Home assignement 2. Deadline 21.12, at 2400

Exercise 1. *Let I be an ideal in a ring A, and let M be an A-module. Show that two different I-stable filtrations on M give the same I-adic topology on M.*

Exercise 2 (AM, 10.9, p. 115)**.** *Let A be a local ring with maximal ideal $\mathfrak{m}$. For any polynomial $f(x) \in A[x]$ we let $\overline{f}(x) \in A/\mathfrak{m}[x]$ denote its reduction modulo $\mathfrak{m}$. Assume that A is $\mathfrak{m}$-adically complete. Prove Hensel's Lemma: Assume that $f(x)$ is monic of degree n, and that $\overline{f}(x) = \overline{g}(x)\overline{h}(x)$, where $\overline{g}(x)$ and $\overline{h}(x)$ are monic and coprime. Then we can find monic polynomials $g(x)$ and $h(x)$ whose reductions modulo $\mathfrak{m}$ are $\overline{g}(x)$ and $\overline{h}(x)$, and where $f(x) = g(x)h(x)$.*

Exercise 3. *Show the following statement. Let A be a ring, I and ideal, and M an A-module. Show that the canonical map $\hat{A} \otimes_A M \to \hat{M}$ is surjective. If furthermore A is Noetherian, then the canonical map $\hat{A} \otimes_A M \to \hat{M}$ is an isomorphism (see Proposition 10.13).*

Exercise 4. *In the ring $A = k[x, y, z]/(xy, xz)$ (k is a field) we have the following three prime ideals $P_1 = (x-1, y, z)$, $P_2 = (x, y-1, z-1)$ and $P_3 = (x, y, z)$. Let $(A_i, \mathfrak{m}_i)$ denote the local ring obtained by localization of A in the prime P_i ($i = 1, 2, 3$). Let d_i denote the dimension of A_i.*

1. *Write the the Hilbert series $P(A_i, t)$ of the associated graded ring $G_{\mathfrak{m}_i}(A) = \bigoplus_{n \geq 0} \mathfrak{m}_i^n/\mathfrak{m}_i^{n+1}$, as $f_i(z)/(1-z)^{d_i}$ (with polynomials f_i, for $i = 1, 2, 3$).*
2. *Compute the characteristic polynomial $\chi_{\mathfrak{m}_i}(z)$ of A_i with respect to the filtration $(\mathfrak{m}_i^n)$ (for $i = 1, 2, 3$).*
3. *Construct a chain of prime ideals in A_i of length d_i.*
4. *Construct a $\mathfrak{m}_i$-primary ideal generated by d_i-elements.*
5. *Compute the vector spaces $\mathfrak{m}_i/\mathfrak{m}_i^2$ and determine which A_i that are regular.*

Exercise 5 (AM 11.1 p. 125)**.** *Let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial over an algebraically closed field k. A point $P = (a_1, \ldots, a_n)$ on the hypersurface $V(f)$ is non-singular if and only not all the partial derivatives $\frac{\partial f}{\partial x_i}$ vanish att P. Let $A = k[x_1, \ldots, x_n/(f))$, and let $\mathfrak{m}$ be the maximal ideal of P. Prove that P is non-singular if and only if $A_{\mathfrak{m}}$ is a regular ring.*