Equality in Green's hyperplane restriction theorem

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Chapter 1

Preliminaries

In the beginning the Universe was created. This has made a lot of people very angry and has been widely regarded as a bad move.

Douglas Adams
The Restaurant at the End of the Universe

1.1. Order theory

Before we can start off we need to define some different orderings of n-tuples occuring later in the paper. We start off with some basic definitions. Most of these definitions are versions of definitions found in the books [**9**], [**22**] and [**10**].

Definition 1.1. A *relation* on a set X is a property that holds for some of the pairs in $X \times X$. If $\sim$ is a relation on X, we write $x \sim y$ to indicate that $\sim$ holds for the pair $(x, y) \in X \times X$

Now we can define ordering relations on sets.

Definition 1.2. A *total order* on a set X is a relation $\leq$ that satisfy the following for given arbitrary elements $x, y, z \in X$

(*i*) $x \leq x$ (*reflexive*)

(*ii*) If $x \leq y$ and $y \leq x$, then $x = y$ (*antisymmetric*)

(*iii*) If $x \leq y$ and $y \leq z$, then $x \leq z$ (*transitive*) (*iv*) $x \leq y$ or $y \leq x$ (*totality*)

A *partial order* on a set X is a relation $\leq$ that only satisfies (*i*), (*ii*) and (*iii*). Thus a partial order has no requirement for totality.
A set together with a partial order on it is called a *partially ordered set* or short a *poset*. A set together with a total order on it is called a *totally ordered set*. We will use the notation $(P, \leq)$ to denote a partially ordered set P together with its partial order $\leq$.
In $(P, \leq)$, then $x < y$ means that x and y satisfy $x \leq y$ but $x \neq y$. If we have either $x \leq y$ or $x \geq y$ in our poset $(P, \leq)$ we say that the two elements are *comparable*. Otherwise the elements are said to be *incomparable*.
Let I be a subset of $(P, \leq)$. An element x of I is *maximal* with respect to the partial order if for any $y \in I$ $x \leq y$ can hold only if $x = y$. The set of all maximal elements of I with respect to $\leq$ will be denoted $\max_{<}(I)$.
Similiarly we have that an element x of I is *minimal* with respect to the partial order if for any $y \in I$ $x \geq y$ can hold only if $x = y$. The set of all minimal elements of I with respect to $\leq$ will be denoted $\min_{<}(I)$

Example 1.3. The set of all integers $\mathbb{Z}$ is a totally ordered set. The total order on $\mathbb{Z}$ is the standard one $(\ldots < -1 < 0 < 1 < 2 < \ldots)$. This total order on $\mathbb{Z}$ will often be written $\leq$ without any subscript.

It will be interesting to consider special subsets of posets called *filters* (or *dual order ideals*)

Definition 1.4. A *filter* of a poset $(P, \leq)$ is a subset I of X such that if $x \in I$ and $y \geq x$, then $y \in I$.

When dealing with filters of finite posets we have this nice property

Proposition 1.5. *A filter of a finite poset is completely defined by its minimal elements. A filter of a finite totally ordered set is defined by a single minimal element.*

Proof. Since P is finite so is I. Thus it must have minimal elements, since otherwise we could find infinitely many distinct elements $x_i \in I,\ i = 1, 2, \ldots$ all satisfying $x_i \geq x_{i+1}$.
Let $x_1, x_2, \ldots, x_r$ be the minimal elements of I. Then we clearly have $I = \bigcup_{i=1}^{r} X_i$ where $X_i = \{y \in P : y \geq x_i\}$.
Now let $\leq$ be a total order and let x and y be two minimal elements of I. Then by totality we must have either $x \geq y$ or $x \geq y$. But then since both elements are minimal this implies that $x = y$, thus we can only have one minimal element. $\square$

A very simple consequence of the proposition is that if $(P, \leq)$ is a finite totally ordered set, then the cardinality of the filter with minimal element

x is given by $|\{y \in P : y \geq x\}|$ which thus also defines the filter. We summarize this in a corollary

Corollary 1.6. *A filter I of a finite totally ordered set $(P, \leq)$ is defined by it's cardinality.*

A related type of subsets is the *order ideals*

Definition 1.7. An *order ideal* of a poset $(P, \leq)$ is a subset I of X such that if $x \in I$ and $y \leq x$, then $y \in I$.

Now we continue with the orderings of n-tuples.

Definition 1.8. A *n-tuple* is a ordered list of n elements from a given set X and is represented as

$$(a_1, a_2, \ldots, a_n)$$

The *length* of a tuple denotes the number of elements in the tuple. Thus a n-tuple has length n. The set of all n-tuples with elements from the set X will be denoted X^n.

The n-tuples that we will be working with in this paper are mostly those consisting of integervalued elements. Thus when talking about n-tuples it will often be understood that we are talking about n-tuples with elements from $\mathbb{Z}$ or $\mathbb{N}$ depending on context. For these n-tuples we have that the *sum* of $(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ (or $\mathbb{N}^n$) is simply the sum $\sum_{i=1}^n a_i$. Further the set of all n-tuples in $\mathbb{Z}^n$ and $\mathbb{N}^n$ with sum equal to d will be denoted $\mathbb{Z}_d^n$ and $\mathbb{N}_d^n$ respectively.
Now we define some orderings on our n-tuples we will be using later on:

Definition 1.9. The *lexicographical order*, denoted $\leq_{\text{lex}}$, on the set $\mathbb{Z}^n$ (or $\mathbb{N}^n$) is a total order defined as follows: Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be two elements of $\mathbb{Z}^n$ (or $\mathbb{N}^n$). Then $\mathbf{a} <_{\text{lex}} \mathbf{b}$ iff $\mathbf{a} \neq \mathbf{b}$ and $a_j < b_j$, where $j = \min\{i : a_i \neq b_i\}$

Example 1.10. In $\mathbb{N}^n$ we have that $(0, 3, 0) <_{\text{lex}} (1, 1, 1)$ since $0 < 1$.

Definition 1.11. The (degree) *reverse lexicographical order*, denoted $\leq_{\text{rlex}}$, on the set $\mathbb{Z}^n$ (or $\mathbb{N}^n$) is a total order defined as follows: Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be two elements of $\mathbb{Z}^n$ (or $\mathbb{N}^n$). Then $\mathbf{a} <_{\text{rlex}} \mathbf{b}$ iff $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or $\mathbf{a} \neq \mathbf{b}$, $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and $a_j > b_j$, where $j = \max\{i : a_i \neq b_i\}$

Example 1.12. In $\mathbb{N}^n$ we have that $(1, 1, 1) <_{\text{rlex}} (0, 3, 0)$ since $1+1+1 = 3$ and $1 > 0$. This combined with Example 1.10 implies that the lexiographical order and the reverse lexiographical order in general differs in $\mathbb{N}^n$ and even in $\mathbb{N}_d^n$. However we will se that if $n \leq 2$ then the lexiographical order and the reverse lexiographical order on $\mathbb{N}_d^n$ will coincide.

Next we want to define a special partial order on $\mathbb{N}^n$. But before we can do this we need to define a family of functions on $\mathbb{N}^n$.

Definition 1.13. A *strongly stable redistributing function* is a function of the form, for given $\mathbf{a} \in \mathbb{N}^n$ and $i, j \in \{1, \dots, n\}$.

$$\mu_{i,j}(\mathbf{a}) = \begin{cases} (a_1, \dots, a_i + 1, \dots, a_j - 1, \dots, a_n) & \text{if } i < j \text{ and } a_j \geq 1 \\ \mathbf{a} & \text{otherwise} \end{cases}$$

One can note that if $i \geq j$ then $\mu_{i,j}$ equals the identity function on $\mathbb{N}^n$. Now we're ready to define our partial order.

