine complex plane. However, our construction of the ideal is very different from that of Haiman, using the formalism of divided powers rather than representation theory.

In the non-flat case we obtain a similar result by replacing the $n$'th symmetric product by the $n$'th divided power product.

The Hilbert scheme, $\mathrm{Hilb}^n_{X/S}$, of length $n$ subschemes of a scheme $X$ over some $S$ is in general not smooth even if $X \to S$ itself is smooth. Even worse, it may not even be (relatively) irreducible. In the case of the affine plane over the complex numbers (where the Hilbert scheme is smooth and irreducible) Haiman (cf. [14]) realised the Hilbert scheme as the blow-up of a very specific ideal of the $n$'th symmetric product of the affine plane. It is the purpose of this article to generalise Haiman's construction. As the Hilbert scheme in general is not irreducible while the symmetric product is (for a smooth geometrically irreducible scheme over a field say) it does not seem reasonable to hope to obtain a Haiman like description of all of $\mathrm{Hilb}^n_{X/S}$ and indeed we will only get a description of the schematic closure of the open subscheme of $n$ distinct points. With this modification we get a general result which seems very close to that of Haiman. The main difference from the arguments of Haiman is that we need to define the ideal that we want to blow up in a general situation and Haiman's construction seems to be too closely tied to the 2-dimensional affine space in characteristic zero.

As a bonus we get that our constructions work very generally. We have thus tried to present our results in a generality that should cover reasonable applications (encouragement from one of the referees has made us make it more general than we did in a previous version of this article).

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There are some rather immediate consequences of this generality. The first one is that we have to work with algebraic spaces instead of schemes as otherwise the Hilbert scheme (as well as the symmetric product) may not exist. A second consequence is that we find ourselves in a situation where existing references do not ensure the existence of $\mathrm{Hilb}^n_{X/S}$ and we give an existence proof in the generality required by us (which is a rather easy patching argument to reduce it to known cases).

It turns out that the key to constructing the ideal to blow up is to use the formalism of divided powers. Recall that if $A$ is a commutative ring and $F$ a flat $A$-algebra, then the subring of $\mathfrak{S}_n$-invariants of $F^{\otimes_A n}$ is isomorphic to the $n$'th divided power algebra $\Gamma^n_A(F)$ (through the map that takes $\gamma^n(r)$ to $r^{\otimes n}$).

Using the fact that $\Gamma^n(F)$ is the degree $n$ component of the divided power algebra $\Gamma^*(F)$ we can define an ideal in $\Gamma^n(F)$ (this graded component of the divided power algebra becomes an algebra using the multiplication of $F$) which is our candidate to be blown up. Note that in the definition of this ideal we are using in an essential way the multiplication in the divided power algebra $\Gamma^*(F)$ forcing us to carefully distinguish between the multiplication in this graded algebra and the multiplication of its graded component $\Gamma^n(F)$ induced by the multiplication on $F$. On the upside it is exactly this interplay that allows us to define, in a generality outside of Haiman's case, the ideal. Furthermore, the excellent formal properties of $\Gamma^n(F)$ allows us to *define* an analogue of the symmetric product of $\mathrm{Spec}(F) \to \mathrm{Spec}(A)$ as $\mathrm{Spec}(\Gamma^n(F)) \to \mathrm{Spec}(A)$ in the case when $A \to F$ is not flat. This makes our arguments go through without problems in the case when $\mathrm{Spec}(F) \to \mathrm{Spec}(A)$ is not necessarily flat. (We also need to extend the construction of $\mathrm{Spec}(\Gamma^n(F))$ to the non-affine case; the gluing argument needed to make this extension uses results of David Rydh, [21].)

In more detail this paper has the following structure: We start with some preliminaries on divided powers and recall of the Grothendieck-Deligne norm map. The main technical result is to be found in Sections 5 and 6. There we first find a (local) formula for the multiplication of the tautological rank $n$-algebra over the configuration space of $n$ distinct points of $X$. We then note that this formula makes sense over the blow-up of a certain ideal in the full symmetric product. This gives us a family of length $n$ subschemes of $X$ over this blow-up and hence a map of it to the Hilbert scheme. Once having constructed it, it is quite easy to show that it gives an isomorphism of the blow-up to the schematic closure of the subspace of $n$ distinct points of the Hilbert scheme. The proof first does this in the case $X \to S$ is affine and then discusses the patching (and limit arguments) needed to extend it to the more general case.

We finish by tying some loose ends. First we generalise the result of Fogarty on the smoothness of $\mathrm{Hilb}^n_{X/S}$ for $X \to S$ smooth of relative dimension 2 removing the conditions on the base $S$ needed by Fogarty. Finally, we discuss how one can, under suitable conditions, embed the blow-up in a Grassmannian as Haiman does.

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## 1. Divided powers and norm

In this section we first recall some properties for the ring of divided powers. The standard reference is Roby [19] and [20], but see also [5] and [9]. Algebras in this note are commutative and unital.

1.1. **The ring of divided powers.** Let $A$ be a commutative ring and $M$ an $A$-module. The ring of divided powers $\Gamma_A M$ is constructed as follows. We consider the polynomial ring over $A[\gamma^n(x)]_{(n,x)\in \mathbf{N}\times M}$, where the variables $\gamma^n(x)$ are indexed by the set $\mathbf{N}\times M$, where $\mathbf{N}$ is the set of non-negative integers. Then the ring $\Gamma_A M$ is obtained by dividing out the polynomial ring by the following relations

$$\gamma^0(x) - 1 \tag{1.1.1}$$

$$\gamma^n(\lambda x) - \lambda^n \gamma^n(x) \tag{1.1.2}$$

$$\gamma^n(x+y) - \sum_{j=0}^{n} \gamma^j(x)\gamma^{n-j}(y) \tag{1.1.3}$$

$$\gamma^n(x)\gamma^m(x) - \binom{n+m}{n}\gamma^{n+m}(x) \tag{1.1.4}$$

for all integers $m, n \in \mathbf{N}$, all $x, y \in M$, and all $\lambda \in A$. The residue class of the variable $\gamma^n(x)$ in $\Gamma_A M$ we denote by $\gamma^n_M(x)$, or simply $\gamma^n(x)$ if no confusion is likely to occur. The ring $\Gamma_A M$ is graded where $\gamma^n(x)$ has degree $n$, and with respect to this grading we write $\Gamma_A M = \bigoplus_{n\geq 0} \Gamma^n_A M$.

1.2. **Polynomial laws.** Let $A$ be a ring, and let $M$ and $N$ be two fixed $A$-modules. Assume that we for each $A$-algebra $B$ have a map *of sets* $g_B \colon M \bigotimes_A B \longrightarrow N \bigotimes_A B$ such that for any $A$-algebra homomorphism

$u\colon B \longrightarrow B'$ the following diagram is commutative

$$\begin{array}{ccc} M \bigotimes_A B & \xrightarrow{g_B} & N \bigotimes_A B \\ \downarrow & & \downarrow \\ M \bigotimes_A B' & \xrightarrow{g_{B'}} & N \bigotimes_A B', \end{array}$$

where the vertical maps are the canonical homomorphisms. Such a collection of maps is called a polynomial law from $M$ to $N$, and we denote the polynomial law with $\{g\}\colon M \longrightarrow N$.

**Definition 1.3** (Norms)**.** Let $A$ be a ring, $M$ and $N$ two $A$-modules.

(1) A polynomial law $\{g\}\colon M \longrightarrow N$ is *homogeneous of degree* $n$ if for any $A$-algebra $B$ we have that $g_B(bx) = b^n g_B(x)$, for any $x \in M \bigotimes_A B$ and any $b \in B$.

(2) A polynomial law $\{g\}\colon F \longrightarrow E$ between two $A$-algebras $F$ and $E$, is *multiplicative* if $g_B(xy) = g_B(x)g_B(y)$ for any $x$ and $y$ in $F \bigotimes_A B$, for any $A$-algebra $B$. Furthermore, we require that $g_B(1) = 1$.

A norm (of degree $n$) from an $A$-algebra $F$ to an $A$-algebra $E$ is a homogeneous multiplicative polynomial law of degree $n$.

1.4. **Universal norms.** Let $n$ be a non-negative integer. For any $A$-algebra $B$ we have that $\Gamma^n_A(M) \bigotimes_A B$ is canonically identified with $\Gamma^n_B(M \bigotimes_A B)$. It follows that we have a polynomial law $\{\gamma^n\}\colon M \longrightarrow \Gamma^n_A M$ and by (1.1.2) the law is homogeneous of degree $n$. The polynomial law $\{\gamma^n\}\colon M \longrightarrow \Gamma^n_A M$ is *universal* in the sense that the assignment $u \mapsto \{u \circ \gamma^n\}$ gives a bijection between the $A$-module homomorphisms $u\colon \Gamma^n_A M \longrightarrow N$ and the set of polynomial laws of degree $n$ from $M$ to $N$.

Furthermore, if $F$ is an $A$-algebra then $\Gamma^n_A F$ is an $A$-algebra and then the polynomial law $\{\gamma^n\}\colon F \longrightarrow \Gamma^n_A F$ is the universal *norm* of degree $n$ ([20, Thm. p. 871], [9, 2.4.2, p. 11] ). The norm $\{\gamma^n\}$ is compatible with the product, that is $\gamma^n_B(xy) = \gamma^n_B(x)\gamma^n_B(y)$, for all $A$-algebras $B$."Universal" here means in the sense as described above, but for $A$-algebra homomorphisms from $\Gamma^n_A F$.

1.5. **The different products.** The product structure on $\Gamma_A F$ we refer to as the external structure. We will denote the external product with $*$ in order to distinguish the external product from the product structure on each graded component $\Gamma^n_A F$ defined in the previous section. (Note that our convention is the reverse of the one used in [9].)

1.6. **The canonical homomorphism.** An important norm is the following. Let $E$ be an $A$-algebra that is locally free of finite rank $n > 0$ as an $A$-module. For any $A$-algebra $B$ we have the determinant map $d_B\colon E \bigotimes_A B \longrightarrow B$ sending $x \in E \bigotimes_A B$ to the determinant of the

$B$-linear endomorphism $e \mapsto ex$ on $E \bigotimes_A B$. It is clear that the determinant maps give a multiplicative polynomial law $\{d\}\colon E \longrightarrow A$, homogeneous of degree $n = \mathrm{rank}_A E$. By the universal properties (1.4) of $\Gamma^n_A E$ we then have an $A$-algebra homomorphism

$$\sigma_E \colon \Gamma^n_A E \longrightarrow A, \tag{1.6.1}$$

such that $\sigma_E(\gamma^n(x)) = \det(e \mapsto ex)$ for all $x \in E$. We call $\sigma_E$ the canonical homomorphism ([7, Section 6.3, p.180], [15, Section 1.4, p.13]).

**Proposition 1.7.** *Let $E$ be an $A$-algebra such that $E$ is free of finite rank $n > 0$ as an $A$-module. For any element $x \in E$ the characteristic polynomial* $\det(t - x) \in A[t]$ *of the endomorphism $e \mapsto ex$ on $E$ is* $t^n + \sum_{j=1}^{n} (-1)^j t^{n-j} \sigma_E(\gamma^j(x) * \gamma^{n-j}(1))$. *In particular we have*

$$\mathrm{Trace}(e \mapsto ex) = \sigma_E(\gamma^1(x) * \gamma^{n-1}(1)).$$

*Proof.* Let $t$ be an independent variable over $A$, and write $E[t] = E \bigotimes_A A[t]$. By the defining property of the canonical homomorphism $\sigma_{E[t]}$ we have that the characteristic polynomial $\det(t-x) = \sigma_{E[t]}(\gamma^n(t-x))$. We now use the defining relations (1.1.2) and (1.1.3) in the $A[t]$-algebra $\Gamma^n_{A[t]} E[t]$ and obtain

$$\begin{aligned} \gamma^n(t - x) &= \sum_{j=0}^{n} (-1)^j \gamma^j(x) * \gamma^{n-j}(t) \\ &= \sum_{j=0}^{n} (-1)^j t^{n-j} \gamma^j(x) * \gamma^{n-j}(1). \end{aligned}$$

We have that $\Gamma^n_A(R) \bigotimes_A B = \Gamma^n_B(R \bigotimes_A B)$ and that $\sigma_{E[t]} = \sigma_E \otimes \mathrm{id}_{A[\lambda]}$. Consequently $\sigma_{E[t]}$ acts trivially on the variable $t$ and that the action otherwise is as $\sigma_E$. Thus we obtain that $\sigma_E(\gamma^j(x) * \gamma^{n-j}(1))$ in $A$ is the $j$'th coefficient of the characteristic polynomial of $e \mapsto ex$ which proves the claim. □

## 2. Discriminant and ideal of norms

In this section we define the important ideal of norms and show their connection with discriminants.

**Definition 2.1.** Let $F$ be an $A$-algebra. For each integer $n \geq 0$ we consider the $A$-module homomorphism

$$\delta \colon \Lambda^n_A F \bigotimes_A \Lambda^n_A F \longrightarrow \Gamma^n_A(F)$$

that sends $x = x_1 \wedge \cdots \wedge x_n$ and $y = y_1 \wedge \cdots \wedge y_n$ to

$$\delta(x, y) := \det{}^*(\gamma^1(x_i y_j)) := \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sign}(\sigma) \gamma^1(x_1 y_{\sigma(1)}) * \cdots * \gamma^1(x_n y_{\sigma(n)}).$$

(Here we use $\det^*$ to denote the determinant with respect to the $*$-product. We also allow $n = 0$, the determinant of a $0 \times 0$-matrix being equal to 1.)

*Remark* 2.2. Note that for each element $z \in F$ the element $\gamma^1(z)$ is in $\Gamma^1_A F = F$, but the product $\gamma^1(z_1) * \cdots * \gamma^1(z_n)$ is in $\Gamma^n_A F$.

*Remark* 2.3. Since $F$ is commutative we have that $\delta(x, y) = \delta(y, x)$.

**2.4.** As a preparation for the next lemma we make the following observation. If $F$ is the product ring $F' \times F''$ then if $e', f' \in F'$ and $e'', f'' \in F''$ and $s', s'', t', t''$ are polynomial variables we may expand $\gamma_n(s'e' + s''e'') \cdot \gamma_n(t'f' + t''f'') = \gamma_n(s't'e'f' + s''t''e''f'')$ and conclude that the decomposition $\Gamma^n_A(F' \times F'') = \prod_{i+j=n} \Gamma^i_A F' \bigotimes_A \Gamma^j_A F''$ is a decomposition as rings and that the ring structure on $\Gamma^i_A F' \bigotimes_A \Gamma^j_A F''$ is the tensor product of the ring structures of $\Gamma^i_A F'$ and $\Gamma^j_A F''$. In particular, for the $A$-algebra $F = \prod_{i=1}^m Ae_i$ we get that $\Gamma^n_A F$ is the product of copies of $A$ with the primitive idempotents being the DP-monomials $\gamma^{k_1}(e_1) * \gamma^{k_2}(e_2) * \cdots * \gamma^{k_n}(e_n)$, where $0 \leq k_i$ and $\sum_i k_i = n$.