Definition 1.14. The *strongly stable order*, denoted $\leq_{\text{str}}$, on the set $\mathbb{N}^n$ is a partial order defined as follows: Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ be two elements of $\mathbb{N}^n$. Then $\mathbf{a} <_{\text{str}} \mathbf{b}$ if and only if $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{b} = \mu_{i_1,j_1} \circ \cdots \circ \mu_{i_r,j_r}(\mathbf{a})$, where μ_{i_k,j_k} are strongly stable redistributing functions for $k = 1, \dots, r$.

Example 1.15. To see that $\leq_{\text{str}}$ does not exhibit the totality property one can check that we have neither $(1,3,0) \leq_{\text{str}} (2,1,1)$ nor $(1,3,0) \geq_{\text{str}} (2,1,1)$ in $\mathbb{N}^3$. The example even shows that we don't have totality in $\mathbb{N}^3_4$ since we have $2+1+1 = 1+3$. It's clear that $\leq_{\text{str}}$ differs from $\leq_{\text{lex}}$ and $\leq_{\text{rlex}}$ in general since it lacks totality.

Motivated by our claim in Example 1.12 we now prove our first proposition

Proposition 1.16. *Let $d, n \in \mathbb{N}$ so that $n \leq 2$. Then the orders $\leq_{lex}$, $\leq_{rlex}$ and $\leq_{str}$ all coincide on the set $\mathbb{N}^n_d$*

Proof. If $n = 1$ then $\mathbb{N}_d = \{(d)\}$. There's only one way you can order a single element, thus all orders on $\mathbb{N}_d$ coincide.
Now let $n = 2$ and let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ be two distinct elements of $\mathbb{N}^2_d$. Since $\mathbf{a} \neq \mathbf{b}$ we have $a_1 \neq b_1$ and $a_2 \neq b_2$ since otherwise we would have $a_1 = b_1$ and thus $a_2 = d - a_1 = d - b_1 = b_2$ or $a_2 = b_2$ and thus $a_1 = d - a_2 = d - b_2 = b_1$, which yields $\mathbf{a} = \mathbf{b}$ in both cases.
We start by showing that the orders $\leq_{\text{lex}}$ and $\leq_{\text{rlex}}$ coincide. Assume that $\mathbf{a} <_{\text{lex}} \mathbf{b}$. Then by definition $a_1 < b_1$ which gives $a_2 = d - a_1 > d - b_1 = b_2$ and $\mathbf{a} <_{\text{rlex}} \mathbf{b}$. By exchanging the indices we get by the exact same argument that $\mathbf{a} <_{\text{rlex}} \mathbf{b}$ gives $\mathbf{a} <_{\text{lex}} \mathbf{b}$.
Now we show that the orders $\leq_{\text{str}}$ and $\leq_{\text{lex}}$ coincide. Assume that $\mathbf{a} <_{\text{lex}} \mathbf{b}$. Then by definition $a_1 < b_1$. Put $\delta = b_1 - a_1$. Then we have that $\mathbf{b} = \mu^{\delta}_{1,2}(\mathbf{a})$ where $\mu^{\delta}_{1,2}$ means $\mu_{1,2}$ composed δ times. Thus $\mathbf{a} <_{\text{str}} \mathbf{b}$. Now assume $\mathbf{a} <_{\text{str}} \mathbf{b}$. Then we must have $\mathbf{b} = \mu^{\delta}_{1,2}(\mathbf{a})$ for some positive δ since $\mu_{1,2}$ is the only strongly stable redistributing function in $\mathbb{N}^2$ not equal to the identity.

But then by definition of the strongly stable redistributing functions we have $a_1 = b_1 - \delta$ which gives us $a_1 < b_1$ and $\mathbf{a} <_{\text{lex}} \mathbf{b}$. □

Later on when dealing with filters with respect to the strongly stable order it will be interesting to consider the set-theorectical complement of the filters

Proposition 1.17. *Let $n \in \mathbb{N}$ and let I be an filter of the poset $(\mathbb{N}^n, \leq_{str})$. Then the set-theoretical complement of I, $\mathbb{N}^n \setminus I$, is denoted I^C and is a order ideal of the poset.*

Proof. Let $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ be arbitrary distinct elements such that $\mathbf{a} \notin I$ and $\mathbf{a} = \mu_{i_1,j_1} \circ \cdots \circ \mu_{i_r,j_r}(\mathbf{b})$, where μ_{i_k,j_k} are strongly stable redistributing functions for $k = 1, \ldots, r$. Now if $\mathbf{b} \in I$ then we would get that $\mathbf{a} \in I$ which is a contradiction. Thus $\mathbf{b} \notin I$ and we get that if $\mathbf{a} \in I^C$ and $\mathbf{b} \leq_{\text{str}} \mathbf{a}$ then $\mathbf{b} \in I^C$. Thus by Defintion 1.7 I^C is a order ideal. □

1.2. Combinatorics

We will need some combinatoric facts later working on the main subject of the paper. Thus we take our time to get acquainted with the underlying combinatorics that make out the foundation of Green's hyperplane restriction theorem. Many definitions we will use are slightly modified versions of those that can be found in [**7**], [**17**],[**1**], [**22**] and [**10**], made to fit our setting. We start by examining the number of ways of writing integers.

Definition 1.18. Let $n \in \mathbb{N}$ be an integer. A *partition* of n is sequence of positive integers $\{\lambda_i\}_{i=1}^k$ such that $\sum \lambda_i = n$ and $\lambda_1 \geq \cdots \geq \lambda_k$. The integers λ_i are called *parts*.

Even though the subject of partitions of an integer is interesting in its own right, the type of partitions we will meet in this paper is a bit more restricted.

Definition 1.19. Let $n \in \mathbb{N}$ be an integer. A *partition of n into distinct parts* is a sequence of positive integers $\{\lambda_i\}_{i=1}^k$ such that $\sum \lambda_i = n$ and $\lambda_1 > \cdots > \lambda_k$. The number of partition into distinct parts of n is given by the function $\hat{p}(n)$. For convenience one sets $\hat{p}(0) = 1$.

One can show, which Leonard Euler did in 1748 (see [**3**]), that the number of partitions with distinct parts is equal to the number of partitions with odd parts. We will not state the proof of this here, but the interested reader is encouraged to read the article [**4**]. Also a detailed list of the first 2000 values of the function $\hat{p}(n)$, and more information can be found at [**19**].

When working with dimensional bounds in graded algebras it is more interesting to represent integers as sums of binomial coefficents. We recall the definition of the binomial coefficient $\binom{k}{l}$.

Definition 1.20. Let $k \in \mathbb{C}$ and $l \in \mathbb{N}$. Then we define the *binomial coefficient* $\binom{k}{l}$ in the following way

$$\binom{k}{l} = \begin{cases} \frac{1}{l!}k(k-1)\cdots(k-l+1) & \text{if } l \geq 1 \\ 1 & \text{if } l = 0 \end{cases}$$

We note that $\binom{k}{l} = 0$ if $k \in \mathbb{N}$ and $0 \leq k < l$. From the definition of the binomial coefficients one can directly acquire the famous Pascal's triangle.

Proposition 1.21. (Pascal's triangle)
Let $k \in \mathbb{N}$ and $l \in \mathbb{N}$ such that $1 \leq l \leq k$. Then

$$\binom{k}{l} = \binom{k-1}{l-1} + \binom{k-1}{l}$$

Proof. If $l = 1$ then we have that $\binom{k}{l} = k = 1 + (k-1) = \binom{k-1}{l-1} + \binom{k-1}{l}$ so assume that $2 \leq l$. We have by definition
$\binom{k}{l} = \frac{1}{l!}k(k-1)\cdots(k-l+1) = \frac{1}{(l-1)!}(k-1)\cdots(k-l+1)\frac{k}{l} =$
$\frac{1}{(l-1)!}(k-1)\cdots(k-l+1)(1+\frac{k-l}{l}) = \frac{1}{(l-1)!}((k-1)\cdots(k-l+1) +$
$(k-1)\cdots(k-l)\frac{1}{l}) = \frac{1}{(l-1)!}(k-1)\cdots(k-l+1) + \frac{1}{l!}(k-1)\cdots(k-l) =$
$\binom{k-1}{l-1} + \binom{k-1}{l}$ □

Now we can define our representation of integers

Definition 1.22. Let d be a positive integer. Then for $c \in \mathbb{N}$ the *d-binomial expansion of c* is the unique expression

$$c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}$$

where $k_d > k_{d-1} > \cdots > k_1 \geq 0$. We call $k_d, k_{d-1}, \ldots, k_1$ the *d-binomial coefficients of c.*

To convince the reader that this representation seems plausible we will include a proof of why such a representation exists and is unique, following the outline of the proof from [**7**, Lemma 4.2.6]

Proposition 1.23. *Let d be a positive integer. Then any $c \in \mathbb{N}$ can be written uniquely in the form*

$$c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}$$

where $k_d > k_{d-1} > \cdots > k_1 \geq 0$.

Proof. In order to prove the existence, we use induction on c. For $c = 1$ we have $c = \binom{d}{d} + \sum_{i=1}^{d-1}\binom{i-1}{i}$. So assume $c > 1$. Now we choose k_d maximal such that $\binom{k_d}{d} \leq c$. If $c = \binom{k_d}{d}$, then $c = \sum_{i=1}^{d}\binom{k_i}{i}$ with $k_i = i-1$ for $i = 1, \ldots, d-1$. So assume that $c' = c - \binom{k_d}{d} > 0$. By the induction hypothesis we may assume that $c' = \sum_{i=1}^{d-1}\binom{k_i}{i}$ with $k_{d-1} > k_{d-2} > \cdots > k_1 \geq 0$. It remains to show that $k_d > k_{d-1}$. But since $\binom{k_d+1}{d} > c$ it follows from Pascal's identity that

$$\binom{k_d}{d-1} = \binom{k_d+1}{d} - \binom{k_d}{d} > c' \geq \binom{k_{d-1}}{d-1}$$

Hence $k_d > k_{d-1}$.
To be able to show uniqueness we start by showing the following: if $c = \sum_{i=1}^{d}\binom{k_i}{i}$ with $k_d > k_{d-1} > \cdots > k_1 \geq 0$, then k_d is the single largest integer with $\binom{k_d}{d} \leq c$. We prove this statement by induction on c. For $c = 1$ the assertion is trivial. Now assume that $c > 1$, and $\binom{k_d+1}{d} \leq c$. Then

$$\sum_{i=1}^{d-1}\binom{k_i}{i} \geq \binom{k_d+1}{d} - \binom{k_d}{d} = \binom{k_d}{d-1} \geq \binom{k_{d-1}+1}{d-1}$$

and this contradicts the induction hypothesis.
Now we can show uniqueness by once again using induction on c. For $c = 1$ the sum $c = \binom{d}{d} + \sum_{i=1}^{d-1}\binom{i-1}{i}$ is unique since k_d must zero or d. But if $k_d = 0$ then we must have $k_{d-1} < 0$ which is wrong by definition. With $k_d = d$ there is only one choice for the k_i's that satisfies the inequalties $k_d > k_{d-1} > \cdots > k_1 \geq 0$. So assume $c > 1$. If $c = \sum_{i=1}^{d}\binom{k_i}{i} = \sum_{i=1}^{d}\binom{k'_i}{i}$ Then we have by our earlier argument that $k_d = k'_d$ is the singe largest integer with $\binom{k_d}{d} \leq c$. Thus $\sum_{i=1}^{d-1}\binom{k_i}{i} = \sum_{i=1}^{d-1}\binom{k'_i}{i}$ and by induction $k_i = k'_i$ for all i. □

Now given a integer c denote $\delta = \min\{i : k_i \geq i\}$ and $\Delta_i = k_i - i$ where k_i denotes the n-binomial coefficients of c for $i = 1, 2, \ldots, n$. Then we have

$$\begin{aligned}
c &= \binom{k_n}{n} + \binom{k_{n-1}}{n-1} + \cdots + \binom{k_1}{1} = \\
&= \binom{k_n}{n} + \binom{k_{n-1}}{n-1} + \cdots + \binom{k_\delta}{\delta} + \binom{\delta-2}{\delta-1} + \cdots + \binom{0}{1} = \\
&= \binom{n+\Delta_n}{n} + \binom{(n-1)+\Delta_{n-1}}{n-1} + \cdots + \binom{\delta+\Delta_\delta}{\delta} + \binom{(\delta-1)-1}{\delta-1} + \\
&+ \cdots + \binom{1-1}{1} = \\
&= \binom{n+\Delta_n}{n} + \binom{(n-1)+\Delta_{n-1}}{n-1} + \cdots + \binom{\delta+\Delta_\delta}{\delta}
\end{aligned}$$

Where we have $\Delta_n \geq \Delta_{n-1} \geq \ldots \geq \Delta_\delta \geq 0$. And since by Proposition 1.23 this expression is unique, thus it makes sense to define the following alternate unique representation

Definition 1.24. Let d be a positive integer and $c \in \mathbb{N}$ as before. Then we define the d^{th} *Macaulay difference tuple of* c to be the tuple of length d:

$$M_d(c) = (\Delta_d, \Delta_{d-1}, \ldots, \Delta_\delta, -1, -1, \ldots, -1)$$

where δ and Δ_i are defined as above. We denote the *positive length* of the d^{th} Macaulay difference tuple of c to be the number of non-negative elements of the tuple. With the above notation we have that the positive length is equal to $d - \delta + 1$.

To simplify the notation of the d^{th} Macaulay difference tuple we will often write

$$(\Delta_d, \Delta_{d-1}, \ldots, \Delta_\delta, -1, -1, \ldots) = (\Delta_d, \Delta_{d-1}, \ldots, \Delta_\delta)$$

where it is understood that we have $\delta - 1$ number of -1 at the end of the tuple. We will also use the notation

$$M_d(0) = (-1, -1, \ldots, -1) = (-)$$

for the d^{th} Macaulay difference tuple of 0 consisting of d elements of -1.
Note that even though there is a indexing used in Definition 1.24 this indexing is different from the one used when viewing the Macaulay difference tuples as ordered tuples and these two should not be confused. The indexing of ordered tuples only depends on which order the elements appear in the tuple.

Remark 1.25. Our Defintion 1.24 differs from its origin given in the article [**1**, p. 3], because it includes a trail of negative ones. This is beacuse this defintion makes lexicographical comparison easier.