**Lemma 2.5.** *Let $x_1, \dots, x_n$ and $y_1, \dots, y_n$, $n \geq 0$, be $2n$-elements in $F$. Then we have*

$$\delta(x, y) = \det(\gamma^1(x_i y_j) * \gamma^{n-1}(1))_{1 \leq i,j \leq n}.$$

*Proof.* We first note that the right hand side has the same transformational properties as $\delta$ giving rise to an $A$-linear map $\Lambda^n_A F \bigotimes_A \Lambda^n_A F \to \Gamma^n_A F$. Furthermore, the statement is compatible with changes in both $A$ and $F$ so we may assume that $A = \mathbb{Z}$ and that $F$ is the polynomial ring in the variables $x_i$ and $y_i$. We may then replace $\mathbb{Z}$ by an algebraically closed field $K$ of characteristic zero. Now, the formula to proven involves only elements of $\Lambda^n_K F'$ where $F' \subseteq F$ is the subspace spanned by the $x_i$ and $y_j$, with $1 \leq i, j \leq n$. This means that we may replace $F$ by any algebra quotient $F \to F''$ into which $F'$ injects. Since $K$ is algebraically closed, we may assume that $F = \prod_{i=1}^m Ke_i$. As we want to show equality of two $K$-linear maps $\Lambda^n_K F \bigotimes_K \Lambda^n_K F \to \Gamma^n_K F$ we may assume that $x_i = e_{r_i}$ and $y_j = e_{s_j}$ for $r_1 < r_2 < \cdots < r_n$ and $s_1 < s_2 < \cdots < s_n$. However, unless $r_i = s_i$ for all $i$ both matrices $(\gamma^1(x_i y_j))$ and $(\gamma^1(x_i y_j) * \gamma^{n-1}(1))$ will contain a zero row or column and hence their determinants will both be zero. Hence we may assume $r_i = s_i$ and then also that $m = n$ and $r_i = s_i = i$. This means that the matrix $(\gamma^1(x_i y_j))$ will be diagonal with diagonal entries $\gamma^1(e_i)$ and its determinant is therefore $\gamma^1(e_1) * \cdots * \gamma^1(e_n)$. On the other hand we have that $1 = e_1 + \cdots + e_n$ and hence $(\gamma^1(x_i y_j) * \gamma^{n-1}(1))$ will also be a diagonal matrix whose $i$'th diagonal entry consists of all the degree $n$ monomials in the $\gamma^j(e_k)$ which contain $\gamma^1(e_i)$. As the determinant is the product of these diagonal entries and these monomials are orthogonal idempotents we see that the only term that survives is the term

$\gamma^1(e_1)*\cdots*\gamma^1(e_n)$ from each diagonal entry and their product is again $\gamma^1(e_1)*\cdots*\gamma^1(e_n)$. □

**Lemma 2.6.** *Let $x_1,\dots,x_n$, $n>0$, and $f$ be elements in an $A$-algebra $F$. Then we have that $\gamma^1(x_1f^n)*\gamma^1(x_2)*\cdots*\gamma^1(x_n)$ equals*

$$\sum_{c=1}^{n}(-1)^{c+1}(\gamma^c(f)*\gamma^{n-c}(1))\cdot(\gamma^1(x_1f^{n-c})*\gamma^1(x_2)*\cdots*\gamma^1(x_n)).$$

*Proof.* Using that $\gamma^n(1)$ is the identity element with respect to the internal product on $\Gamma^nF$, the equality above is equivalent to

$$0=\sum_{c=0}^{n}(-1)^{c+1}(\gamma^c(f)*\gamma^{n-c}(1))\cdot(\gamma^1(x_1f^{n-c})*\gamma^1(x_2)*\cdots*\gamma^1(x_n)).$$

As in the proof of Lemma (2.5) we may assume that $F=\prod_{i=1}^m Ae_i$, that $x_1=e_1$ and each $x_i$, $i>1$, is equal to some $e_j$ and we may further write $f=\sum_{i=1}^m\lambda_ie_i$. Then, for $0\le c\le n$, $\gamma^1(x_1f^{n-c})*\gamma^1(x_2)*\cdots*\gamma^1(x_n)$ equals $\lambda_1^{n-c}\gamma^1(x_1)*\gamma^1(x_2)*\cdots*\gamma^1(x_n)$ and hence the sum to be shown to be equal to zero equals

$$(-1)^{n+1}(\gamma^1(x_1)*\gamma^1(x_2)*\cdots*\gamma^1(x_n))\cdot\sum_{c=0}^{n}(\gamma^c(f)*\gamma^{n-c}(-\lambda_1)).$$

The right multiplicand equals $\gamma^n(f-\lambda_1)$ and as $f-\lambda_1=\sum_{i=2}^m(\lambda_i-\lambda_1)e_i$ we get that $\gamma^n(f-\lambda_1)$ is a linear combination of DP-monomials $\gamma^{k_1}(e_1)*\gamma^{k_2}(e_2)*\cdots*\gamma^{k_n}(e_n)$ with $k_1=0$. On the other hand, as $x_1=e_1$, $\gamma^1(x_1)*\gamma^1(x_2)*\cdots*\gamma^1(x_n)$ is an integer multiple of a DP-monomial $\gamma^{k_1}(e_1)*\gamma^{k_2}(e_2)*\cdots*\gamma^{k_n}(e_n)$ with $k_1>0$ and as different DP-monomials have internal product equal to zero we conclude. □

**Definition 2.7** (The ideal of norms)**.** Let $n>0$ be a fixed integer, and let $V\subseteq F$ be an $A$-submodule of an $A$-algebra $F$. We define $I_V\subseteq\Gamma^n_AF$, the ideal of norms associated to $V$, as the ideal generated by

$$\delta(x,y)\in\Gamma^n_AF$$

for any $2n$-elements $x=x_1,\dots,x_n$ and $y=y_1,\dots,y_n$ in $V\subseteq F$.

*Remark* 2.8. Both the symmetric product and the Hilbert scheme make sense when $n=0$. However, our results become trivial in that case so we shall from now assume that $n>0$.

**Lemma 2.9.** *Let $A\longrightarrow B$ be a homomorphism of rings, and let $V\subseteq F$ be an $A$-submodule of an $A$-algebra $F$. The extension of the ideal $I_V$ by the $A$-algebra homomorphism $\Gamma^n_AF\longrightarrow\Gamma^n_A(F)\bigotimes_AB$ equals the ideal $I_{V_B}$; the ideal of norms associated to the $B$-submodule $Im(V\bigotimes_AB\longrightarrow F\bigotimes_AB)$.*

*Proof.* Via the canonical identification $\Gamma^n_A(F)\bigotimes_AB=\Gamma^n_B(F\bigotimes_AB)$ the element $\delta(x,y)\otimes1_B$ is identified with $\delta(x\otimes1_B,y\otimes1_B)$, from which the lemma follows. □

**Lemma 2.10.** *Let $F = A[T_1, \dots, T_r]$ be the polynomial ring in $r > 0$ variables, and let $V \subset F$ be the $A$-module spanned by those monomials whose degree in each of the variables is less than $n$. Then the ideals of norms associated to $V$ and $F$ are equal; that is $I_V = I_F$. Furthermore, if $n!$ is invertible in $A$ then $I_W = I_F$, where $W \subset F$ is the $A$-module spanned by monomials of degree less than $n$.*

*Proof.* Given $x_1, \dots, x_n$ and $f$ in $F$ we write $x(c) = x_1 f^c, x_2, \dots, x_n$. For any $y_1, \dots, y_n$ we then obtain from the equality given in Lemma (2.6) that

$$\delta(x(n), y) = \sum_{c=1}^{n} (-1)^{c+1} (\gamma^c(f) * \gamma^{n-c}(1)) \cdot \delta(x(n-c), y).$$

The first assertion of the lemma follows from the above equality. When $n!$ is invertible, the $n$'th powers of linear forms span the module generated by degree $n$ monomials, and the above equality then also yields the second assertion. □

**2.11. Discriminant.** Let $E$ be an $A$-algebra that is free of finite rank $n$ as an $A$-module. The trace map $E \to A$ sends an element $x \in E$ to the trace of the endomorphism $e \mapsto ex$ of the $A$-module $E$. There is an associated map $E \longrightarrow \mathrm{Hom}_A(E, A)$ taking $y \in E$ to the trace $\mathrm{tr}(xy)$, for any $x \in E$.

The discriminant ideal $D_{E/A} \subseteq A$ is defined (see e.g. [1, p. 124]) as the ideal generated by the determinant of the associated map $E \longrightarrow \mathrm{Hom}(E, A)$.

**Proposition 2.12.** *Let $E$ be an $A$-algebra that is free of finite rank $n$ as an $A$-module. Then we have for any elements $x = x_1, \dots, x_n$ and $y = y_1, \dots, y_n$ in $E$ that*

$$\sigma_E(\delta(x, y)) = \det(\mathrm{tr}(x_i y_j)),$$

*where $\sigma_E$ is the canonical homomorphism $\sigma_E \colon \Gamma_A^n E \longrightarrow A$, and $(\mathrm{tr}(x_i y_j))$ is the $(n \times n)$ matrix with entries $\mathrm{tr}(x_i y_j)$. In particular the extension of $I_V$, the ideal of norms associated to $V = E$, by $\sigma_E$ is the discriminant ideal and we have that the extension $\sigma_E(I_V)A = A$ if and only if $\mathrm{Spec}(E) \longrightarrow \mathrm{Spec}(A)$ is étale.*

*Proof.* Let $x = x_1, \dots, x_n$ be an $A$-module basis of $E = V$. We have that the ideal $I_V$ is generated by the single element $\delta(x, x)$. By Lemma (2.5) we have the identity $\delta(x, x) = \det(\gamma^1(x_i x_j) * \gamma^{n-1}(1))$ in $\Gamma_A^n F$. As $\sigma_E$ is an algebra homomorphism we have

$$\sigma_E \det(\gamma^1(x_i x_j) * \gamma^{n-1}(1)) = \det(\sigma_E(\gamma^1(x_i x_j) * \gamma^{n-1}(1))).$$

By Proposition (1.7) we have $\sigma_E(\gamma^1(x_i x_j) * \gamma^{n-1}(1)) = \mathrm{Trace}(e \mapsto e x_i x_j)$. Thus we have a matrix with entries $\mathrm{Trace}(e \mapsto e x_i x_j)$, and the determinant is then the discriminant. □

## 3. Connection with symmetric tensors

**3.1. A norm vector.** Let $F$ be an $A$-algebra, and let $n$ be a fixed positive integer. We let $T^n_A F = F \bigotimes_A \cdots \bigotimes_A F$ be the tensor product with $n$ copies of $F$. For any element $x \in F$ we use the following notation

$$x_{[j]} = 1 \otimes \cdots \otimes x \otimes \cdots \otimes 1,$$

where the $x$ occurs at the $j$'th component of $T^n_A F$. The group $\mathfrak{S}_n$ of permutations of $n$ letters acts on $T^n_A F$ by permuting the factors. For any $n$-elements $x = x_1, \dots, x_n$ in $F$ we define the norm vector

$$\nu(x) = \nu(x_1, \dots, x_n) = \det((x_i)_{[j]}) \in T^n_A F.$$

Expanding the determinant we also get that $\nu(x) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$. It is clear that $\nu$ extends to a linear map $\nu \colon \Lambda^n_A F \to T^n_A F$ and that the image lies in the vectors that are anti-symmetric with respect to the action of $\mathfrak{S}_n$ given by permutation of factors of $\mathrm{T}^n_A F$.

**3.2.** Let $\mathrm{TS}^n_A F$ denote the invariant ring of $\mathrm{T}^n_A F$ by the natural action of the symmetric group $\mathfrak{S}_n$ in $n$-letters that permutes the factors. We have the map $F \longrightarrow \mathrm{T}^n_A F$ sending $x \mapsto x \otimes \cdots \otimes x$, and it is clear that the map factors through the invariant ring $\mathrm{TS}^n_A F$. The map $F \longrightarrow \mathrm{TS}^n_A F$ determines a norm of degree $n$, as one readily verifies, hence there exist an $A$-algebra homomorphism

$$\alpha_n \colon \Gamma^n_A F \longrightarrow \mathrm{TS}^n_A F \tag{3.2.1}$$

such that $\alpha_n(\gamma^n(x)) = x \otimes \cdots \otimes x$, for all $x \in F$.

**3.3. The shuffle product.** When $F$ is an $A$-algebra that is flat as an $A$-module, or if $n!$ is invertible in $A$, then the $A$-algebra homomorphism $\alpha_n$ (3.2.1) is an isomorphism ([19, IV, §5. Proposition IV.5], [5, Exercise 8(a), AIV. p.89]). In those cases we can identify $\Gamma_A F$ as the graded sub-module

$$\Gamma_A F = \bigoplus_{n \geq 0} \mathrm{TS}^n_A F \subseteq \bigoplus_{n \geq 0} \mathrm{T}^n_A F = \mathrm{T}_A F.$$

The external product structure on $\Gamma_A F$ is then identified with the shuffle product on the full tensor algebra $\mathrm{T}_A F$. The shuffle product of an $n$-tensor $x \otimes \cdots \otimes x$ and an $m$-tensor $y \otimes \cdots \otimes y$ is the $m+n$-tensor given as the sum of all possible different shuffles of the $n$ copies of $x$ and $m$ copies of $y$ ([5, Exercise 8 (b), AIV. p.89]).

**Proposition 3.4.** *Let $F$ be an $A$-algebra, and let $x, y \in \Lambda^n_A F$. The $A$-algebra homomorphism $\alpha_n \colon \Gamma^n_A F \longrightarrow \mathrm{TS}^n_A F$ (3.2.1) has the property*

$$\alpha_n(\delta(x, y)) = \nu(x)\nu(y).$$

*Proof.* We may assume that $x = x_1 \wedge \cdots \wedge x_n$ and $y = y_1 \wedge \cdots \wedge y_n$ and then we have by Lemma (2.5) that $\delta(x, y)$ is the determinant of the matrix $(\gamma^1(x_i y_j) * \gamma^{n-1}(1))$. Hence $\alpha_n(\delta(x, y))$ is the determinant of $(\alpha_n(\gamma^1(x_i y_j) * \gamma^{n-1}(1)))$. This matrix is the product $((x_i)_{[j]})((y_i)_{[j]})^t$ (where $((y_i)_{[j]})^t = ((y_j)_{[i]})$ denotes the transpose) and using multiplicativity of determinants we get the formula. □

**Corollary 3.5.** *For any $x, y, z$ and $w$ in $\Lambda^n_A F$ we have $\delta(x, y)\delta(z, w) = \delta(x, z)\delta(y, w)$. In particular we have $\delta(x, y)^2 = \delta(x, x)\delta(y, y)$.*

*Proof.* We may reduce to the case when $F$ is flat over $A$, and then we have that $\alpha_n \colon \Gamma^n_A F \to \mathrm{TS}^n_A F$ is injective. By the proposition we have $\alpha_n(\delta(x, y)\delta(z, w)) = \alpha_n(\delta(x, y))\alpha_n(\delta(z, w)) = \nu(x)\nu(y)\nu(z)\nu(w)$ and rearranging the last product and working backwards we get the desired formula. □

*Remark* 3.6. We have used two methods to prove universal relations in $\Gamma^n_A F$ and $\Lambda^n_A F$; reducing to the case when $F$ is a finite product of copies of $A$ and explicit computation using primitive idempotents, and reducing to a computation in $T^n_A F$. It would have been possible to only use the first (and no doubt to only use the second) but we felt that both techniques were worthwhile illustrating. It should also be mentioned that in a version of this article we used a third method of computing directly in $\Gamma^n_A F$. However, it led to rather non-transparent combinatorial calculations which we ultimately felt obscured the underlying arguments too much.

**3.7.** We have a map $\alpha_n + \nu \colon \Gamma^n_A F \bigoplus \Lambda^n_A F \to \mathrm{T}^n_A F$ whose image is a subring under the product induced from that of $F$. Even though we shall not use it we can use $\delta$ to define a commutative ring structure on the source making the map a ring homomorphism. Indeed the ring structure will be $\mathbb{Z}/2$-graded with respect to the direct sum decomposition, the product $\Gamma^n_A F \times \Gamma^n_A F \to \Gamma^n_A F$ the interior product, the product $\Lambda^n_A F \times \Lambda^n_A F \to \Gamma^n_A F$ will be given by $\delta$ and the map $\Gamma^n_A F \times \Lambda^n_A F \to \Lambda^n_A F$ by $\gamma^n(x) \cdot y_1 \wedge \cdots \wedge y_n := xy_1 \wedge \cdots \wedge xy_n$. With the aid of Proposition (3.4) it is easy to verify that $\alpha_n + \nu$ is multiplicative and when $A = \mathbb{Z}$ and $F$ is $A$-flat it is also injective and as one can reduce to that case we get associativity for the operation.