The d-binomial coefficients of integers and their respective Macaulay difference tuples have the following nice properties

Proposition 1.26. *Let $k_d, \ldots, k_1$, respectively $k'_d, \ldots, k'_1$ be the d-binomial coefficients of a, respectively a'.*
Then the following statements are equivalent:

(*i*) $a > a'$

(*ii*) $(k_d, \ldots, k_1) >_{lex} (k'_d, \ldots, k'_1)$

(*iii*) $M_d(a) >_{lex} M_d(a')$

Proof. The equvialence of (i) and (ii) is proven in [**7**, Lemma 4.2.7.] and in [**17**, Proposition 5.5.4] so it's enough to prove the equivalence of (ii) and (iii). We denote $M_d(a) = (\Delta_d, \Delta_{d-1}, \ldots, \Delta_1)$ and $M_d(a') = (\Delta'_d, \Delta'_{d-1}, \ldots, \Delta'_1)$. Now if we have (ii) then there exists a j such that $j = \max\{i : k_i \neq k'_i\}$, so we get:

$$k_j > k'_j \implies k_j - j > k'_j - j \implies \Delta_j > \Delta'_j \implies M_d(a) >_{\text{lex}} M_d(a')$$

But if we have (iii) then there exists a j such that $j = \max\{i \in \{d, d-1, \ldots, 1\} : \ \Delta_i \neq \Delta'_i\}$, so we get:

$$\Delta_j > \Delta'_j \implies k_j - j > k'_j - j \implies k_j > k'_j \implies (k_d, \ldots, k_1) >_{\text{lex}} (k'_d, \ldots, k'_1)$$

□

Now we recall two important combinatoric constructions when dealing with bounds of Hilbert functions:

Definition 1.27. Let d be a positive integer and $c \in \mathbb{N}$ as before. Then we define

$$c^{<d>} = \binom{k_d + 1}{d + 1} + \binom{k_{d-1} + 1}{d} + \cdots + \binom{k_1 + 1}{2}$$

where $k_d, k_{d-1}, \ldots, k_1$ denotes the d-binomial coefficients of c.

Definition 1.28. Let d be a positive integer and $c \in \mathbb{N}$ as before. Then we define

$$c_{<d>} = \binom{k_d - 1}{d} + \binom{k_{d-1} - 1}{d - 1} + \cdots + \binom{k_\delta - 1}{\delta} + \binom{k_{\delta-1}}{\delta - 1} + \cdots + \binom{k_1}{1} =$$

$$= \binom{k_d - 1}{d} + \binom{k_{d-1} - 1}{d - 1} + \cdots + \binom{k_\delta - 1}{\delta} + \binom{\delta - 2}{\delta - 1} + \cdots + \binom{0}{1}$$

where $k_d, k_{d-1}, \ldots, k_1$ denotes the d-binomial coefficients of c and $\delta = \min\{i : k_i \geq i\}$.

Remark 1.29. The reason why we only subtract from the d-binomial coefficients with index bigger than δ in Definition 1.28 is because the numbers are meant to represent dimensions of graded algebras. If one chooses to subtract from all coefficients then we would have the counterexample

$$1 = \binom{2}{2} + \binom{0}{1} \Rightarrow 1_{<2>} = \binom{1}{2} + \binom{-1}{1} = -1$$

With such a definition we would have to allow negative dimensions. Thus we only subtract from the d-binomial coefficients with index bigger than δ. An interesting note is that the book [**7**, p. 162] unfortunately falls into this trap of defining possible "negative dimensions", even though it uses the correct definiton later on.

Remark 1.30. Let $k_d, k_{d-1}, \ldots, k_1$ denote the d-binomial coefficients of c. We then have

$$\begin{aligned}
&M_{d+1}(c^{<d>}) = \\
&= ((k_d + 1) - (d + 1), (k_{d-1} + 1) - d, \ldots, (k_1 + 1) - (1 + 1), -1) = \\
&= (k_d - d, k_{d-1} - (d - 1), \ldots, k_1 - 1, -1) = \\
&= (M_d(c), -1)
\end{aligned}$$

Further if $\delta = \min\{i : k_i \geq i\}$ and $\Delta_i = k_i - i$ then

$$M_d(c_{<d>}) = (\Delta_d - 1, \Delta_{d-1} - 1, \ldots, \Delta_\delta - 1, -1, \ldots, -1)$$

Now we're ready to prove a combinatoric fact about integer-valued functions that will come in handy later when dealing with Hilbert-functions. The following proposition is based on a unproved claim from the article [**1**, p. 6] which stated that the proposition is *"not difficult to see"* since it follows from a *"good look at Pascal's triangle"*.

Proposition 1.31. *Let $h : \mathbb{N} \to \mathbb{N}$ be a numerical function that has $h(0) = 1$ and satisfies*

$$h(n + 1) \leq h(n)^{<n>}$$

for all $n \geq 1$. Then there exists an integer d such that for all $j \geq d$ we have

$$h(j + 1) = h(j)^{<j>}$$

Proof. We prove by contradiction. So lets assume the proposition is false. Thus given any $n \in \mathbb{N}$ we can find infinitely many distinct integers $l_1, l_2, \ldots$ such that $\ldots > l_2 > l_1 \geq n$ and $h(l_i + 1) < h(l_i)^{<l_i>}$ for all $i = 1, 2, \ldots$. We can choose our set $\{l_i\}_i$ such that $h(k + 1) = h(k)^{<k>}$ for all $k \notin \{l_i\}_i$ whenever $k \geq n$.

Now fix an arbitrary $n \geq 1$ such that $h(n) \neq 0$ and let $l_1, l_2, \ldots$ be defined as above. Denote $M_n(h(n)) = (a_n, a_{n-1}, \ldots, a_\delta)$. We want to prove that

$$\exists n_0 > n \Rightarrow M_{n_0}(h(n_0)) \leq (a_n, a_{n-1}, \ldots, a_{\delta+1}) \tag{1.1}$$

lexicographically. We prove this by induction on a_δ:

For $a_\delta = 0$ we have that $n_0 = l_1 + 1$ since by construction and Remark 1.30 we have

$$\begin{aligned} h(l_1+1) &< h(l_1)^{<l_1>} \Rightarrow h(l_1+1) \leq h(l_1)^{<l_1>} - 1 = \\ &= h(l_1)^{<l_1>} - \binom{(l_1+1-n)+\delta+a_\delta}{(l_1+1-n)+\delta} = \\ &= \cdots + \binom{(l_1+1-n)+\delta+a_\delta}{(l_1+1-n)+\delta} - \binom{(l_1+1-n)+\delta+a_\delta}{(l_1+1-n)+\delta} = \\ &= \binom{l_1+1+a_n}{l_1+1} + \cdots + \binom{(l_1+1-n)+\delta+1+a_{\delta+1}}{(l_1+1-n)+\delta+1} \end{aligned}$$

By Proposition 1.26 we thus get $M_{l_1+1}(h(l_1+1)) \leq (a_n, a_{n-1}, \ldots, a_{\delta+1})$. So assume such an integer n_0 exists for $a_\delta - 1 \geq 0$. Now we have by construction

$$h(l_1+1) \leq h(l_1)^{<l_1>} - 1 = \binom{l_1+1+a_n}{l_1+1} + \cdots + \binom{(l_1+1-n)+\delta+a_\delta}{(l_1+1-n)+\delta} - 1$$

We set $\delta_0 = l_1 + 1 - n + \delta$. Then by repetitive use of Pascal's identity we get

$$\begin{aligned} &\binom{\delta_0+a_\delta}{\delta_0} = \binom{\delta_0-1+a_\delta}{\delta_0} + \binom{\delta_0-2+a_\delta}{\delta_0-1} + \cdots + \binom{a_\delta}{1} + 1 \\ &\Rightarrow \binom{\delta_0+a_\delta}{\delta_0} - 1 = \binom{\delta_0+a_\delta-1}{\delta_0} + \binom{\delta_0+a_\delta-2}{\delta_0-1} + \cdots + \binom{a_\delta}{1} \\ &\Rightarrow h(l_1)^{<l_1>} - 1 = \binom{l_1+1+a_n}{l_1+1} + \cdots + \binom{\delta_0+a_\delta-1}{\delta_0} + \cdots + \binom{a_\delta}{1} \\ &\Rightarrow M_{l_1+1}(h(l_1+1)) \leq (a_n, \ldots, a_{\delta+1}, a_\delta - 1, a_\delta - 1, \ldots, a_\delta - 1) \end{aligned}$$

where we have δ_0 copies of $a_\delta - 1$ in the tail. But by our induction step we know that there exists a $n_0 > l_1$ such that

$$M_{n_0}(h(n_0)) \leq (a_n, \ldots, a_{\delta+1}, a_\delta - 1, a_\delta - 1, \ldots, a_\delta - 1)$$

where we now instead have $\delta_0 - 1$ copies of $a_\delta - 1$ in the tail. By repetitive use of our induction step we thus can find a $n_1 > n_0$ such that

$$M_{n_1}(h(n_1)) \leq (a_n, \ldots, a_{\delta+1})$$

And thus we're finished with the induction.