**Corollary 3.8.** *Let $\tilde{\alpha} \colon \Gamma^n_A F \longrightarrow \mathrm{T}^n_A F$ denote the composition of the map $\alpha_n$ and the inclusion $\mathrm{TS}^n_A F \subseteq \mathrm{T}^n_A F$. Let $I \subseteq \mathrm{T}^n_A F$ denote the extension of the ideal of norms $I_F$ by $\tilde{\alpha}$, and let $J \subseteq \mathrm{T}^n_A F$ denote the ideal of the schematic union of the diagonals. Then we have $\sqrt{I} = \sqrt{J}$.*

*Proof.* Let $\varphi \colon \mathrm{T}^n_A F \longrightarrow L$ be a morphism with $L$ a field, and let $\varphi_i \colon F \longrightarrow L$ be the composition of $\varphi$ and the $i$'th co-projection $F \longrightarrow \mathrm{T}^n_A F$, where $i = 1, \ldots, n$. If $\varphi$ corresponds to a point in the open complement of the diagonals then all the maps $\varphi_i$ are different. That is, no $\mathfrak{p}_i = \ker(\varphi_i)$ is contained in another $\mathfrak{p}_j$. Furthermore, since the kernels also are prime ideals there exists, for each $i$, an element $x_i$ not in $\mathfrak{p}_i$, but where $x_i \in \mathfrak{p}_j$ when $j \neq i$. We then have that $\varphi_j(x_i) = 0$ for $j \neq 0$, and that $\varphi_i(x_i) \neq 0$. Hence there are elements $x_1, \dots, x_n$ in $F$ such that $\det(\varphi_j(x_i)) \neq 0$. Then also the image of $\nu(x_1, \dots, x_n)$ is non-zero in $L$, and we have that the point $\varphi$ is in the open complement of the scheme defined by $I \subseteq \mathrm{T}^n_A F$.

Conversely, if $\varphi$ corresponds to a point on the diagonals then at least two of the maps $\varphi_i$ are equal. Consequently, for any elements $x_1, \dots, x_n$ in $F$ we have that $\varphi(\nu(x_1, \dots, x_n)) = 0$. It follows that $I \subseteq \ker \varphi$, proving the claim. □

## 4. Grothendieck-Deligne norm map

In this section we recall the Grothendieck-Deligne norm map following Deligne ([7]), and we discuss briefly the related Hilbert-Chow morphism. Furthermore we define the notion of sufficiently big submodules.

**4.1. The Hilbert functor of $n$-points.** We fix an $A$-algebra $F$, and a positive integer $n$. We let $\mathrm{Hilb}^n_F$ denote the covariant functor from the category of $A$-algebras to sets, that sends an $A$-algebra $B$ to the set

$$\mathrm{Hilb}^n_F(B) = \{\text{ideals in } F \bigotimes_A B \text{ such that the quotient } E \text{ is locally free of rank } n \text{ as a } B\text{-module}\}.$$

**4.2. The Grothendieck-Deligne norm.** If $E$ is an $B$-valued point of $\mathrm{Hilb}^n_F$ we have the sequence

$$F \longrightarrow F \bigotimes_A B \longrightarrow E,$$

from where we obtain the $A$-algebra homomorphisms $\Gamma^n_A F \longrightarrow \Gamma^n_B E$ that sends $\gamma^n(x)$ to $\gamma^n(\bar{x} \otimes 1)$, where $\bar{x} \otimes 1$ is the residue class of $x \otimes 1$ in $E$. Furthermore, when we compose the homomorphism $\Gamma^n_A F \longrightarrow \Gamma^n_B E$ with the canonical homomorphism $\sigma_E \colon \Gamma^n_B E \longrightarrow B$ we obtain an assignment that is functorial in $B$; that is we have a morphism of functors

$$\mathrm{n}_F \colon \mathrm{Hilb}^n_F \longrightarrow \mathrm{Hom}_{A\text{-alg}}(\Gamma^n_A F, -). \tag{4.2.1}$$

The natural transformation $\mathrm{n}_F$ we call the Grothendieck-Deligne norm map.

*Remark* 4.3. The Hilbert functor $\mathrm{Hilb}^n_F$ can in a natural way be viewed as a contra-variant functor from the category of schemes (over $\mathrm{Spec}(A)$) to sets. In that case the functor $\mathrm{Hilb}^n_F$ is representable by a scheme (see e.g. [13]). If $X = \mathrm{Spec}(F) \longrightarrow S = \mathrm{Spec}(A)$ we write $\mathrm{n}_X \colon \mathrm{Hilb}^n_{X/S} \longrightarrow \mathrm{Spec}(\Gamma^n_A F)$ for the morphism that corresponds to the natural transformation (4.2.1).

**4.4. The geometric action.** Let $A = K$ be an algebraically closed field, and let $E$ be a finitely generated Artinian $K$-algebra. As $E$ is Artinian it is a product of local rings $E = \prod_{i=1}^{p} E_i$, and we let $\rho_i \colon E \longrightarrow K$ denote the residue class map that factors via $E_i$. Let $m_i = \dim_K(E_i)$, and let $n = \dim_K(E) = m_1 + \cdots + m_p$. Iversen ([15, Proposition 4.7]) shows that the canonical homomorphism $\sigma_E \colon \Gamma^n_K E = \mathrm{TS}^n_K E \longrightarrow K$ factors via the homomorphism $\rho\colon T^n_K E \longrightarrow K$, where

$$\rho = (\rho_1, \ldots, \rho_1, \ldots, \rho_p, \ldots, \rho_p),$$

and where each factor $\rho_i$ is repeated $m_i$-times.

**4.5. Hilbert-Chow morphism.** Assume that the base ring $A = K$ is a field, and let $X = \mathrm{Spec}(F)$. Then we can identify $\mathrm{Spec}(\Gamma^n_K F)$ with the symmetric quotient $\mathrm{Sym}^n(X) := \mathrm{Spec}(\mathrm{TS}^n_K F)$. Furthermore we have that the $\mathrm{Spec}(K)$-valued points of $\mathrm{Hilb}^n_X$ correspond to closed zero-dimensional subschemes $Z \subseteq X$ of length $n$. When $K$ is algebraically closed we have by (4.4) that the Grothendieck-Deligne norm map sends an $K$-valued point $Z \subseteq X$ to the "associated" zero-dimensional cycle

$$\mathrm{n}_X(Z) = \sum_{P \in |Z|} \dim_K(\mathscr{O}_{Z,P})[P],$$

where the summation runs over the points in the support of $Z$. Hence we see that the norm morphism $\mathrm{n}_X$ has the same effect on geometric points as the Hilbert-Chow morphism. The Hilbert-Chow morphism that appears in [10] and [8] requires that the Hilbert scheme is reduced, whereas the Hilbert-Chow morphism that appears in [17] requires that the Hilbert scheme is (semi-) normal. As the morphism $\mathrm{n}_X$ does not require any hypothesis on the source we have chosen to refer to that morphism with a different name; the Grothendieck-Deligne norm map.

**Lemma 4.6.** *Let $A = K$ be a field of characteristic zero, and let $F = K[T]$ be the polynomial ring in a finite set of variables $T_1, \ldots, T_r$. For $n > 0$ the $K$-algebra $\Gamma^n_K F$ is generated by*

$$\gamma^1(m) * \gamma^{n-1}(1),$$

*for monomials $m \in K[T]$ of degree deg$(m) \leq n$.*

*Proof.* The identification $\alpha_n \colon \Gamma^n_K K[T] \longrightarrow \mathrm{TS}^n_K K[T]$ identifies, for any $m \in K[T]$, the element $\gamma^1(m) * \gamma^{n-1}(1)$ with the shuffled product of $\alpha_1(m) = m$ and $\alpha_{n-1}(1) = 1 \otimes \cdots \otimes 1$. That is

$$\alpha_n(\gamma^1(m) * \gamma^{n-1}(1)) = m \otimes 1 \cdots \otimes 1 + \cdots + 1 \otimes \cdots 1 \otimes m = P(m).$$

By a well-known result of Weyl ([22, II 3]) the invariant ring $\mathrm{TS}^n_K F$ is generated by the power sums $P(m)$ of monomials $m \in K[T]$ of degree less or equal to $n$. $\square$

**Definition 4.7** (Sufficiently big modules)**.** Let us fix an $A$-algebra $F$. An $A$-submodule $V \subseteq F$ is $n$-sufficiently big if the composite $B$-module homomorphism

$$V \bigotimes_A B \longrightarrow F \bigotimes_A B \longrightarrow E$$

is surjective for all $A$-algebras $B$, and all $B$-valued points $E$ of the Hilbert functor $\mathrm{Hilb}^n_F$.

*Remark* 4.8. Sufficiently big submodules always exist as we can take $V = F$.

*Remark* 4.9. If $V$ is sufficiently big then we clearly have a morphism of functors

$$\mathrm{Hilb}^n_F \longrightarrow \mathrm{Grass}^n_V$$

from the Hilbert functor of rank $n$-families, to the Grassmannian of locally free rank $n$-quotients of $V$.

**Theorem 4.10.** *Let $F$ be an $A$-algebra, $n$ a positive integer and let $V \subseteq F$ be an $n$-sufficiently big submodule. Then we have for any $A$-algebra $B$, and any $B$-valued point $E$ of $\mathrm{Hilb}^n_F$ that the extension of $I_V$, the ideal of norms associated to $V$, by the Grothendieck-Deligne norm map $\mathrm{n}_F \colon \Gamma^n_A F \longrightarrow B$ is the discriminant ideal of $E$ over $B$. That is*

$$\mathrm{n}_F(I_V)B = D_{E/B} \subseteq B.$$

*Proof.* As discriminant ideals are compatible with base change, we may assume that $B$ is a local ring. Let $K$ denote the residue field of $B$. By assumption the composite map

$$V \bigotimes_A K \longrightarrow F \bigotimes_A K \longrightarrow E \bigotimes_A K$$

is surjective as $K$-vector spaces. Let $x_1, \dots, x_n$ in $V$ be such that the residue classes of $x_1 \otimes \mathrm{id}_K, \dots, x_n \otimes \mathrm{id}_K$ in $E \otimes_A K$ form a $K$-vector space basis. It then follows from Nakayama's Lemma that the residue classes of $x_1 \otimes \mathrm{id}_B, \dots, x_n \otimes \mathrm{id}_B$ form a $B$-module basis of $E \bigotimes_A B = E_B$. By Lemma (2.9) we have that the extension of $I_V$ by the composition $\Gamma^n_A F \longrightarrow \Gamma^n_B(E_B)$ is the ideal of norms associated to $E_B$. The result then follows from Proposition (2.12).

□

## 5. Families of distinct points

**5.1. The canonical morphism.** The map $F \longrightarrow F \bigotimes_A \Gamma^{n-1}_A F$ sending $z$ to $z \otimes \gamma^{n-1}(z)$ determines a norm of degree $n$. Consequently there is a unique $A$-algebra homomorphism $\Gamma^n_A F \longrightarrow F \bigotimes_A \Gamma^{n-1}_A F$ that takes $\gamma^n(z)$ to $z \otimes \gamma^{n-1}(z)$. Let

$$\pi_n \colon \operatorname{Spec}(F) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(\Gamma^{n-1}_A F) \longrightarrow \operatorname{Spec}(\Gamma^n_A F) \quad (5.1.1)$$

denote the corresponding morphism of schemes. Furthermore, we let $\Delta \subseteq \operatorname{Spec}(\Gamma^n_A F)$ denote the closed subscheme corresponding to the ideal of norms associated to $F$.

**Proposition 5.2.** *Let $U = \operatorname{Spec}(\Gamma^n_A F) \setminus \Delta$ denote the open set where the ideal sheaf of norms equals the structure sheaf. Then the induced morphism*

$$\pi_{n|} \colon \pi_n^{-1}(U) \longrightarrow U$$

*is étale of rank $n$.*

*Proof.* Let $U_n \subseteq \operatorname{Spec}(\mathrm{T}^n_A F)$ denote the open complement of the diagonals. The group of permutations of $n$ letters, $\mathfrak{S}_n$, acts freely on $U_n$ and the quotient map $U_n \longrightarrow U_n/\mathfrak{S}_n$ is étale of rank $n! = |\mathfrak{S}_n|$. The morphism $\operatorname{Spec}(\alpha_n)\colon \operatorname{Spec}(\mathrm{TS}^n_A F) \longrightarrow \operatorname{Spec}(\Gamma^n_A F)$ is an isomorphism when restricted to $U_n/\mathfrak{S}_n$ (see e.g. [21, Prop. 4.2.6]). It follows from Corollary (3.8) that $\operatorname{Spec}(\mathrm{T}^n_A F) \longrightarrow \operatorname{Spec}(\Gamma^n_A F)$ is étale over $\operatorname{Spec}(\Gamma^n_A F) \setminus \Delta$. Furthermore, after a faithfully flat base change $A \longrightarrow A'$ we can assume that $\Gamma^n_A(F) \bigotimes_A A' = \Gamma^n_{A'}(F \bigotimes_A A')$ is generated by elements of the form $\gamma^n(z)$ ([9, Lemma 2.3.1]). Then clearly the diagram

$$\begin{array}{ccc}
\operatorname{Spec}(\mathrm{T}^n_A F) & \xrightarrow{\quad\tilde{\alpha}_n\quad} & \operatorname{Spec}(\Gamma^n_A F) \\
& {\scriptstyle 1\times\tilde{\alpha}_{n-1}} \searrow \qquad \nearrow {\scriptstyle \pi_n} & \\
& \operatorname{Spec}(F) \times \operatorname{Spec}(\Gamma^{n-1}_A F) &
\end{array}$$

is commutative. As $1 \times \tilde{\alpha}_{n-1}$ is étale of rank $(n-1)!$ on the complement of $\pi_n^{-1}(\Delta)$, it follows that $\pi_n$ is étale of rank $n$ over $U$. □

**5.3. Notation.** We have the ordered sequence $x = x_1, \dots, x_n$ of elements in $F$ fixed. Let $U_A(x)$ be $\Gamma^n_A F$ localized at the element $\delta(x, x)$, and consider the induced map

$$U_A(x) \longrightarrow (F \bigotimes_A \Gamma^{n-1}_A F) \bigotimes_{\Gamma^n_A F} U_A(x) = M_A(x)$$

obtained by localization of (5.1.1).

**Lemma 5.4.** *The images of the elements $x = x_1, \dots, x_n$ by the map $F \longrightarrow F \bigotimes_A U_A(x) \longrightarrow M_A(x)$ form an $U_A(x)$-module basis for $M_A(x)$.*

*Proof.* By Proposition (5.2) we have that $M_A(x)$ is Zariski locally free of rank $n$ over $U_A(x)$. To show that $M_A(x)$ is free it suffices to show that the images of $x_1, \dots, x_n$ form a basis locally. Hence we may assume that $M$ is a free $U$-module, where $U$ is some localization of $U_A(x)$. Let $e = e_1, \dots, e_n$ be a basis of $M$, and let $q(z)$ denote the image of $z \in F$ in $M$. There exist scalars $a_{i,j} \in U$ such that $q(x_i) = \sum_{j=1}^n a_{i,j} e_j$ for $i = 1, \dots, n$. Let $q(x) = q(x_1), \dots, q(x_n)$, and let $A = (a_{i,j})$ denote the matrix of the scalars. From the Definition (2.1) we obtain

$$\delta(q(x), q(x)) = \det(A^2)\delta(e, e) \quad \text{in} \quad \Gamma^n_U M.$$

The element $\delta(x,x)\otimes 1$ in $\Gamma^n_A(F)\bigotimes_{\Gamma^n_A F} U = \Gamma^n_U(F\bigotimes_A U)$ is invertible by definition. The natural morphism $F\bigotimes_A U \longrightarrow M$ induces a morphism $\Gamma^n_U(F\bigotimes_A U) \longrightarrow \Gamma^n_U M$ sending $\delta(x,x)\otimes 1$ to the invertible element $\delta(q(x),q(x))$. Then also $\det(A)$ must be invertible, and consequently we have that $q(x_1),\dots,q(x_n)$ form a basis of $M_A(x)$. □

**Definition 5.5.** The functor $\mathscr{H}^{\mathrm{et}}_F(x)$ is the covariant functor from the category of $A$-algebras to sets that maps an $A$-algebra $B$ to the set of ideals in $F\bigotimes_A B$ such that corresponding quotients $Q$ satisfy the following

(1) The elements $q(x_1),\dots,q(x_n)$ in $Q$ form a $B$-module basis, where $q: F \longrightarrow F\bigotimes_A B \longrightarrow Q$ is the composite map.
(2) The algebra homomorphism $B \longrightarrow Q$ is étale.

**Lemma 5.6.** *Let $B$ be an $A$-algebra, and $Q$ a $B$-valued point of $\mathscr{H}^{\mathrm{et}}_F(x)$. Then we have the following commutative diagram of algebras*

(5.6.1)
$$\begin{array}{ccc} \Gamma^n_A F & \longrightarrow & \Gamma^n_B Q \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \sigma_Q} \\ U_A(x) := (\Gamma^n_A F)_{\delta(x,x)} & \longrightarrow & B. \end{array}$$

*Proof.* The composite morphism $F \longrightarrow F\bigotimes_A B \longrightarrow Q$ induces a morphism of $A$-algebras $\Gamma^n_A F \longrightarrow \Gamma^n_B Q$ that sends the element $\delta(x,x)$ to $\delta(q(x),q(x))$, where $q(x) = q(x_1),\dots,q(x_n)$ in $Q$. By assumption the elements $q(x)$ form a basis of $Q$ and that $Q$ is étale. Then, by Proposition (2.12) we that the image of $\delta(q(x),q(x))$ by the canonical map $\sigma_Q\colon \Gamma^n_B Q \longrightarrow Q$ is a unit, and the commutativity of the diagram (5.6.1) follows. □

**5.7. Universal coefficients.** For each pair of indices $1\le i,j\le n$ we look at the product $x_ix_j$ in $F$, and for each $k=1,\dots,n$ we consider the sequence

$$x^{i,j}_k = x_1,\dots,x_{k-1},x_ix_j,x_{k+1},\dots,x_n \tag{5.7.1}$$

where the $k$'th element is replaced with the product $x_ix_j$. We now define the universal coefficient

$$\alpha^{i,j}_k = \frac{\delta(x,x^{i,j}_k)}{\delta(x,x)} \quad \text{in} \quad U_A(x) = (\Gamma^n_A F)_{\delta(x,x)}.$$

**Proposition 5.8.** *Let $Q$ be a* $\mathrm{Spec}(B)$*-valued point of $\mathscr{H}^{\mathrm{et}}_F(x)$, and let $q\colon F \longrightarrow F\bigotimes_A B \longrightarrow Q$ denote the composite map. For each $k=1,\dots,n$, let $b^{i,j}_k$ be the unique elements in $B$ such that*

$$q(x_ix_j) = \sum_{k=1}^n b^{i,j}_k q(x_k)$$

*in $Q$. Then $b_k^{i,j}$ is the specialization of the element $\alpha_k^{i,j}$ under the natural map $U_A(x) \longrightarrow B$ of Lemma (5.6), for each $i,j,k = 1,\dots,n$. In particular we have that $M_A(x) \bigotimes_{U_A(x)} B = Q$ as quotients of $F \bigotimes_A B$.*

*Proof.* Having the triplet $i,j,k$ fixed, we let $x_k^{i,j}$ denote the sequence (5.7.1) of elements in $F$. Consider the element $\delta(q(x), q(x_k^{i,j}))$ in $\Gamma_B^n Q$. We replace the element $q(x_i x_j)$ in $Q$ with $\sum b_k^{i,j} q(x_k)$, and obtain

$$\delta(q(x), q(x_k^{i,j})) = b_k^{i,j} \delta(q(x), q(x)) \quad \in \Gamma_B^n Q.$$

The element $\delta(q(x), q(x))$ is the image of $\delta(x,x)$ by the induced map $\Gamma_A^n F \longrightarrow \Gamma_B^n Q$. It follows from the commutative diagram (5.6.1) that $b_k^{i,j}$ in $B$ is the image of $\alpha_k^{i,j}$. □

**Corollary 5.9.** *The pair $(U_A(x), M_A(x))$ represents $\mathscr{H}_F^{et}(x)$.*

*Proof.* It follows from Proposition (5.2) and Lemma (5.4) that $M := M_A(x)$ is a $U := U_A(x)$-valued point of $\mathscr{H}_F^{et}(x)$. If $Q$ is any $B$-valued point of $\mathscr{H}_F^{et}(x)$ we have by Proposition (5.8) one morphism $U \longrightarrow B$ with the desired property, and we need to establish uniqueness of that map. Therefore, let $\varphi_i : U \longrightarrow B$ $(i = 1,2)$, be two $A$-algebra homomorphisms such that both extensions $M \bigotimes_U B$ equal $Q$ as quotients of $F \bigotimes_A B$. We then have that the natural map

$$\Gamma_U^n M \longrightarrow \Gamma_U^n(M) \bigotimes_U B = \Gamma_B^n Q$$

is independent of the maps $\varphi_i : U \longrightarrow B$. And in particular the canonical section $\sigma_Q = \sigma_M \otimes 1 : \Gamma_B^n Q \longrightarrow B$ is independent of the maps $\varphi_i, (i = 1,2)$. For any element $u \in U$ we have that $\sigma_M(u\gamma^n(1)) = u$, and then also that $\sigma_Q(u\gamma^n(1) \otimes 1_B) = \varphi_i(u)$. Thus $\varphi_1 = \varphi_2$, and we have proven uniqueness. □

**5.10. Étale families.** We let $\mathscr{H}_F^{et,n}$ denote the functor of étale families of the Hilbert functor $\mathrm{Hilb}_F^n$ of $n$-points on $F$. That is, we consider the co-variant functor from $A$-algebras to sets whose $B$-valued points are

$$\mathscr{H}_F^{et,n}(B) = \{I \in \mathrm{Hilb}_F^n(B) \mid B \longrightarrow F \bigotimes_A B/I \text{ is étale}\} .$$

It is clear that $\mathscr{H}_F^{et,n}$ is an open subfunctor of $\mathrm{Hilb}_F^n$ and we will end this section by describing the corresponding open subscheme of the Hilbert scheme.

**Proposition 5.11.** *Let $F$ be an $A$-algebra. Let $\Delta \subseteq \mathrm{Spec}(\Gamma_A^n F)$ be the closed subscheme defined by the ideal of norms $I_F$, and let $U = \mathrm{Spec}(\Gamma_A^n F) \setminus \Delta$ denote its open complement. The family $\pi_{n|} : \pi_n^{-1}(U) \longrightarrow U$ of Proposition (5.2) represents $\mathscr{H}_F^{et,n}$.*

*Proof.* Clearly the functors $\mathscr{H}_F^{et}(x)$, for different choices of elements $x = x_1,\dots,x_n$ in $F$, give an open cover of $\mathscr{H}_F^{et,n}$. By Corollary (5.9) the restriction of the family $\pi_{n|} \colon \pi_n^{-1}(U) \longrightarrow U$ to the open subscheme $\mathrm{Spec}(U_A(x)) \subseteq U$ represents $\mathscr{H}_F^{et}(x)$. From Proposition (2.12) we get that the intersection $\mathrm{Spec}(U_A(x)) \cap \mathrm{Spec}(U_A(y))$, for $x = x_1, \dots, x_n$ and $y = y_1, \dots, y_n$, equals $\mathscr{H}_F^{et}(x) \cap \mathscr{H}_F^{et}(y)$. And finally by Corollary (3.5) we have that he union of the schemes $\mathrm{Spec}(U_A(x))$, for different $x = x_1, \dots, x_n$, is the scheme $U$. □

## 6. Closure of the locus of distinct points

We will continue with the notation from the preceding sections. In this section we will construct universal families, not for the locus of distinct points as in Section 5, but for its closure.

**6.1. Notation.** Let $F$ be an $A$-algebra, and let $R = \bigoplus_{m \geq 0} I_F^m$ denote the graded ring where $I_F \subseteq \Gamma_A^n F$ is the ideal of norms associated to $V = F$. We let $x = x_1, \dots, x_n$ be $n$-elements in $F$, and we denote by $R(x) = R_{(\delta(x,x))}$ the degree zero part of the localization of $R$ at $\delta(x,x) \in I_F$. Finally we let $\mathscr{E}$ denote the free $R(x)$-module of rank $n$. We will write

$$\mathscr{E} = \bigoplus_{i=1}^{n} R(x)[x_i], \tag{6.1.1}$$

where $[x_i]$ is our notation for a basis element pointing out the $i$'th component of the direct sum $\mathscr{E}$. As $\Gamma_A^n F$ is an $A$-algebra we have that $\mathscr{E}$ is an $A$-module. We define the $A$-module homomorphism

$$[\;] \colon F \longrightarrow \mathscr{E}$$

in the following way. For any $y \in F$, and any $i = 1, \dots, n$, we let

$$x_y^i = x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n$$

denote the $n$-elements in $F$ where the $i$'th element $x_i$ is replaced with $y$. Then we define the value of the map (6.1.3) on the element $y \in F$ as

$$[y] = \sum_{i=1}^{n} \frac{\delta(x, x_y^i)}{\delta(x,x)} [x_i] \quad \text{in} \quad \mathscr{E}. \tag{6.1.2}$$

Note that when $y = x_i$ the notation of (6.1.1) is consistent with the notation of (6.1.2). As determinants are linear in its columns (and rows) it follows that the map $[\;] \colon F \longrightarrow \mathscr{E}$ defined above is an $A$-module homomorphism. We get a $R(x)$-module homomorphism $R(x) \longrightarrow \mathscr{E}$ that sends $r \mapsto r \cdot [1]$, and then also an $A$-module homorphism

$$F \bigotimes\nolimits_A R(x) \longrightarrow \mathscr{E} \tag{6.1.3}$$

sending $y \otimes r \mapsto r \cdot [y]$.

**6.2. Universal multiplication.** With the notation as above we define now the $R(x)$-bilinear map $\mathscr{E} \times \mathscr{E} \longrightarrow \mathscr{E}$ by defining its action on the basis as

$$[x_i][x_j] := [x_i x_j] \quad \text{for} \quad i, j \in \{1, \ldots, n\}. \tag{6.2.1}$$

We will show that the above defined bilinear map gives $\mathscr{E}$ the structure of a commutative $R(x)$-algebra. We first observe the following simple but important fact. Consider $\mathscr{E}$ as a sheaf on $\operatorname{Spec}(R(x))$, and let $U \subset \operatorname{Spec}(R(x))$ be a quasi-compact subscheme of $\operatorname{Spec}(R(x))$. Assume furthermore that the bilinear map (6.2.1) restricted to $\mathscr{E}_U$ gives a ring structure on $\mathscr{E}_U$. That is the product (6.2.1) is associative, has an multiplicative identity and is distributive, then we also have a ring structure on $\mathscr{E}_{\bar{U}}$, where $\bar{U}$ is the scheme theoretic closure of $U \subseteq \operatorname{Spec}(R(x))$. We will apply this observation to a scheme theoretic dense open subset $U \subseteq \operatorname{Spec}(R(x))$.

**Proposition 6.3.** *Let $F$ be an $A$-algebra. We have that (6.1.2) defines an algebra structure on $\mathscr{E}$ and that the map (6.1.3) is a surjective $R(x)$-algebra homomorphism.*

*Proof.* Let $R = \bigoplus_{n \geq 0} I_F^n$, where $I_F \subseteq \Gamma_A^n F$ is the ideal of norms. We have that $\operatorname{Spec}(R(x))$ is an affine open subset of $\operatorname{Proj}(R)$, where

$$\rho\colon \operatorname{Proj}(R) \longrightarrow \operatorname{Spec}(\Gamma_A^n F)$$

is the blow-up with center $\Delta = \operatorname{Spec}((\Gamma_A^n F)/I_F)$. The open complement $\operatorname{Proj}(R) \setminus \rho^{-1}(\Delta)$ of the effective Cartier divisor $\rho^{-1}(\Delta)$ is schematically dense. Hence

$$U := \operatorname{Spec}(R(x)) \setminus \rho^{-1}(\Delta) \cap \operatorname{Spec}(R(x))$$

is schematically dense in $\operatorname{Spec}(R(x))$. By (6.2) it suffices to show the statements over $U$. However we have that $U = \operatorname{Spec}(U_A(x))$ as defined in (5.6.1), and that the restriction of $\mathscr{E}_{|U}$ coincides with the family $\operatorname{Spec}(M_A(x))$. In other words, we have that restriction of the multiplication map (6.1.3) to the open $U$ coincides with the universal multiplication map of Proposition (5.11).