Now by repetitive use of (1.1) we get that there exists a d such that

$$M_d(h(d)) \leq (-) \Rightarrow h(d) \leq 0 \Rightarrow h(d) = 0$$

But then for all $j \geq d$ we have

$$h(j+1) \leq h(j)^{<j>} = h(d)^{<d>} = 0 \Rightarrow h(j+1) = h(j)^{<j>}$$

But this is a contradiction to our assumption that the proposition was false, thus the proposition must be true. □

Whether or not this proof serves as a *"good look at Pascal's triangle"*, we leave to the reader to decide.

1.3. Commutative algebra

We need to start off with some basic notation and definitions from commutative algebra before we can continue to the interesting results. The reader is assumed to be familiar with basic abstract algebra. Most definitions are versions of those from the books [**10**], [**16**], [**17**], [**18**] and [**13**].
For the remainder of the text we will have that, unless stated otherwise, $\mathbb{K}$ is an arbitrary infinite field and $\mathbb{K}[x_1, x_2, \ldots, x_n]$ will denote the polynomial ring in n variables (or indeterminates) over the field $\mathbb{K}$.

Since we will be dealing alot with ideal in the paper we remind us what a ideal is

Definition 1.32. A *ideal* of a ring R is a additative subgroup I of R such that if $r \in R$ and $s \in I$ then $rs \in I$. An ideal I is said to be *generated* by a subset $\mathcal{S} \subset I$ if every element $t \in I$ can be written in the form

$$t = \sum_{i=1}^{m} r_i s_i$$

where $r_i \in R$ and $s_i \in \mathcal{S}$. We write $\langle \mathcal{S} \rangle$ for the ideal generated by $\mathcal{S}$ or equivalently $\langle s_1, \ldots, s_m \rangle$ if $\mathcal{S} = \{s_1, \ldots, s_m\}$.

Later when dealing with the geometric background of the thesis we will see that is useful to define the the radical of an ideal.

Definition 1.33. Let I be an ideal of a ring R. The radical of I, denoted $\sqrt{I}$, is defined as

$$\sqrt{I} = \{f \in R : f^d \in I \text{ for some } d > 0\}$$

If $\sqrt{I} = I$ then I is said to be *radical*.

In this paper we're mostly interested in polynomial rings over fields and, as we will see, they will often prove to have a nice property of being *Noetherian*.

Definition 1.34. A ring is said to be *Noetherian* if all its ideals are finitely generated.

Proposition 1.35. *If a ring R is Noetherian, then the polynomial ring $R[x_1, x_2, \ldots, x_n]$ is Noetherian. Especially we get that $\mathbb{K}[x_1, x_2, \ldots, x_n]$ is Noetherian, whenever $\mathbb{K}$ is a field.*

Proof. See [**10**, Theorem 1.2] for a proof of why $R[x]$ is Noetherian if R is, then use induction on the number of variables and use the fact that $R[x_1,\dots,x_n] = R[x_1,\dots,x_{n-1}][x_n]$.
If $\mathbb{K}$ is a field then the only possible ideals of $\mathbb{K}$ is $\mathbb{K}$ itself and $\{0\}$, since every non-zero element is a generator of $\mathbb{K}$. □

Thus in our investigation of ideals in $\mathbb{K}[x_1,x_2,\dots,x_n]$ we only need to consider finitely generated ideals. Before continuing we need a notion of rings that will be used a lot in the paper

Definition 1.36. A *graded ring* (or $\mathbb{N}$-*graded ring*) is a ring R together with a decomposition

$$R = \bigoplus_{i=0}^{\infty} R_i$$

such that $R_0, R_1, \dots$ are additative subgroups of R and satisfy

$$R_i R_j \subseteq R_{i+j} \quad \forall\, i,j \in \mathbb{N}$$

The elements $f \in R_i$ are called *homogeneous of degree i* or *i-forms*. A 1-form will be called a *linear form*. The degree of f is denoted $\deg f$. An arbitrary element $f \in R$ has a unique representation $f = \sum_i f_i$ as a sum of homogeneous elements $f_i \in R_i$. The elements f_i are called the *homogeneous components* of f. A ideal, I, is called a *homogeneous ideal* if it is generated by only homogenous elements.

One can find many different gradings that would make $\mathbb{K}[x_1,x_2,\dots,x_n]$ into a graded ring (see e.g. [**7**, Examples 1.5.3]), but the one we will use is the intuitive one, namely that degree of a homogeneous polynomial $f \in \mathbb{K}[x_1,x_2,\dots,x_n]$ coincide with the degree of f when viewing it as a element of a graded ring. Thus if $R = \mathbb{K}[x_1,\dots,x_n]$ in Definition 1.36 then R_d denotes the vector space of homogeneous polynomials of degree d (when viewed as polynomials). Especially we get that $R_0 = \mathbb{K}$ and $\deg x_i = 1$ for all $i = 1,\dots,n$. This is called the *standard grading* of $\mathbb{K}[x_1,x_2,\dots,x_n]$. Let's make this explicit.

Definition 1.37. A *monomial* in $\mathbb{K}[x_1,x_2,\dots,x_n]$ is a product $\mathbf{x}^{\mathbf{a}} = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$, where $(a_1,\dots,a_n) \in \mathbb{N}^n$. If $a_i \geq 1$ we say that $\mathbf{x}^{\mathbf{a}}$ is *divisble* by x_i. This is written $\mathbf{x}^{\mathbf{a}}|x_i$. If $a_i = 0$ we write $\mathbf{x}^{\mathbf{a}} \nmid x_i$. The *degree* of $\mathbf{x}^{\mathbf{a}}$ is the sum $\sum_{i=1}^n a_i$ and is written $\deg(\mathbf{x}^{\mathbf{a}})$.

With this definition we get that a polynomial $f \in \mathbb{K}[x_1,x_2,\dots,x_n]$ can be represented as a sum of monomials,

$$f = \sum_{\mathbf{a}\in\mathbb{N}^n} c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}} \tag{1.2}$$

where we have $c_{\mathbf{a}} \in \mathbb{K}$ and only finitely many of the coefficients $c_{\mathbf{a}}$ can be non-zero.

Definition 1.38. The *standard grading* of $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ is the representation

$$R = \bigoplus_{i=0}^{\infty} R_i$$

where we have $R_d = \{\sum_{\mathbf{a} \in \mathbb{N}^n} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in R : \deg(\mathbf{x}^{\mathbf{a}}) = d \text{ whenever } c_{\mathbf{a}} \neq 0\}$.