□

**Corollary 6.4.** *We have that $\mathscr{E}(x)$ is an $R(x)$-valued point of the Hilbert functor $\operatorname{Hilb}_F^n$.*

*Proof.* The proposition gives that $\operatorname{Spec}(\mathscr{E})$ is a closed subscheme of $\operatorname{Spec}(F \bigotimes_A R(x))$. By construction the $R(x)$-module $\mathscr{E}$ is free of rank $n$. □

**Corollary 6.5.** *The schemes $\operatorname{Spec}(R(x))$, for different choices of $x = x_1, \ldots, x_n$ in $F$, form an affine open cover of $\operatorname{Proj}(R)$, and the families $\operatorname{Spec}(\mathscr{E}(x)) \longrightarrow \operatorname{Spec}(R(x))$ glue together to a $\operatorname{Proj}(R)$-valued point of the Hilbert functor $\operatorname{Hilb}_F^n$.*

*Proof.* The first statement follows from Lemma (3.5). To prove the second assertion it suffices to see that the families glue over an open schematically dense set. Let $U = \mathrm{Proj}(R) \setminus \rho^{-1}(\Delta)$, where $\rho \colon \mathrm{Proj}(R) \longrightarrow \mathrm{Spec}(\Gamma^n_A F)$ is the blow-up with center $\Delta$. Then we have that $\mathrm{Spec}(R(x)) \cap U = \mathrm{Spec}(U_A(x))$ for any $n$-elements $x = x_1, \dots, x_n$ in $F$, and the result follows. □

## 7. The good component

**7.1.** When $X \longrightarrow S$ is an algebraic space we have the Hilbert functor $\mathrm{Hilb}^n_{X/S}$ of closed subspaces of $X$ that are flat and finite of rank $n$ over the base. If $U \longrightarrow X$ is an étale map we define the subfunctor $\mathscr{H}^n_{U\to X}$ of $\mathrm{Hilb}^n_{U/S}$ by assigning to any $S$-scheme $T$ the set

$$\mathscr{H}^n_{U\to X}(T) = \{Z \in \mathrm{Hilb}^n_{U/S}(T) \text{ such that the composite map } Z \subseteq U \times_S T \longrightarrow X \times_S T \text{ is a closed immersion}\}.$$

**Proposition 7.2.** *Let $X \longrightarrow S$ be a separated quasi-compact algebraic space over an affine scheme $S$, and let $U \longrightarrow X$ be an étale representable cover with $U$ an affine scheme, and let $R = U \times_X U$. Then we have the following*

1. *The functor $\mathscr{H}^n_{U\to X}$ is representable by a scheme.*
2. *The natural map $\mathscr{H}^n_{U\to X} \longrightarrow \mathrm{Hilb}^n_{X/S}$ is representable, étale and surjective.*
3. *The two maps $\mathscr{H}^n_{R\to X} \rightrightarrows \mathscr{H}^n_{U\to X}$ form an étale equivalence relation, and the quotient is $\mathrm{Hilb}^n_{X/S}$.*

*Proof.* Since $X \longrightarrow S$ is separated the composition $Z \longrightarrow U \times_S T \longrightarrow X \times_S T$ will be finite, for any $Z \in \mathrm{Hilb}^n_{U/S}(T)$, any $S$-scheme $T$. It is then clear that $\mathscr{H}^n_{U\to X}$ is an open subfunctor of $\mathrm{Hilb}^n_{U/S}$ where the latter is known to be representable ([13]). This shows the first assertion. To see that the map $\mathscr{H}^n_{U\to X} \longrightarrow \mathrm{Hilb}^n_{X/S}$ is representable we let $T \longrightarrow \mathrm{Hilb}^n_{X/S}$ be a morphism, with $T$ some $S$-scheme. Let $Z \subseteq X \times_S T$ denote the corresponding closed subscheme, and let $Z_U = Z \times_X U$. It is easily verified that the set of $T$-points of the fiber product $\mathscr{H}^n_{U\to X} \times_{\mathrm{Hilb}^n_{X/S}} T$ equals the set of sections of $Z_U \longrightarrow Z$. Thus the fibred product equals the Weil restriction of scalars $\mathfrak{R}_{Z/T}(Z_U)$ of $Z_U$ with respect to $Z \longrightarrow T$. If $T$ is an affine scheme then so is $U \times_S T$ and $Z_U$. In particular the fiber of $Z_U \longrightarrow T$ over any point $t \in T$ is contained in some affine open subscheme of $Z_U$. Therefore [4, Thm. 7:4] applies, and the Weil restriction $\mathfrak{R}_{Z/T}(Z_U)$ is representable by a scheme. Hence the map $\mathscr{H}^n_{U\to X} \longrightarrow \mathrm{Hilb}^n_{X/S}$ is representable. Étaleness of the map follows from [4, Prop. 7:5], and surjectivity follows as any $T$-valued point of $\mathrm{Hilb}^n_{X/S}$ étale locally lifts to $U$. It is easy to see that the natural map $\mathscr{H}^n_{R\to X} \longrightarrow \mathscr{H}^n_{U\to X} \times_{\mathrm{Hilb}^n_{X/S}} \mathscr{H}^n_{U\to X}$ is an isomorphism. The result then follows from (2). □

**Corollary 7.3.** *Let $X \longrightarrow S$ be a separated map of algebraic spaces. Then* $\mathrm{Hilb}^n_{X/S}$ *is an algebraic space.*

*Proof.* It suffices to show the statement for affine base $S$. Let $X' \subseteq X$ be an open immersion. Then as $X \longrightarrow S$ is assumed separated we have a map $\mathrm{Hilb}^n_{X'/S} \longrightarrow \mathrm{Hilb}^n_{X/S}$ which is a representable open immersion. Furthermore, as

$$\mathrm{Hilb}^n_{X/S} = \operatorname*{ind.lim}_{\substack{X' \subseteq X \\ \text{open, q-compact}}} \mathrm{Hilb}^n_{X'/S}$$

we may assume that $X \longrightarrow S$ is quasi-compact as well. Then the result follows from the proposition. □

*Remark* 7.4. For a quasi-projective scheme $X \longrightarrow S$ over a Noetherian base scheme $S$ it was proven by Grothendieck that the Hilbert functor $\mathrm{Hilb}^n_{X/S}$ is representable by a scheme ([12]). For a separated algebraic space $X \longrightarrow S$ locally of finite presentation, Artin proved that $\mathrm{Hilb}^n_{X/S}$ is an algebraic space ([2]). The proof of the general result above showing that $\mathrm{Hilb}^n_{X/S}$ is an algebraic space for any separated algebraic space $X \longrightarrow S$ was suggested to us by one of the referees.

**7.5. The good component.** Let $X \longrightarrow S$ be a separated algebraic space, and let $Z \longrightarrow \mathrm{Hilb}^n_{X/S}$ be the universal family, which by definition is finite, flat of rank $n$. The discriminant $D_Z \subseteq \mathrm{Hilb}^n_{X/S}$ is a closed subspace with the open complement $U^{et}_{X/S}$ parameterizing length $n$ étale subspaces of $X$. We define $\mathrm{G}^n_{X/S} \subseteq \mathrm{Hilb}^n_{X/S}$ as the schematic closure of the open subspace $U^{et}_{X/S}$. We call $\mathrm{G}^n_{X/S}$ the *good* or *principal component.*

*Remark* 7.6. Let $f\colon Z \longrightarrow H$ be a morphism of algebraic spaces which is a finite and flat morphism of rank $n$. Then the set $U \subseteq H$ above which $f$ is étale is an open subset being the complement of the discriminant $D_{Z/H}$. The scheme theoretic closure of $U \subseteq H$ is then the largest closed subspace of $H$ over which the discriminant of $f$ is a non-zero-divisor.

**Theorem 7.7.** *Let $X = \mathrm{Spec}(F) \longrightarrow S = \mathrm{Spec}(A)$ be a morphism of affine schemes, and let $\Delta \subseteq \mathrm{Spec}(\Gamma^n_A F)$ be the closed subscheme defined by the ideal of norms. Then we have that the good component $\mathrm{G}^n_{X/S}$ is isomorphic to the blow-up $\mathrm{Bl}(\Delta)$ of $\mathrm{Spec}(\Gamma^n_A F)$ along $\Delta$. The isomorphism*

$$\mathrm{b}_X\colon \mathrm{G}^n_{X/S} \xrightarrow{\simeq} \mathrm{Bl}(\Delta),$$

*is induced from restricting the norm map $\mathrm{n}_X\colon \mathrm{Hilb}^n_{X/S} \longrightarrow \mathrm{Spec}(\Gamma^n_A F)$ to the good component $\mathrm{G}^n_{X/S}$.*

*Proof.* By Theorem (4.10) we have that the inverse image $\mathrm{n}_X^{-1}(\Delta)$ is the discriminant $D_Z \subseteq \mathrm{Hilb}^n_{X/S}$ of the universal family $Z \longrightarrow \mathrm{Hilb}^n_X$.

Consequently we have that the local equation of the closed immersion

$$\mathrm{G}^n_{X/S} \cap \mathrm{n}_X^{-1}(\Delta) \subseteq \mathrm{G}^n_{X/S},$$

is not a zero divisor. Therefore, by the universal properties of the blow-up, we get an induced morphism $\mathrm{b}_X \colon \mathrm{G}^n_{X/S} \longrightarrow \mathrm{Bl}(\Delta)$. A morphism we will show is an isomorphism.

We have by Corollary (6.5) the $\mathrm{Bl}(\Delta)$-valued point $\mathscr{E}$ of the Hilbert functor $\mathrm{Hilb}^n_F$. From the defining properties of the Hilbert scheme we then have a morphism $f_{\mathscr{E}} \colon \mathrm{Bl}(\Delta) \longrightarrow \mathrm{Hilb}^n_{X/S}$ such that the pull-back of the universal family is $\mathscr{E}$. When restricting $\mathscr{E}$ to the open set $U = \mathrm{Spec}(\Gamma^n_A F) \setminus \Delta$ we have an étale family – by construction of $\mathscr{E}$. Hence the image $f_{\mathscr{E}}(U)$ is contained in $U^{et}_{X/S}$. It follows that the schematic closure $\overline{U^{et}_{X/S}} = \mathrm{G}^n_{X/S}$ contains the image of the schematic closure of $\overline{U} = \mathrm{Bl}(\Delta)$. Consequently we have a morphism $f_{\mathscr{E}} \colon \mathrm{Bl}(\Delta) \longrightarrow \mathrm{G}^n_{X/S}$, a morphism we claim is the inverse to the map $\mathrm{b}_X \colon \mathrm{G}^n_{X/S} \longrightarrow \mathrm{Bl}(\Delta)$.

By Proposition (5.11) we have that the restriction of $f_{\mathscr{E}}$ to $U$ is the inverse of the restriction of $\mathrm{b}_X$ to $U^{et}_{X/S}$. As both $U$ in $\mathrm{Bl}(\Delta)$ and $U^{et}_{X/S}$ in $\mathrm{G}^n_{X/S}$ are open complements of effective Cartier divisors it follows that $f_{\mathscr{E}}$ is the inverse of $\mathrm{b}_X$. □

**7.8.** For a separated map of algebraic spaces $X \longrightarrow S$ there exists an algebraic space $\Gamma^n_{X/S}$ that naturally globalize the affine situation with $\mathrm{Spec}(\Gamma^n_A F)$ ([21]). For the convenience of the reader we will give a description of this space for $X$ quasi-compact over an affine base. Not only is the quasi-compact case technically easier to handle, but it turns out to be sufficient in order to generalize Theorem (7.7).

**7.9. Pro-equivalence.** We will say that two sequences (indexed by the non-negative integers) of ideals $\{I_m\}$ and $\{J_m\}$ in a ring $B$ are *pro-equivalent* if for each $m$ there exists an integer $m' \geq 0$ such $I_{m'} \subseteq J_m$, and $J_{m'} \subseteq I_m$.

**Lemma 7.10.** *Let $G$ be a finite group acting on a Noetherian ring $B$ and let $\mathfrak{a} \subseteq B$ be an invariant ideal. Assume furthermore that the invariant ring $B^G$ is Noetherian, and that $B$ is a finite module over the invariant ring. Then, as ideals in $B^G$, we have that $\{(\mathfrak{a}^G)^m\}$ is pro-equivalent with $\{(\mathfrak{a}^m)^G\}$.*

*Proof.* Clearly $(\mathfrak{a}^G)^{m'} \subseteq (\mathfrak{a}^m)^G$ for all $m' \geq m$, and consequently it suffices to show that $(\mathfrak{a}^{m'})^G \subseteq (\mathfrak{a}^G)^m$ for some $m'$. An element $x \in B$ is a root of the monic polynomial $m_x(t) = \prod_{g \in G}(t - gx)$. Since $\mathfrak{a}$ is $G$-invariant this gives that for any $x \in \mathfrak{a}$ we have $x^{|G|} \in \mathfrak{a}^G$. If now $\mathfrak{a}$ is generated by $r$ elements this implies that

$$\mathfrak{a}^{m'} \subseteq (\mathfrak{a}^G)^m B,$$

where $m' = (r(|G|-1)+1)m$. By assumption $B$ is a finitely generated $B^G$-module, and consequently by the Artin-Rees Lemma ([3, Cor. 10.10]) there exists an integer $k \geq 0$ such that for $m \geq k$ we have that

$$(\mathfrak{a}^G)^m B \cap B^G = (\mathfrak{a}^G)^{m-k}\big((\mathfrak{a}^G)^k B \cap B^G\big) \subseteq (\mathfrak{a}^G)^{m-k}.$$

Hence $(\mathfrak{a}^{m'+k})^G \subseteq (\mathfrak{a}^G)^m$. □

**Lemma 7.11.** *Let $F$ be an $A$-algebra of finite type, and let $I \subseteq F$ be a finitely generated ideal. For each $m > 0$ we let $J_m$ denote the kernel of the natural map $\Gamma^n_A(F) \longrightarrow \Gamma^n_A(F/I^m)$. Then $\{J_m\}$ is pro-equivalent with $\{J_1^m\}$.*

*Proof.* We first show a special case. Let $X = x_1, \dots, x_r$ and $T = t_1, \dots, t_s$ be variables over $A = \mathbf{Z}$, the integers, and let $F = \mathbf{Z}[X,T]$, and $I = (T)$. Let $\mathfrak{a}_m$ denote the kernel of $\mathrm{T}^n_A F \longrightarrow \mathrm{T}^n_A(F/I^m)$. It is easily checked that $\{\mathfrak{a}_m\}$ is pro-equivalent with $\{\mathfrak{a}_1^m\}$. The group $\mathfrak{S}_n$ acts on $\mathrm{T}^n_A F$, and it follows that $\{(\mathfrak{a}_1^m)^{\mathfrak{S}_n}\}$ is pro-equivalent with $\{\mathfrak{a}_m^{\mathfrak{S}_n}\}$. By Lemma (7.10) we have that $\{(\mathfrak{a}_1^m)^{\mathfrak{S}_n}\}$ is pro-equivalent with $\{(\mathfrak{a}_1^{\mathfrak{S}_n})^m\}$. As $F/I^m$ is free, and in particular flat $\mathbf{Z}$-module for all $m > 0$, we have that $\Gamma^n_A(F/I^m) = \mathrm{TS}^n_A(F/I^m)$. In particular we get that

$$\ker(\Gamma^n_A F \longrightarrow \Gamma^n_A(F/I^m)) = (\mathfrak{a}_m)^{\mathfrak{S}_n},$$

and we have proven the lemma in the special case. Since we have that $\Gamma^n_{\mathbf{Z}}\mathbf{Z}[X,T] \bigotimes_{\mathbf{Z}} A = \Gamma^n_A A[X,T]$ the lemma is also proven for $F = A[X,T]$, and $I = (T)$. In the general case we let $\varphi\colon A[X,T] \longrightarrow F$ denote the $A$-algebra homomorphism that sends $X$ to a set of generators of $F$, and $T$ to a set of generators of the ideal $I \subseteq F$. For each $m > 0$ we have induced surjective maps $\varphi_m\colon A[X,T]/(T)^m \longrightarrow F/I^m$ and $\Gamma(\varphi_m)\colon \Gamma^n_A A[X,T]/(T)^m \longrightarrow \Gamma^n_A F/I^m$. An element in $\ker(\Gamma(\varphi_m))$ is of the form ([19, Prop. IV.8, p. 284])

$$\gamma^c(\bar{f}) * \gamma^{n-c}(\bar{g})$$

where $\bar{g} \in A[X,T]/(T)^m$ and $\bar{f} \in \ker(\varphi_m)$. Clearly we can find elements $f$ and $g$ in $A[X,T]$, with $f \in \ker(\varphi)$, that restricts to $\bar{f}$ and $\bar{g}$ by the canonical map. Thus the induced map $\ker(\Gamma^n(\varphi)) \longrightarrow \ker(\Gamma^n(\varphi_m))$ is surjective for all $m > 0$. It follows that the induced map from

$$\mathfrak{a}_m = \ker\left(\Gamma^n_A A[X,T] \longrightarrow \Gamma^n_A(A[X,T]/(T)^m)\right)$$

to $J_m = \ker(\Gamma^n_A F \longrightarrow \Gamma^n_A(F/I^m))$ is surjective. In particular $\mathfrak{a}_1$ surjects to $J_1$, so $\mathfrak{a}_1^m$ surjects to $J_1^m$. The lemma now follows by lifting elements to $\mathfrak{a}_m$ and $\mathfrak{a}_1^m$, where the result holds. □

**7.12. FPR-sets.** Let $G$ be a finite group acting on a separated algebraic space $X$. By a result of Deligne the geometric quotient $X/G$ exists as an algebraic space. We will make use of that result, but we need also to recall the notion of *fixed-point-reflecting* (abbreviated FPR) sets.

Following ([16, p.183]) we say that an equivariant map $f\colon X \longrightarrow Y$ is FPR at a point $\varphi\colon \operatorname{Spec}(L) \longrightarrow X$, where $L$ is a field, if for all $\sigma \in G$ we have that $\sigma(f\varphi) = f\varphi$ implies that $\sigma(\varphi) = \varphi$. An equivalent condition is that we have an equality of stabilizer groups $G_\varphi = G_{f\varphi}$.