A special ideal that we will meet often in $\mathbb{K}[x_1, x_2, \ldots, x_n]$ is the *irrelevant ideal*

Definition 1.39. The *irrelevant ideal* of $\mathbb{K}[x_1, x_2, \ldots, x_n]$, denoted $\mathfrak{m}$, is defined as $\mathfrak{m} = \langle x_1, x_2, \ldots, x_n \rangle$.

Remark 1.40. Note that we have for $k \in \mathbb{N}$ that $\mathfrak{m}^k = \langle \mathfrak{M}_k \rangle$, where $\mathfrak{M}_k$ denotes all monomials of degree k, since $\mathfrak{m}^k$ denotes all possible elements when choosing k arbitrary elements from $\mathfrak{m}$ and multiplying them.

When studying homgeneous ideals it's interesting to study their *saturation*. But to define the saturation we need a notion of ideal quotients

Definition 1.41. Let $I, J \subseteq \mathbb{K}[x_1, x_2, \ldots, x_n]$ be homogeneous ideals. The *ideal quotient* of I with J is defined as

$$(I : J) = \{f \in \mathbb{K}[x_1, x_2, \ldots, x_n] : fs \in I \text{ for all } s \in J\}$$

Now we can define the saturation of a ideal I

Definition 1.42. Let I be an ideal of $\mathbb{K}[x_1, x_2, \ldots, x_n]$. If $(I : \mathfrak{m}) = I$ then I is said to be *saturated*. The *saturation* of I, denoted I^{sat} is defined as

$$I^{\text{sat}} = \bigcup_{k=0}^{\infty} (I : \mathfrak{m}^k)$$

I is *m-saturated* if $I_d = I_d^{\text{sat}}$ for all $d \geq m$. The *satiety* of I, denoted $\text{sat}(I)$, is the smallest m for which I is m-saturated.

1.4. Monomial orders

It's interesting to consider the set of monomials in our polynomial ring. Thus we let $\mathfrak{M}$ denote the set of all monomials in $\mathbb{K}[x_1, x_2, \ldots, x_n]$. Further we let $\mathfrak{M}_d \subset \mathfrak{M}$ denote all monomials of degree d. Now by Definition 1.37 we see that each monomial can represented by a element in $\mathbb{N}^n$. Thus it comes as no suprise that the orders defined for $\mathbb{N}^n$ in our first section carries over to the monomials

Proposition 1.43. *Let $\leq_P$ be a relation on $\mathbb{N}^n$. Then we define the relation $\leq_{P'}$ on $\mathfrak{M}$ for elements $\mathbf{x}^\mathbf{a}, \mathbf{x}^\mathbf{b} \in \mathfrak{M}$ as: $\mathbf{x}^\mathbf{a} \leq_{P'} \mathbf{x}^\mathbf{b}$ if and only if $\mathbf{a} \leq_P \mathbf{b}$ in $\mathbb{N}^n$.*
Now if $\leq_P$ is a partial order on $\mathbb{N}^n$ then $\leq_{P'}$ is a partial order on $\mathfrak{M}$. If $\leq_P$ exhibits totality so does $\leq_{P'}$.

Proof. The proposition follows from the bijection $\mathbf{x}^\mathbf{a} \mapsto \mathbf{a}$. □

Thanks to the discussion in our first section we now have three different orderings of the monomials at our disposal namely $\leq_{\text{lex}'}$ $\leq_{\text{rlex}'}$ and $\leq_{\text{str}'}$. We get rid of the cumbersome notation and just write $\leq_{\text{lex}}$ $\leq_{\text{rlex}}$ and $\leq_{\text{str}}$ for the orders on $\mathfrak{M}$ as well. It will be understood by the context which type of order we mean.
Armed with our orders on the set of monomials we are ready to define some special subspaces of $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$. Remember that R_d denotes the vector space of homogeneous polynomials of degree d.

Definition 1.44. A filter of the poset $(\mathfrak{M}_d, \leq_{\text{lex}})$ is called a *lex-segment*.
A $\mathbb{K}$-vector subspace V of R_d is called a *lex-segment space* if $V \cap \mathfrak{M}_d$ is a $\mathbb{K}$-basis of V and a lex-segment.

The lex-segment spaces are extremly interesting in this paper since they will be seen to always exhibit equality in Green's hyperplane restriction theorem.
In Chapter 3 we will reduce Green's hyperplane restriction theorem to a simpler case when $\mathbb{K}$ has characteristic zero. In this reduced form Green's theorem will be seen to have a strong connection with *strongly stable* subsets.

Definition 1.45. A non-empty subset V of R is called *strongly stable* if $V \cap \mathfrak{M}$ is a filter of the poset $(\mathfrak{M}, \leq_{\text{str}})$.

Now we have the following nice connection between lex-segments and strongly stable subsets:

Proposition 1.46. *Every lex-segment is strongly stable.*

Proof. Let V denote a lex-segement of $\mathfrak{M}_d$ and let $\mathbf{x}^\mathbf{a} \in V$ and $\mathbf{x}^\mathbf{b} \in \mathfrak{M}_d$ such that $\mathbf{x}^\mathbf{a} <_{\text{str}} \mathbf{x}^\mathbf{b}$. Then we have that $\mathbf{b} = \mu_{i_1,j_1} \circ \cdots \circ \mu_{i_r,j_r}(\mathbf{a})$ where μ_{i_k,j_k} are strongly stable redistributing functions for $k = 1, \ldots, r$. We can assume that all our strongly stable redistributing functions differs from the identity since we can remove all those functions from the composition that don't. Let $i = \min\{i_1, \ldots, i_r\}$. Then we have that $a_i < b_i$ since there is at least one function $\mu_{i,j}$ for some j, acting on $\mathbf{a}$ such that $b_i \geq a_i + 1 > a_i$. And since i was the smallest integer of the i_k's we have that $b_k = a_k$ for $k < i$. Thus we get that $i = \min\{k : a_k \neq b_k\}$ and thus since $a_i < b_i$ we get $\mathbf{x}^\mathbf{a} <_{\text{lex}} \mathbf{x}^\mathbf{b}$ so $\mathbf{x}^\mathbf{b} \in V$. Since $\mathbf{x}^\mathbf{b}$ was arbitrary V is strongly stable. □

Since monomials inherit algebraic structure from their connection with the polynomial ring one often demand that the orderings of monomials in some way preserve this structure. Orders on the set of monomials that do this are called *monomial orderings*.

Definition 1.47. A *monomial ordering* is a total order $\leq_P$ on $\mathfrak{M}$ that also satisfies the following conditions for all $\mathbf{x}^\mathbf{a}, \mathbf{x}^\mathbf{b}, \mathbf{x}^\mathbf{c} \in \mathfrak{M}$, with $1 = \mathbf{x}^\mathbf{0}$:

(*i*) If $\mathbf{x}^\mathbf{a} \leq_P \mathbf{x}^\mathbf{b}$ then $\mathbf{x}^\mathbf{a} \cdot \mathbf{x}^\mathbf{c} \leq_P \mathbf{x}^\mathbf{b} \cdot \mathbf{x}^\mathbf{c}$

(*ii*) $1 \leq_P \mathbf{x}^\mathbf{a}$

Proposition 1.48. *$\leq_{lex}$ and $\leq_{rlex}$ are both monomial orderings.*

Proof. Let $\mathbf{x}^\mathbf{a} \neq \mathbf{x}^\mathbf{b}$ and $\mathbf{x}^\mathbf{c}$ be arbitrary. Denote $c = \sum c_i$
Now if $\mathbf{x}^\mathbf{a} <_{\text{lex}} \mathbf{x}^\mathbf{b}$ then we have that $a_j < b_j$, where $j = \min\{i : a_i \neq b_i\}$. Now we have $c_j + a_j < c_j + b_j$, where $j = \min\{i : c_i + a_i \neq c_i + b_i\}$. Thus $\mathbf{x}^\mathbf{a} \cdot \mathbf{x}^\mathbf{c} <_{\text{lex}} \mathbf{x}^\mathbf{b} \cdot \mathbf{x}^\mathbf{c}$. Now if $\mathbf{x}^\mathbf{a} \neq 1$ then there exists a $a_j > 0$, where $j = \min\{i : a_i \neq 0\}$. Thus $\mathbf{x}^\mathbf{a} >_{\text{lex}} 1$.
Now let $\mathbf{x}^\mathbf{a} <_{\text{rlex}} \mathbf{x}^\mathbf{b}$. Then we either have that $\sum a_i < \sum b_i$ which gives $c \sum a_i < c \sum b_i$ and thus $\mathbf{x}^\mathbf{a} \cdot \mathbf{x}^\mathbf{c} <_{\text{rlex}} \mathbf{x}^\mathbf{b} \cdot \mathbf{x}^\mathbf{c}$, or $a_j > b_j$, where $j = \max\{i : a_i \neq b_i\}$. Now we have $c_j + a_j > c_j + b_j$, where $j = \max\{i : c_i + a_i \neq c_i + b_i\}$. Thus $\mathbf{x}^\mathbf{a} \cdot \mathbf{x}^\mathbf{c} <_{\text{rlex}} \mathbf{x}^\mathbf{b} \cdot \mathbf{x}^\mathbf{c}$. Now if $\mathbf{x}^\mathbf{a} \neq 1$ then $\sum a_i > 0$. Thus $\mathbf{x}^\mathbf{a} >_{\text{rlex}} 1$. □

Equipped with monomial orderings we can define a special construction, called the *inital ideal*, which we will need when reducing Green's theorem in Chapter 3. We start by defining the *initial term* or *leading term* of a polynomial with respect to monomial ordering $\leq_P$

Definition 1.49. Let $f \in \mathbb{K}[x_1, x_2, \ldots, x_n]$ and let $c_\mathbf{a}$ be defined as in 1.2. The *initial term* of f with respect to a monomial ordering $\leq_P$, denoted $\text{in}_P(f)$, is defined

$$\text{in}_P(f) = \max_{<_P}(\{\mathbf{x}^\mathbf{b} \in \mathfrak{M} : c_\mathbf{b} \neq 0\})$$

In words the inital term of a polynomial is the biggest monomial with respect to the given monomial ordering in the representation of said polynomial as a sum of monomials.
Now we can define the initial ideal

Definition 1.50. Let $I \subset \mathbb{K}[x_1, x_2, \ldots, x_n]$ be a ideal. The *initial ideal* of I with respect to the monomial ordering $\leq_P$, denoted $\text{in}_P(I)$, is defined

$$\text{in}_P(I) = \langle \text{in}_P(f) : f \in I \rangle$$

1.5. Algebraic geometry

The most important reason as to why one wants to study polynomial rings comes from geometry. To fully motivate and to understand the results presented in this paper we thus need to represent the structures of our polynomial rings in a geometric way, so that one can use all the machinery developed here to apply in on geometric objects. That is the goal for algebraic geometry. Most definitions in this section are versions of those from the books [**10**], [**15**] and [**20**]. The reader is assumed to be familiar with basic knowledge about vector-spaces. In this section we will assume that the field $\mathbb{K}$ is algebraically closed.

The geometric representation in algebraic geometry is achieved by looking on the zero sets of ideals of polynomials

Definition 1.51. Given an ideal I of $\mathbb{K}[x_1, x_2, \ldots, x_n]$ the *set of zeros* or *zero-set* of I is defined as

$$\mathcal{Z}_{\mathbb{K}}(I) = \{(a_1, \ldots, a_n) \in \mathbb{K}^n : f(a_1, \ldots, a_n) = 0 \; \forall f \in I\}$$

When viewed as a geometric object $\mathcal{Z}_{\mathbb{K}}(I)$ is called an *affine algebraic variety* or just in short affine variety.

The geometry of affine varities is beautiful in its own right but it's often more rewarding to study another type of varities that behaves more nicely, namely the projective varities. Before the definition we remind ourselves of what a projective space is (as a set).

Definition 1.52. The *Projective n-space* over $\mathbb{K}$, denoted $\mathbb{P}^n_{\mathbb{K}}$, is the set of all one-dimensional subspaces of the vector space $\mathbb{K}^{n+1}$.

Remark 1.53. It's often convenient to view the points in projective space as equivalence classes defined, given a fixed representative point $(a_1, a_2, \ldots, a_{n+1}) \in \mathbb{K}^{n+1} \setminus \{0\}$, as

$$[a_1 : a_2 : \cdots : a_{n+1}] = \{(\lambda a_1, \lambda a_2 \ldots, \lambda a_{n+1}) \in \mathbb{K}^{n+1} : \lambda \in \mathbb{K}\}$$

With this interpretation the points of $\mathbb{P}^n_{\mathbb{K}}$ can be viewed as lines in $\mathbb{K}^{n+1}$ through the origin.

Now if f is an arbitrary polynomial in $\mathbb{K}[x_1, x_2, \ldots, x_n]$ and L is a point of $\mathbb{P}^{n-1}_{\mathbb{K}}$, there exist no logical way of evaluating f at the point L, since the value $f(a_1, \ldots, a_n)$ depends on the representative $(a_1, \ldots, a_n)$ of L. But if we demand that f is a homogeneous polynomial of degree d, then we get for $\lambda \in \mathbb{K}$

$$f(\lambda a_1, \lambda a_2, \ldots, \lambda a_n) = \lambda^d f(a_1, a_2, \ldots, a_n)$$

Thus if $f(a_1, a_2, \ldots, a_n) = 0$ for some representative $(a_1, a_2, \ldots, a_n)$ of L, then for $\lambda \in \mathbb{K}$ we have $f(\lambda a_1, \lambda a_2, \ldots, \lambda a_n) = 0$. Thus f can be evaluated as being zero, or vanish, at L if f is a homogeneous polynomial, since as we saw the evaluation was independent of the representative. Thus the following defintion makes sense.

Definition 1.54. Given a homogeneous ideal I of $\mathbb{K}[x_1, x_2, \ldots, x_n]$, the *projective set of zeros* or *projective zero-set* of I is defined as

$$\mathcal{Z}_{\mathbb{P}}(I) = \{[a_1 : \cdots : a_n] \in \mathbb{P}^{n-1}_{\mathbb{K}} : f(a_1, \ldots, a_n) = 0 \\ \text{for all homogeneous } f \in I\}$$

When viewed as geometric object $\mathcal{Z}_{\mathbb{P}}(I)$ is called a *projective algebraic variety* or just in short projective variety. If $V \subset \mathbb{P}^{n-1}_{\mathbb{K}}$ is a projective variety and $W \subset \mathbb{P}^{n-1}_{\mathbb{K}}$ is projective variety such that $W \subset V$ then W is said to be a projective *subvariety* of V.