An open invariant subspace $U \subseteq X$ is called a FPR set if $f\colon X \longrightarrow Y$ is FPR at all points $x$ of $U$.

Let $\mathscr{A}$ be a directed set. A subset $S \subseteq \mathscr{A}$ is *eventually upwards closed* if there exists an index $\alpha \in \mathscr{A}$ such that for all $\beta \geq \alpha$ we have $\beta \in S$. Note that if $\mathscr{A}$ is non-empty, then a subset $S \subseteq \mathscr{A}$ that is eventually upwards closed is also non-empty.

**Lemma 7.13.** *i) Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are $G$-morphisms, $h$ their composite and $x$ a point of $X$. Then $h$ is FPR at $x$ precisely when $f$ is FPR at $x$ and $g$ is FPR at $f(x)$.*

*ii) Suppose that $\{X_\alpha\}$ is an inverse system with affine transition maps of $G$-spaces. For every point $x$ of $X := \varprojlim_\alpha X_\alpha$ the set $S_x := \{\alpha \mid p_\alpha \text{ is FPR at } x\}$, where $p_\alpha\colon X \to X_\alpha$ is the structure map, is eventually upwards closed.*

*iii) Suppose that also $\{Y_\alpha\}$ is an inverse system with affine transition maps of $G$-spaces over the same index set and that $\{f_\alpha\colon X_\alpha \to Y_\alpha\}$ is a $G$-morphism of directed systems. Set $Y := \varprojlim_\alpha Y_\alpha$, $f := \varprojlim_\alpha f_\alpha$ and assume that $f$ is FPR at the point $x$ of $X$. Then $\{\alpha \mid f_\alpha \text{ is FPR at } x_\alpha\}$ is eventually upwards closed.*

*Proof.* For the first part we always have that $G_x \subseteq G_{f(x)} \subseteq G_{h(x)}$ so that if $h$ is FPR at $x$, i.e., $G_x = G_{h(x)}$, then $f$ is FPR at $x$ and $g$ is FPR at $f(x)$ and clearly conversely.

Assume that $\alpha \notin S_x$. By definition the structure map $p_\alpha\colon X \longrightarrow X_\alpha$ is not FPR at $x$, and therefore we have $G_x \neq G_{x_\alpha}$, where $x_\alpha = p_\alpha(x)$. For any $g \notin G_x$ we have $gx \neq x$, hence there is an index $\alpha_g$ such that $gx_{\alpha_g} \neq x_{\alpha_g}$. Since $G_x$ is finite we can find an $\beta \geq \alpha_g$ such that $gx_\alpha \neq x_\alpha$, for all $g \neq G_x$. Then $G_{x_\beta} = G_x$, and we have $\beta \in S_x$. It follows by (i) that for any $\beta' \geq \beta$ we have $\beta' \in S_x$, so $S_x$ is eventually upwards closed.

Finally, we get from ii) that there exists an $\alpha$ such that for all $\beta \geq \alpha$ we have that $p'_\beta\colon Y \to Y_\beta$ is FPR at $f(x)$. If $f$ is FPR at $x$, then by i) we have that $p'_\beta \circ f$ is FPR at $x$. Since $p'_\beta \circ f = f_\beta \circ p_\beta$ it follows by i) again, that $f_\beta$ is FPR at $x_\beta$, for all $\beta \geq \alpha$. □

**7.14.** We have the induced map $(\mathrm{id}_X, \sigma)\colon X \longrightarrow X \times X$, for any group element $\sigma \in G$. By taking the inverse image of the diagonal of a separated algebraic space $X \longrightarrow S$, via the map $(\mathrm{id}_X, \sigma)$ we get a closed subspace $X^\sigma \subseteq X$. If $f\colon X \longrightarrow Y$ is a $G$ equivariant map, we have a closed immersion $X^\sigma \subseteq f^{-1}(Y^\sigma)$.

**Definition-Lemma 7.15.** *If the equivariant map $f\colon X \longrightarrow Y$ is separated and unramified, then $X^\sigma$ is both open and closed in $f^{-1}(Y^\sigma)$.*

*Hence if $Y$ is also separated over some $S$ on which $G$ acts trivially, there is a maximal open FPR-subspace of $X$, which we call the* FPR-locus *of $f$.*

*In the particular case when $U \longrightarrow X$ is an unramified separated map and $X$ is separated over $S$, we will denote the FPR-locus of the induced $\mathfrak{S}_n$-map $U^n_S \longrightarrow X^n_S$ by $\Omega_{U\to X} \subseteq U^n_S$.*

*Proof.* We have a map $f^{-1}(Y^\sigma) \to X \times_Y X$ given by $x \mapsto (x, \sigma x)$ and $X^\sigma$ is the inverse image of the diagonal. As $f$ is unramified and separated, the diagonal is open and closed in $X \times_Y X$ and hence so is $X^\sigma$ in $f^{-1}(Y^\sigma)$. If $Y$ is also separated, then $f^{-1}(Y^\sigma)$ is closed in $X$ and hence the complement of $X^\sigma$ in $f^{-1}(Y^\sigma)$ is closed in $X$ and removing such subsets for all $\sigma$ gives the FPR-locus. □

**Lemma 7.16.** *Let $F \longrightarrow F'$ be an étale morphism of $A$-algebras, where $F$ and $F'$ are of finite type over a Noetherian, strictly Henselian local ring $A$. Let $\varphi\colon \mathrm{T}^n_A F' \longrightarrow L$ be a map to a field $L$, and let $\varphi_i\colon F' \longrightarrow L$ be the co-projections of $\varphi$ (with $i = 1, \dots, n$). Define the ideals $J = \cap \ker \varphi_i$ in $F'$ and $I = \cap \ker \varphi_{i|F}$ in $F$. Assume that $\varphi$ is a closed point in the FPR-locus $\Omega_{F\to F'}$ of* $\mathrm{Spec}(\mathrm{T}^n_A F') \longrightarrow \mathrm{Spec}(\mathrm{T}^n_A F)$*, lying above the closed point of* $\mathrm{Spec}(A)$*. Then the induced map*

$$F/I^m \longrightarrow F'/J^m$$

*is an isomorphism, for all $m > 0$.*

*Proof.* Since $A/\mathfrak{m}_A$ is separably closed, we have for each maximal ideal $\mathfrak{m}$ of $F$, lying above $\mathfrak{m}_A$, that the field extension $F/\mathfrak{m}$ is purely inseparable. Consequently, since $F \longrightarrow F'$ is étale, we have that $F'/\mathfrak{m}F'$ is a product of trivial extensions of $F/\mathfrak{m}$. In particular, for each maximal ideal $\mathfrak{m}'$ of $F'/J$ that contracts to $\mathfrak{m}$ in $F/I$, we have that the $\mathfrak{m}$-adic completion of $F/I$ is isomorphic to the $\mathfrak{m}'$-adic completion of $F'/J$. To prove the result we need only show that we have a bijection between the maximal ideals in $F'/J$ and the maximal ideals in $F/I$.

Since $F'$ is of finite type over $A$, and the point $\varphi\colon \mathrm{T}^n_A F' \longrightarrow L$ is closed, we may assume that $L$ is a finite field extension of the residue field $A/\mathfrak{m}_A$. It follows that the ideals $\ker \varphi_i \subset F'$, and similarily the ideals $\ker \varphi_{i|F} \subset F$, are maximal ideals $(i = 1, \dots, n)$. As the point $\varphi$ is in the FPR-locus $\Omega_{F\to F'}$ we have that $\varphi_i = \varphi_j$ if and only if $\varphi_{i|F} = \varphi_{j|F}$. Hence, there is a bijection between the maximal ideals of $F/I$ and the maximal ideals of $F'/J$. □

**7.17. Notation.** Assume now that the base scheme $S = \mathrm{Spec}(A)$ is affine, and that $X$ is a quasi-compact, separated, algebraic space. Let $U = \mathrm{Spec}(F) \longrightarrow X$ be an étale cover. We have the FPR-locus $\Omega_{U\to X} \subseteq U^n_S$, and we let

$$\Omega'_{U\to X} \subseteq U^n_S/\mathfrak{S}_n$$

denote the image of $\Omega_{U\to X}$ by the quotient map $U^n_S \longrightarrow U^n_S/\mathfrak{S}_n$. Moreover, the morphism $\mathrm{Spec}(\alpha)\colon \mathrm{Spec}(\mathrm{TS}^n_A F) \longrightarrow \mathrm{Spec}(\Gamma^n_A F)$ is a homeomorphism (see e.g. [21, Corollary 4.2.5]), and we let

$$\Omega''_{U\to X} \subseteq \mathrm{Spec}(\Gamma^n_A F)$$

denote the open set given as the image of $\Omega'_{U\to X}$ by the morphism $\mathrm{Spec}(\alpha)$.

**Proposition 7.18.** *Let $F \longrightarrow F'$ be an étale morphism of $A$-algebras, with $F$ and $F'$ of finite type over a Noetherian, strictly Henselian local ring $A$. Let $\xi \in \Omega''_{F\to F'}$ be a closed point lying over the closed point of $\mathrm{Spec}(A)$. Then the induced map of completions*

$$(\Gamma^n_A F)_{\widehat{f(\xi)}} \longrightarrow (\Gamma^n_A F')_{\widehat{\xi}}$$

*is an isomorphism, where $f(\xi)$ is the image of $\xi$ by the induced map $\mathrm{Spec}(\Gamma^n_A F') \longrightarrow \mathrm{Spec}(\Gamma^n_A F)$.*

*Proof.* It suffices to show that there are ideals $I_1 \subset \Gamma^n_A F$ and $J_1 \subset \Gamma^n_A F'$ contained in the ideals corresponding to the points $f(\xi)$ and $\xi$, respectively, such that $I_1$ maps to $J_1$ and the induced map of formal neighborhoods

$$\varprojlim(\Gamma^n_A F)/I_1^m \longrightarrow \varprojlim(\Gamma^n_A F')/J_1^m \tag{7.18.1}$$

is an isomorphism. As the morphism $\mathrm{Spec}(\mathrm{TS}^n_A F') \longrightarrow \mathrm{Spec}(\Gamma^n_A F')$ is a homeomorphism the point $\xi$ lifts to a point of $\mathrm{Spec}(\mathrm{T}^n_A F')$. Let $\varphi\colon \mathrm{T}^n_A F' \longrightarrow L'$ be a lifting of $\xi = \mathrm{Spec}(L)$, with $L'$ some field extension of $L$. Write $\varphi = (\varphi_1, \dots, \varphi_n)$, and define the ideal $J = \cap \ker(\varphi_i)$ in $F$. We let $J_m = \ker(\Gamma^n_A F' \longrightarrow \Gamma^n_A(F'/J^m))$. As the map $\Gamma^n_A F' \longrightarrow L$ factors via $\Gamma^n_A(F/J)$ we have that $J_1$ is contained in the ideal $\ker(\Gamma^n_A F' \longrightarrow L)$. We let $I_m = \ker(\Gamma^n_A F \longrightarrow \Gamma^n_A(F/I^m))$ where $I = \cap \ker(\varphi_{i|F})$, and we consider the induced map (7.18.1).

By Lemma (7.11) we have the limit of the system $\{(\Gamma^n_A F)/I_1^m\}$ equals the limit of the system $\{(\Gamma^n_A F)/I_m = \Gamma^n_A(F/I^m)\}$. By Lemma (7.16) we have that $F/I^m = F'/J^m$, and it follows that the map (7.18.1) is an isomorphism. □

**Corollary 7.19.** *Let $F \longrightarrow F'$ be an étale morphism of $A$-algebras, and let $I_F \subseteq \Gamma^n_A F$ and $I_{F'} \subseteq \Gamma^n_A F'$ be the ideals of norms associated to $F$ and $F'$, respectively. These two ideals, $I_F \Gamma^n_A F'$ and $I_{F'}$, are equal when restricted to the open subscheme $\Omega''_{F\to F'} \subseteq \mathrm{Spec}(\Gamma^n_A F')$.*

*Proof.* Assume first that the result is true when $F$ (and hence $F'$) is a finitely presented $A$-algebra. We can write $f\colon F \longrightarrow F'$ as a limit by a directed set of étale maps $f_\alpha\colon F_\alpha \longrightarrow F'_\alpha$ of finitely presented $A$-algebras, such that $F'_\alpha \bigotimes_{F_\alpha} F_\beta \simeq F'_\beta$ for all $\alpha$ and all $\beta \geq \alpha$. This means that $\mathrm{Spec}(\mathrm{T}^n_A F') \to \mathrm{Spec}(\mathrm{T}^n_A F)$ can be thought of as $\varprojlim_\beta \mathrm{Spec}(\mathrm{T}^n_A F'_\beta) \to \varprojlim_\beta \mathrm{Spec}(\mathrm{T}^n_A F_\beta)$ and similarly for $\mathrm{T}^n$ replaced by

$\mathrm{TS}^n$ (as directed direct limits commute with taking invariants) and $\Gamma^n$. The equality to be proven is one of equality of stalks so we may focus on a particular point $x'' \in \Omega''_{F\to F'}$ which is the image of some point $x \in \Omega_{F\to F'}$. Let $y \in \mathrm{Spec}(\mathrm{T}^n_A F)$ denote the image of $x$ under the map $\mathrm{Spec}(\mathrm{T}^n_A F') \longrightarrow \mathrm{Spec}(\mathrm{T}^n_A F)$. By Lemma (7.13) ii) we may assume that all projection maps $p_\alpha\colon \mathrm{Spec}(\mathrm{T}^n_A F) \longrightarrow \mathrm{Spec}(\mathrm{T}^n_A F_\alpha)$ are FPR at $y$. Then by Lemma (7.13) i) the two compositions

$$\begin{array}{ccc} \mathrm{Spec}(\mathrm{T}^n_A F') & \longrightarrow & \mathrm{Spec}(\mathrm{T}^n_A F) \\ \downarrow{\scriptstyle p'_\alpha} & & \downarrow{\scriptstyle p_\alpha} \\ \mathrm{Spec}(\mathrm{T}^n_A F'_\alpha) & \longrightarrow & \mathrm{Spec}(\mathrm{T}^n_A F_\alpha) \end{array}$$

are FPR at $x$ for all $\alpha$. By Lemma (7.13) i) again, we have that $\mathrm{Spec}(\mathrm{T}^n_A F'_\alpha) \longrightarrow \mathrm{Spec}(\mathrm{T}^n_A F_\alpha)$ is FPR at $p'_\alpha(x)$, which means that $p'_\alpha(x) \in \Omega_{F_\alpha \to F'_\alpha}$. But then $x''_\alpha$, the image of $x''$ under the projection map $\mathrm{Spec}(\Gamma^n_A F') \longrightarrow \mathrm{Spec}(\Gamma^n_A F'_\alpha)$, is in $\Omega''_{F_\alpha\to F'_\alpha}$. Hence we get the equality $I_{F_\alpha}\Gamma^n_A F'_\alpha = I_{F'_\alpha}\Gamma^n_A F'_\alpha$ at $x''_\alpha$ and taking the direct limit of sheaves in $\alpha$ gives the corollary at $x''$ and hence in $\Omega''_{F\to F'}$.

We are therefore left with the case when $F$ is a finitely presented $A$-algebra. By another (simpler) limit argument we reduce to the case when $A$ is Noetherian. Assume, by way of contradiction, that we have a closed point $\xi \in \Omega''_{F\to F'}$ at which $I_{F'}$ and $I_F\Gamma^n_A F'$ differ. Let $\hat{A}$ denote the localizing and strictly Henseliziation of $A$ at the image of $\xi$. Let $\hat{F} = F \otimes_A \hat{A}$ and let $\hat{F}' = F' \otimes_A \hat{A}$. By the proposition the two ideals $I_{\hat{F}'}$ and $I_{\hat{F}}\Gamma^n_{\hat{A}}\hat{F}'$ are equal at the completion of every closed point of $\Omega''_{\hat{F}\to\hat{F}'}$, hence equal on $\Omega''_{\hat{F}\to\hat{F}'}$. But, then it follows that also the ideals $I_{F'}$ and $I_F\Gamma^n_A F'$ are equal at $\xi$, and therefore equal at $\Omega''_{F\to F'}$. □

**Corollary 7.20.** *Let $F \longrightarrow F'$ be an étale morphism of $A$-algebras. The induced maps $\Omega'_{F\to F'} \longrightarrow \mathrm{Spec}(\mathrm{TS}^n_A F)$ and $\Omega''_{F\to F'} \longrightarrow \mathrm{Spec}(\Gamma^n_A F)$ are étale.*

*Proof.* We only show that $\Omega''_{F\to F'} \longrightarrow \mathrm{Spec}(\Gamma^n_A F)$ is étale, the case with $\Omega'_{F\to F'}$ is similar. Assume first that $F$ and $F'$ are finite type over a Noetherian ring $A$. Étaleness can be checked at a point, and by localization and Henselization we may assume that $A$ is strictly Henselian and that the point lies in the special fiber. We can then reduce the question of étaleness to the case with the point being closed, and then the result follows from the proposition.

In the general case we write, as in the previous proof, $f\colon F \longrightarrow F'$ as a limit of a directed set $f_\alpha\colon F_\alpha \longrightarrow F'_\alpha$ of finitely presented algebras, where $F'_\alpha \bigotimes_{F_\alpha} F_\beta = F'_\beta$. We may assume that $A$ is Noetherian.

Let $p_{\alpha,\beta}\colon \operatorname{Spec}(\Gamma^n_A F'_\beta) \longrightarrow \operatorname{Spec}(\Gamma^n_A F'_\alpha)$ denote the induced map. We have the commutative diagram

(7.20.1)
$$\begin{array}{ccc} p^{-1}_{\alpha,\beta}(\Omega''_{F_\alpha \to F'_\alpha}) & \longrightarrow & \operatorname{Spec}(\Gamma^n_A F_\beta) \\ \downarrow & & \downarrow \\ \Omega''_{F_\alpha \longrightarrow F'_\alpha} & \longrightarrow & \operatorname{Spec}(\Gamma^n_A F_\alpha). \end{array}$$

The corollary is proven if we show that the diagram is Cartesian. The lower horizontal map in (7.20.1) is étale by what just proven above. One checks that we have an inclusion of open sets $p^{-1}_{\alpha,\beta}(\Omega''_{F_\alpha\to F'_\alpha}) \subseteq \Omega''_{F_\beta \to F'_\beta}$. And consequently the upper horizontal map in (7.20.1) is also étale. As the horizontal maps in the diagram (7.20.1) are étale, it suffices to check Cartesianity for maps from $\operatorname{Spec}(k)$, with $k$ algebraically closed fields. Since the map $\operatorname{Spec}(\mathrm{TS}^n_A F) \longrightarrow \operatorname{Spec}(\Gamma^n_A F)$ is a homeomorphism, we can reduce the question of Cartesianity to the corresponding statement with $\Omega'$ and $\mathrm{TS}^n_A$ replacing $\Omega''$ and $\Gamma^n_A$, respectively, in (7.20.1). But, to check that we lift everything to $\Omega$ and $\mathrm{T}^n_A$, where it is clear. □

**Corollary 7.21.** *Let $X \longrightarrow S$ be a quasi-compact separated algebraic space over an affine base $S$. Write $X$ as a quotient $R \rightrightarrows U$, with affine schemes $U$ and $R$. Then we have that $\Omega''_{R\to X} \rightrightarrows \Omega''_{U\to X}$ is an étale equivalence relation.*

*Proof.* The étale maps $\Omega_{R\to X} \underset{t}{\overset{s}{\rightrightarrows}} \Omega_{U\to X}$ are $\mathfrak{S}_n$-equivariant, and form an equivalence relation. After taking the quotients modulo the $\mathfrak{S}_n$-action, we get induced maps $\Omega'_{R\to X} \underset{t'}{\overset{s'}{\rightrightarrows}} \Omega'_{U\to X}$. It is readily checked that the two projections $s'$ and $t'$ will satisfy the reflexivity and symmetry condition. To verify the transitivity condition we first form the fiber product $R_2 = R \times_U R$ given by the maps defining the equivalence relation on $U$. We then obtain the commutative diagram

(7.21.1)
$$\begin{array}{ccc} \Omega_{R_2\to X} & \longrightarrow & \Omega_{R\to X} \\ \downarrow & & \downarrow s \\ \Omega_{R\to X} & \xrightarrow[t]{} & \Omega_{U\to X}, \end{array}$$

which one can verify is Cartesian. Thus $\Omega_{R_2\to X}$ consists of pairs $(x, y)$ with $x$ and $y$ in $\Omega_{R\to X}$ such that $t(x) = s(y)$. Transitivity of $s'$ and $t'$ is reduced to showing that the commutative diagram

(7.21.2)
$$\begin{array}{ccc} \Omega'_{R_2\to X} & \longrightarrow & \Omega'_{R\to X} \\ \downarrow & & \downarrow s' \\ \Omega'_{R\to X} & \xrightarrow{t'} & \Omega'_{U\to X} \end{array}$$

obtained by taking the $\mathfrak{S}_n$-quotients of the Cartesian diagram (7.21.1) remains Cartesian. It follows by Corollary (7.20) that the arrows in (7.21.2) are étale. Thus one checks that the diagram (7.21.2) is Cartesian by looking at geometric points, where it is clear. Since the projections $s'$ and $t'$ are étale, the morphism

$$\Omega'_{R\to X} \longrightarrow \Omega'_{U\to X} \times \Omega'_{U\to X} \tag{7.21.3}$$

is unramified. We have that (7.21.3) is injective on field valued points, hence it is a monomorphism ([11, Proposition 17.2.6]). That is, we have an étale equivalence relation $\Omega'_{R\to X} \underset{t'}{\overset{s'}{\rightrightarrows}} \Omega'_{U\to X}$.

We invoke the same arguments again: Applying the morphism $\operatorname{Spec}(\alpha)$ gives induced maps $\Omega''_{R\to X} \underset{t''}{\overset{s''}{\rightrightarrows}} \Omega''_{U\to X}$. By Corollary (7.20) the arrows in the corresponding diagram with $\Omega''$ replacing $\Omega'$ in (7.21.2), are étale. By looking at geometric points one then obtains that $\Omega''_{R_2\to X}$ equals the fiber product of $\Omega''_{R\to X} \times_{\Omega''_{U\to X}} \Omega''_{R\to X}$ via the two projections $s''$ and $t''$. This proves the transitivity axiom, and reflexitivity and symmetry is clear. Finally, the

$$\Omega''_{R\to X} \longrightarrow \Omega''_{U\to X} \times \Omega''_{U\to X} \tag{7.21.4}$$

is unramfied. Since $\operatorname{Spec}(\alpha)$ is a universal homeomorphism we have that (7.21.4) is radical, hence a monomorphism, and we have proven the claim. □

**Proposition 7.22.** *Let $X \longrightarrow S$ be a separated quasi-compact algebraic space over an affine scheme $S = \operatorname{Spec}(A)$. Let $U = \operatorname{Spec}(F) \longrightarrow X$ be an étale affine cover, and let $R = U \times_X U$. Define $\Gamma^n_{X/S}$ as the quotient of the étale equivalence relation $\Omega''_{R\to X} \rightrightarrows \Omega''_{U\to X}$.*

(1) *We have a cartesian diagram*

$$\begin{array}{ccc} \mathscr{H}^n_{U\to X} & \longrightarrow & \operatorname{Hilb}^n_{U/S} \\ \big\downarrow {\scriptstyle \mathrm{n}_U} & & \big\downarrow {\scriptstyle \mathrm{n}_U} \\ \Omega''_{U\to X} & \longrightarrow & \Gamma^n_{U/S} = \operatorname{Spec}(\Gamma^n_A(F)). \end{array}$$

(2) *In the diagram below we have $n_U \circ p_i = q_i \circ n_R$ , $i = 1, 2$, and consequently there is an induced map $\mathrm{n}_X \colon \operatorname{Hilb}^n_{X/S} \longrightarrow \Gamma^n_{X/S}$:*

$$\begin{array}{ccccc} \mathscr{H}^n_{R\to X} & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \mathscr{H}^n_{U\to X} & \xrightarrow{p} & \operatorname{Hilb}^n_{X/S} \\ \big\downarrow {\scriptstyle \mathrm{n}_R} & & \big\downarrow {\scriptstyle \mathrm{n}_U} & & \big\downarrow {\scriptstyle \mathrm{n}_X} \\ \Omega''_{R\to X} & \overset{q_1}{\underset{q_2}{\rightrightarrows}} & \Omega''_{U\to X} & \xrightarrow{q} & \Gamma^n_{X/S} \end{array}$$

*Moreover, the commutative diagrams above are cartesian.*

*Proof.* Let us first consider the special case with $S = \mathrm{Spec}(k)$, where $k$ is an algebraically closed field. A $k$-valued point $Z \subseteq U$ of the Hilbert functor $\mathrm{Hilb}^n_{U/S}$ has support at a finite number of points $\xi_1, \dots, \xi_p$. By (4.4) the associated cycle $\mathrm{n}_U(Z)$ consists of the points $\xi_1, \dots, \xi_p$ counted with multiplicities $m_1, \dots, m_p$. We have that the cycle $\mathrm{n}_U(Z)$ is in the FPR-set $\Omega''_{U \to X}$ if and only if the closed subscheme $Z \subseteq U$ also is a closed subscheme of $X$.

Now, let us prove the proposition. In the first diagram (1) the horizontal maps are open immersions. To see that it is commutative and cartesian it suffices to establish the equality of the two open sets $\mathscr{H}^n_{U \to X}$ and $\mathrm{n}_U^{-1}(\Omega''_{U \to X})$ of $\mathrm{Hilb}^n_{U/S}$. This we can be checked by reducing to $S = \mathrm{Spec}(k)$, with $k$ algebraically closed. Then we are in the special case considered above from which Assertion (1) follows.

In particular we have proven that the restriction of the norm map $\mathrm{n}_U$ to the open subset $\mathscr{H}^n_{U \to X}$ has $\Omega''_{U \to X}$ as range. We therefore obtain the two leftmost diagrams in (2). Since the horizontal maps in these diagram are étale (Proposition (7.2) and Corollary (7.21)) we can prove the diagrams are cartesian by evaluation over algebraically closed points. We are then again reduced to the special case considered above, which proves assertions in (2). □

**Proposition 7.23** (Rydh)**.** *Let $X \longrightarrow S$ be a separated map of algebraic spaces. Then there exists an algebraic space $\Gamma^n_{X/S} \longrightarrow S$ such that*

1. *When $X \longrightarrow S$ is quasi-compact with $S$ an affine scheme, the space $\Gamma^n_{X/S}$ coincides with the one constructed above (7.22).*
2. *For any base change map $T \longrightarrow S$ we have a natural identification $\Gamma^n_{X/S} \times_S T = \Gamma^n_{X \times_S T/T}$.*
3. *For any open immersion $X' \subseteq X$ we have an open immersion $\Gamma^n_{X'/S} \subseteq \Gamma^n_{X/S}$, and moreover*
$$\Gamma^n_{X/S} = \lim_{\substack{X' \subseteq X \\ \text{open, q-compact}}} \Gamma^n_{X'/S}.$$
4. *There is a universal homeomorphism $X^n_S/\mathfrak{S}_n \longrightarrow \Gamma^n_{X/S}$, which is an isomorphism when $X \longrightarrow S$ is flat, or when the characteristic is zero.*

*Proof.* All results can be found in ([21]): Existence of the space $\Gamma^n_{X/S}$ is Theorem (3.4.1), whereas Assertion (4) is Corollary (4.2.5), and the statement about open immersions in (3) is a special case of Proposition (3.1.7). The functorial description of $\Gamma^n_{X/S}$ given by David Rydh immediately gives Assertion (2) and that $\Gamma^n_{X/S}$ is the union of $\Gamma^n_{X'/S}$ with quasi-compact $X' \subseteq X$. Assertion (1) follows as our $\Omega''_{U \to X}$ is what Rydh denotes with $\Gamma^n(U/S)_{|\mathrm{reg}/f}$ (see Proposition (4.2.4), and the proof of Theorem (3.4.1), loc. cit.). □

**7.24. The ideal sheaf of norms.** For $X \longrightarrow S$ quasi-compact and separated over an affine base we have by Corollary (7.19) that the ideals of norms patch together to form an ideal sheaf $\mathscr{I}_X$ on $\Gamma^n_{X/S}$. As these ideals clearly commute with open immersions and base change we obtain by (3) and (1) of Proposition (7.23), an ideal sheaf of norms $\mathscr{I}_X$ on $\Gamma^n_{X/S}$, for any separated algebraic space $X \longrightarrow S$. Let

$$\Delta_X \subseteq \Gamma^n_{X/S}$$

denote the closed subspace defined by the ideal sheaf of norms.

**Theorem 7.25.** *Let $X \longrightarrow S$ be a separated morphism of algebraic spaces. Then the good component $\mathrm{G}^n_{X/S}$ of $\mathrm{Hilb}^n_{X/S}$ is isomorphic to the blow-up of $\Gamma^n_{X/S}$ along the closed subspace $\Delta_X \subseteq \Gamma^n_{X/S}$, defined by the ideal of norms associated to $X \longrightarrow S$. Moreover, if $X \longrightarrow S$ is flat then $\mathrm{G}^n_{X/S}$ is obtained by blowing-up the geometric quotient $X^n_S/\mathfrak{S}_n$.*

*Proof.* The Hilbert scheme $\mathrm{Hilb}^n_{X/S}$ and $\Gamma^n_{X/S}$ commute with arbitrary base change. The good component $\mathrm{G}^n_{X/S}$ as well as blow-ups, commute with flat, and in particular étale base change. We may therefore assume that the base $S$ is an affine scheme.

For any open immersion $X' \subseteq X$, with $X'$ quasi-compact, we have a norm map $\mathrm{n}_{X'} \colon \mathrm{Hilb}^n_{X'/S} \longrightarrow \Gamma^n_{X'/S}$ which, by varying $X'$, form a norm map $\mathrm{n}_X \colon \mathrm{Hilb}^n_{X/S} \longrightarrow \Gamma^n_{X/S}$. We claim now that the inverse image $\mathrm{n}_X^{-1}(\Delta_X)$ is locally principal, which we can verify on an open cover. Moreover, given that we obtain an induced map from the good component $\mathrm{G}^n_{X/S}$ to the blow-up of $\Gamma^n_{X/S}$ along $\Delta_X$. To verify that the induced map is an isomorphism, we also reduce to an open cover. Consequently we may assume that $X$ itself is quasi-compact.

When $X$ is quasi-compact we choose an étale affine cover $U \longrightarrow X$. Then by using the cartesian diagrams (2) and (1) of Proposition (7.22) one establishes using Theorem (4.10) that $\mathrm{n}_X^{-1}(\Delta_X)$ is locally principal. By Theorem (7.7) we have that the blow-up of $\Delta_U \subseteq \Gamma^n_{U/S}$ yields the good component $\mathrm{G}^n_{U/S}$, and the isomorphism is induced by the norm map $\mathrm{n}_U$. It then follows by the two cartesian diagrams (2) and (1) of Proposition (7.22), that the map induced map from $\mathrm{G}^n_{X/S}$ to the blow-up of $\Delta_X \subseteq \Gamma^n_{X/S}$ is an isomorphism. □

**7.26. The case of surfaces.** Before we give a corollary to this result we need a generalisation of a result of Fogarty on the smoothness of the Hilbert scheme ([10, Theorem 2.9]). Fogarty proves that the Hilbert scheme of a smooth map $X \longrightarrow S$ of relative dimension 2 is smooth provided that $S$ is a Dedekind scheme. As the Hilbert scheme commutes with base change and flatness can be verified in the integral Noetherian case by pulling back to Dedekind bases it follows that the result of Fogarty is valid when the base $S$ is integral. However, as we will see, no conditions on the base is needed for that statement.

We shall give a direct proof by proving formal smoothness using the infinitesimal lifting criterion and the Hilbert-Burch theorem.

**Proposition 7.27.** *Let $X \longrightarrow S$ be a smooth and separated morphism of relative dimension* 2*. Then* $\mathrm{Hilb}^n_{X/S} \longrightarrow S$ *is smooth for all* $n$*.*

*Proof.* As $\mathrm{Hilb}^n_{X/S}$ commutes with base change we can assume that the base is Noetherian. It is enough to show formal smoothness so the statement would follow if we could show that for every small thickening $T \subset T'$ of local Artinian $S$-schemes, any $T$-flat finite subscheme $Z \subseteq X \times_S T$ can be extended to a $T'$-flat finite subscheme of $X \times_S T'$. Let $s$ be the closed point in $S$. The obstruction for the existence of such a lifting is an element $\alpha \in \mathrm{Ext}^1_{\mathscr{O}_{X_s}}(\mathscr{I}_{Z_s}, \mathscr{O}_{X_s}/\mathscr{I}_{Z_s})$. We have an exact "local-to-global" sequence

$$H^1(X_s, \mathscr{H}om_{\mathscr{O}_{X_s}}(\mathscr{I}_{Z_s}, \mathscr{O}_{X_s}/\mathscr{I}_{Z_s})) \to \mathrm{Ext}^1_{\mathscr{O}_{X_s}}(\mathscr{I}_{Z_s}, \mathscr{O}_{X_s}/\mathscr{I}_{Z_s}) \to H^0(X_s, \mathscr{E}xt^1_{\mathscr{O}_{X_s}}(\mathscr{I}_{Z_s}, \mathscr{O}_{X_s}/\mathscr{I}_{Z_s})).$$

As $\mathscr{H}om_{\mathscr{O}_{X_s}}(\mathscr{I}_{Z_s}, \mathscr{O}_{X_s}/\mathscr{I}_{Z_s})$ has finite support, the left term of the above sequence is 0, and consequently it suffices to show that the image of the obstruction element $\alpha$ in $H^0(X_s, \mathscr{E}xt^1_{\mathscr{O}_{X_s}}(\mathscr{I}_{Z_s}, \mathscr{O}_{X_s}/\mathscr{I}_{Z_s}))$ is zero. As $Z$ is a disjoint union of points we have that $\alpha = \prod \alpha_{z_i}$, where at a point $z \in Z$ the factor $\alpha_z$ is the obstruction for lifting $\mathrm{Spec}\, \mathscr{O}_{Z,z}$, which is a closed flat subscheme of $\mathrm{Spec}\, \mathscr{O}_{X \times_S T,z}$, to a flat subscheme of $\mathrm{Spec}\, \mathscr{O}_{X \times_S T',z}$. It is thus enough to show that these local obstructions vanish. Hence our situation is as follows: We have a surjection of local Artinian rings $R' \longrightarrow R$ whose kernel is 1-dimensional over the residue field, an essentially smooth 2-dimensional local $R'$-algebra $S'$, and a quotient $S := S' \bigotimes_{R'} R \longrightarrow T$ such that $T$ is a finite flat $R$-module. We then want to lift $T$ to a quotient $S' \longrightarrow T'$ which is a flat $R'$-module. We first claim that $T$ has projective dimension 2 over $S$. As $T$ is $R$-flat it is enough to check $\overline{T}$ has projective dimension 2 over $\overline{S}$, where $\overline{(-)}$ denotes reduction modulo the maximal ideal of $R$. In that case we have that $\overline{T}$ is a Cohen-Macaulay module over the regular local ring $\overline{S}$ with support of codimension 2 and the result follows.

By [18, Thm. 7.15] (cf. also the original proof in [6]) it then follows that the ideal $I_T$ defining $T$ is the determinant ideal of $n \times n$-minors of an $n+1 \times n$-matrix $M$ and that the grade (the maximal length of $S$-regular sequence contained in $I_T$) of $I_T$ is 2. We then (arbitrarily) lift $M$ to a matrix $M'$ over $S'$ and let $T'$ be defined by $n \times n$-minors of $M'$. What remains to show is that $T'$ is $R'$-flat. The grade of $I_{T'}$ is also 2 as we may lift an $S$-regular sequence in $I_T$ to elements of $I_{T'}$ which then give an $S'$-regular sequence and hence by [18, Thm. 7.16], the sequence

$$0 \longrightarrow (S')^n \longrightarrow (S')^{n+1} \longrightarrow S' \longrightarrow T' \longrightarrow 0$$

is exact, where $(S')^n \longrightarrow (S')^{n+1}$ is given by the lifted matrix and $(S')^{n+1} \longrightarrow S'$ by its minors (with appropriate signs). For the same reason this sequence tensored with the residue field of $R'$ remains exact which shows that $T'$ is $R'$-flat. □

**Corollary 7.28.** *Let $X \longrightarrow S$ be a smooth, separated morphism of pure relative dimension 2. Then we have that the Hilbert scheme $\mathrm{Hilb}^n_{X/S}$ is the blow-up of $\Gamma^n_{X/S}$ along $\Delta_X$.*

*Proof.* As in the proof of Corollary (7.3) we may reduce to the case when $S$ is affine and $X \longrightarrow S$ is quasi-compact. If we can prove that the open locus $U^{et}$ of $\mathrm{Hilb}^n_{X/S}$ is schematically dense then we are finished by the Theorem. As the defining ideal of the complement of $U^{et}$ is locally principal and as $\mathrm{Hilb}^n_{X/S} \longrightarrow S$ is flat by the proposition this can be checked fibre by fibre so we may assume that $S$ is the spectrum of a field $k$. Now, in that case $\mathrm{Hilb}^n_{X/S}$ is smooth by the proposition or by Fogarty's result. For the density statement we may reduce to the base field $k$ being algebraically closed. Write $X = \sqcup_{i=1,\dots,p} X_i$ as a disjoint union of integral surfaces. We then have that $\mathrm{Hilb}^n_{X/S}$ is the disjoint union $\sqcup_{n_1+\cdots+n_p=n} \prod_i \mathrm{Hilb}^{n_i}(X_i)$. As $U^{et}$ is non-empty in each of the components $\mathrm{Hilb}^{n_i}(X_i)$ that are irreducible ([10, Propositions 2.3-4]), this implies that it is schematically dense in $\mathrm{Hilb}^n_{X/S}$. □

*Remark* 7.29. As pointed out by the referee, there is a small inaccuracy in ([10, Propositions 2.3-4]) concerning the connectedness of the Hilbert scheme in that the Hilbert scheme of a connected scheme is not necessarily connected. The proof had to take that into account.

## 8. The good component for affine varieties

We will in this last section generalize the approach Haiman gives in [14], using the fact that the Hilbert scheme $\mathrm{Hilb}^n_Y$, for a projective scheme $Y$, can be embedded as a closed subscheme of the Grassmannian of rank $n$-quotients of $H^0(Y, \mathscr{O}_Y(N))$, when $N$ is large enough. To simplify we assume that our base $\mathrm{Spec}(A)$ is Noetherian.

**Proposition 8.1.** *Let $X = \mathrm{Spec}(F) \longrightarrow S = \mathrm{Spec}(A)$ be a finite type morphism of affine schemes, and let $V \subseteq F$ be an $n$-sufficiently big $A$-submodule. Let $I_V$ and $I_F$ be the ideals of norms associated to $V$ and $F$, respectively. The natural morphism $\bigoplus_{m\geq 0} I_V^m \to \bigoplus_{m\geq 0} I_F^m$ induces a morphism*

$$\varphi\colon \mathrm{G}^n_{X/S} = \mathrm{Proj}(\bigoplus_{m\geq 0} I_F^m) \longrightarrow \mathrm{Bl}_{I_V}(\Gamma^n_A F) = \mathrm{Proj}(\bigoplus_{m\geq 0} I_V^m)$$

*which is finite.*

*Proof.* Let $U$ respectively $U'$, be the complement of $\mathrm{Spec}(\Gamma^n_A F)$ in $\mathrm{Spec}(\bigoplus_{m\geq 0} I_F^m)$ respectively $\mathrm{Spec}(\bigoplus_{m\geq 0} I_V^m)$. That the map on Proj's is well-defined means that the map on spectra maps $U$ into $U'$. Assume therefore, by way of contradiction, that we have a point $x$ of $U$ that does not map into $U'$. This gives us a field valued point of $\mathrm{Hilb}^n_{\mathrm{Spec}(F)/\mathrm{Spec}(A)}$, i.e., an $n$-dimensional quotient $F \bigotimes_A k \to R$. However, the assumption that the image of $x$ does not lie in $U'$ means that the image of $V$ does not span $R$. This however contradicts the assumption that $V$ is $n$-sufficiently big.

For graded elements $f$ in a graded ring $R$ we let $D_+(f)$ denote the basic open affine given as the spectrum of the degree zero part of the localized ring $R_f$. We have, for any $f \in I_V$ that $\varphi^{-1}(D_+(f)) = D_+(f)$, hence the morphism $\varphi$ is an affine morphism. Since $F$ is assumed of finite type it follows from Lemma (2.10) that $I_F$ is of finite type, and consequently $\mathrm{G}^n_{X/S}$ is proper over $\mathrm{Spec}(\Gamma^n_A F)$. Since $\mathrm{Bl}_{I_V}(\Gamma^n_A F)$ is separated it follows that $\varphi$ is proper. Thus the morphism $\varphi$ is both proper and affine, hence finite. □

When $V \subseteq F$ is $n$-sufficiently big we have an induced morphism

$$h\colon \mathrm{Hilb}^n_{X/S} \longrightarrow \mathrm{Grass}^n_V$$

from the Hilbert scheme to the Grassmannian.

**Lemma 8.2.** *Let $X = \mathrm{Spec}(F) \longrightarrow S = \mathrm{Spec}(A)$ be of finite type, and let $V \subset F$ be $n$-sufficiently big, finitely generated $A$-module. We have a commutative diagram*

$$\begin{array}{ccc} \mathrm{G}^n_{X/S} & \longrightarrow & \mathrm{Hilb}^n_{X/S} \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle h} \\ \mathrm{Bl}_{I_V}(\Gamma^n_A F) & \longrightarrow & \mathrm{Grass}^n_V. \end{array}$$

*Proof.* Since $V$ is finitely generated we can use the Plücker coordinates to embed $\mathrm{Grass}^n_V$ as a closed subscheme of $\mathbf{P}(\Lambda^n V)$. Composition with the diagonal embedding and the Segre embedding yields the closed immersion $\iota_1$ given as the composite

$$\mathrm{Grass}^n_V \subset \mathbf{P}(\Lambda^n_A V) \subset \mathbf{P}(\Lambda^n_A V) \times \mathbf{P}(\Lambda^n_A V) \subset \mathbf{P}(\Lambda^n_A V \bigotimes_A \Lambda^n_A V)\ .$$

The natural map of $A$-modules $\Lambda^n V \bigotimes_A \Lambda^n V \longrightarrow I_V$ will by definition hit all the generators for the ideal $I_V$, and consequently determine a closed immersion $\iota_2\colon \mathrm{Bl}_{I_V}(\Gamma^n_A F) \longrightarrow \mathbf{P}(\Lambda^n_A V \bigotimes_A \Lambda^n_A V) \times \mathrm{Spec}(\Gamma^n_A(F))$. We now have the commutative diagram

$$\begin{array}{ccccc} \mathrm{G}^n_{X/S} & \longrightarrow & \mathrm{Hilb}^n_{X/S} & \xrightarrow{h} & \mathrm{Grass}^n_V \\ \downarrow{\scriptstyle\varphi} & & & & \downarrow{\scriptstyle\iota_1} \\ \mathrm{Bl}_{I_V}(\Gamma^n_A F) & & \xrightarrow{\quad p_1 \circ \iota_2 \quad} & & \mathbf{P}(\Lambda^n_A V \bigotimes_A \Lambda^n_A V), \end{array}$$

where $p_1$ is the projection on the first factor. The inverse image $\varphi^{-1}(E)$ of the exceptional divisor $E \subseteq \mathrm{Bl}_{I_V}(\Gamma^n_A F)$ is the exceptional divisor of

$\mathrm{G}^n_{X/S}$, and on the open complement we have that $\varphi$ is an isomorphism. Consequently $p_1 \circ \iota_2 \colon \mathrm{Bl}_{I_V}(\Gamma^n_A F) \longrightarrow \mathbf{P}(\Lambda^n_A V \bigotimes_A \Lambda^n_A V)$ factors through $\mathrm{Grass}^n_V$ since it does so on the complement of a Cartier divisor. $\square$

**8.3.** Consider now $Y = \mathbf{P}^r_S$, and let $g \colon Y \longrightarrow S$ denote the structure map. For any closed subscheme $Z \subseteq Y$ that is flat, locally free of rank $n$ over $S$, the induced map

$$g_* \mathscr{O}_Y(N) \longrightarrow g_* \mathscr{O}_Z(N)$$

is easily seen to be surjective for $N \geq n-1$. Furthermore, the ideal sheaf $\mathscr{I}_Z$ twisted with $N \geq n$ is regular, that is $R^p g_* \mathscr{I}_Z(N-p) = 0$ for $p > 0$ when $N \geq n$. It follows ([12]) that the induced morphism

$$\mathrm{Hilb}^n_{Y/S} \longrightarrow \mathrm{Grass}^n_{g_* \mathscr{O}_Y(N)} \tag{8.3.1}$$

is a closed immersion for $N \geq n$.

**Proposition 8.4.** *Let $F$ be an $A$-algebra generated by $t_1, \dots, t_r$, let $V \subseteq F$ be spanned by the monomials of degree $\leq n$ in the $t_1, \dots, t_r$. Then the morphism*

$$\varphi \colon \mathrm{G}^n_{X/S} \longrightarrow \mathrm{Bl}_{I_V}(\Gamma^n_A F)$$

*is an isomorphism.*

*Proof.* We embed $X = \mathrm{Spec}(F)$ in $Y = \mathbf{P}^r_S$ using $(1 \colon t_1 \colon \cdots \colon t_r)$. We have natural maps $h \colon \mathrm{Hilb}^n_{X/S} \longrightarrow \mathrm{Grass}^n_V$ and $\mathrm{Grass}^n_V \to \mathrm{Grass}^n_{g_*(\mathscr{O}_Y(N))}$, $N \geq n$, where the latter is a closed immersion. As $\mathrm{Hilb}^n_{X/S}$ immerses into $\mathrm{Hilb}^n_{Y/S}$, and the map (8.3.1) is an immersion, it follows that the map $h \colon \mathrm{Hilb}^n_{X/S} \longrightarrow \mathrm{Grass}^n_V$ is an immersion.

By Lemma (8.2) we have that the restriction of $h$ to $\mathrm{G}^n_{X/S}$ factors through

$$\varphi \colon \mathrm{G}^n_{X/S} \longrightarrow \mathrm{Bl}_{I_V}(\Gamma^n_A F),$$

hence $\varphi$ must be an immersion as well. However, by Proposition (8.1) the map $\varphi$ is proper, and consequently we have that the map $\varphi$ must be a closed immersion. Furthermore, since $\varphi$ is an isomorphism over the complement of a Cartier divisor, it is an isomorphism. $\square$

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