We have the following special cases of projective vareties

Definition 1.55. Let be F be any homogeneous polynomial of $\mathbb{K}[x_1, x_2, \ldots, x_n]$ and denote I_F the ideal generated by F. Thus $I_F = \langle F \rangle$. Then the projective variety $\mathcal{Z}_{\mathbb{P}}(I_F)$ is called a *hypersurface*. If F is a linear form of $\mathbb{K}[x_1, x_2, \ldots, x_n]$ then $\mathcal{Z}_{\mathbb{P}}(I_F)$ is called a *hyperplane*.
Let $f_1, f_2, \ldots, f_r$ all be linear homogeneous forms of $\mathbb{K}[x_1, x_2, \ldots, x_n]$ and denote $I_f = \langle f_1, f_2, \ldots, f_r \rangle$. Then the projective variety $\mathcal{Z}_{\mathbb{P}}(I_f)$ is called a *projective linear variety* or in short linear variety.
Let be F be any homogeneous polynomial and let $f_1, f_2, \ldots, f_r$ all be linear homogeneous polynomials in $\mathbb{K}[x_1, x_2, \ldots, x_n]$. Denote $I_F = \langle F, f_1, f_2, \ldots, f_r \rangle$. Then the projective variety $\mathcal{Z}_{\mathbb{P}}(I_F)$ is called a *hypersurface in a linear subspace of* $\mathbb{P}^{n-1}_{\mathbb{K}}$.

Given a projective variety V we can define the related ring-structure of V called the *coordinate ring*. First we need the to define a related construction to the zero-set

Definition 1.56. Let $V \subseteq \mathbb{P}^{n-1}_{\mathbb{K}}$ be a projective variety. Then the *homogeneous vanishing ideal* of V is defined to be

$$\mathbb{I}(V) = \{f \in \mathbb{K}[x_1, x_2, \ldots, x_n] : f(p) = 0 \ \forall p \in V\}$$

It is clear that the homogeneous vanishing ideal of V actually is a homogeneous ideal (see [**20**, pp. 19 and 40] to convince you otherwise). Now there exists a deep connection between the homogeneous vanishing ideal of V and its defining ideal

Theorem 1.57. (Hilbert's Homogeneous Nullstellensatz) *For any homogeneous ideal $I \subset \mathbb{K}[x_1, x_2, \ldots, x_n]$, except $\mathfrak{m}$, we have*

$$\mathbb{I}(\mathcal{Z}_{\mathbb{P}}(I)) = \sqrt{I}$$

Proof. See [**10**, Theorem 1.6 pp.134-135] combined with the discussion at [**20**, pp. 39-40]. □

Remark 1.58. The reason why we don't allow our ideal to be $\mathfrak{m}$ is since the zero-set of $\mathfrak{m}$ in $\mathbb{K}^n$ is the origin, 0, which is not an element of $\mathbb{P}^{n-1}_{\mathbb{K}}$. This is the reason why $\mathfrak{m}$ is called the irrelevant ideal.

Now we're ready to define the coordinate ring of V, which will be one of our prime subjects of investigation in the paper

Definition 1.59. Let $V \subseteq \mathbb{P}^{n-1}_{\mathbb{K}}$ be a projective variety. Then the *homogenous coordinate ring* of V is defined to be the ring

$$R[V] = \frac{\mathbb{K}[x_1, x_2, \ldots, x_n]}{\mathbb{I}(V)}$$

Later when we will apply Green's hyperplane restriction theorem to the homogenous coordinate rings of a projective varieties we will see that a lot of geometric information about the projective varieties and their embeddings in projective space can be read from the cases where we have equality in Green's theorem. Thus we take some time to get aquinted with the geometric properties of projective varieties.
The dimension of vector spaces can easily be defined by counting the cardinality of the basis of the vector space. When dealing with varieties the setting is different and we need the notion of irreducibility to define dimension

Definition 1.60. Let $V \subseteq \mathbb{P}^{n-1}_{\mathbb{K}}$ be a projective variety. Then V is *irreducible* if it cannot be written as the nontrivial union of two projective subvarieties.

Remark 1.61. In some literature the defintion of a *projective variety* involves it being irreducible. Here the word *projective algebraic set* instead defines what we have choosen to call a projective variety. In this thesis we've chosen the same naming convention as the book [**20**]. This should be taken in to consideration when reading other literature e.g. [**15**].

We have a nice decomposition of varieties into irreducible components.

Proposition 1.62. *Let $V \subseteq \mathbb{P}^{n-1}_{\mathbb{K}}$ be a projective variety. Then V can be written as the finite union of irreducible projective subvarieties, $V = \cup_{i=1}^{r} V_i$. The V_i's are called the* irreducible components *of V.*

Proof. Use the proof of [**10**, Theorem 3.10] combined with the discussion [**10**, p. 88] or use [**15**, Proposition 1.5] together with [**15**, Exercise 2.5]. □

Now we can define the dimension of a projective variety

Definition 1.63. Let $V \subseteq \mathbb{P}_{\mathbb{K}}^{n-1}$ be a projective variety. Then the *dimension* of V, denoted $\dim V$, is defined as

$$\dim V = \max\{d \in \mathbb{N} : V \supseteq V_d \supsetneq V_{d-1} \supsetneq \cdots \supsetneq V_1 \supsetneq V_0\}$$

where we demand that all V_i's are nonempty irreducible subvarieties of V.

An interesting special case of varieties are the *equidimensional* varieties.

Definition 1.64. Let $V \subseteq \mathbb{P}_{\mathbb{K}}^{n-1}$ be a projective variety. Then V is *equidimensional* if all of it's irreducible components have the same dimension.

Another property that tells us something about the projective varieties is the *degree*

Definition 1.65. Let $V \subseteq \mathbb{P}_{\mathbb{K}}^{n-1}$ be a projective variety. Then the *degree* of V, denoted $\deg V$, is defined as

$$\deg V = \max\{|V \cap L| < \infty : L \text{ is a linear variety }, \dim L + \dim V = n-1\}$$

Thus the degree of V is the greatest possible finite number of intersection points of V with a linear variety of dimension equal to the codimension of V.

Remark 1.66. If I is a homogeneous ideal of $\mathbb{K}[x_1, x_2, \ldots, x_n]$ then Hilbert's Nullstellensatz puts a restriction on what possible rings $\mathbb{K}[x_1, x_2, \ldots, x_n]/I$ will be able to give a geometric interpretation as the coordinate rings for projective varieties, namely that I must be radical. Fortunatly there exists a generalization of the concept of a projective variety called *schemes* that makes it possible to give a geometric interpretation to any ring of the form $\mathbb{K}[x_1, x_2, \ldots, x_n]/I$ where I is any homogeneous saturated ideal (see [**15**, Exercise 5.10(d)]), which is a much larger class of ideals. Unfortunatly the treatment of schemes would make this paper even more lengthy than it already is, and thus we only note that the generalization exists and thus also the geometric interpretation of $\mathbb{K}[x_1, x_2, \ldots, x_n]/I$ where I is any homogeneous saturated ideal. The interested reader is encouraged to read the book [**15**] for more information about schemes.
Later in the paper when we study arbitrary homogeneous ideals this is well supported geometrically since the saturation of a arbitrary homogeneous ideal can only give rise to one (closed) scheme, even though the motivation is not in this paper. Also the theorems presented from articles on the subject in Chapter 2 will often be originally written in the languages of schemes, but

here they will be presented as theorems revolving around projective varieties so that an overview of this vast subject can be achieved. In the language of schemes the projective varieties are reduced closed subschemes (see [**15**, Proposition 4.10]).

1.6. The Zariski topology

To understand the underlaying geometry in commutative algebra we need to define the topology used in commutative algebra. First we recall what a topology is

Definition 1.67. A *topology* on a set X is a collection of subsets $\mathcal{U}$ of X satisfy the following

(*i*) $\emptyset, X \in \mathcal{U}$

(*ii*) If $\{U_i\}_{i=0}^{\infty}$ is a collection of sets from $\mathcal{U}$, then $\bigcup_{i=0}^{\infty} U_i \in \mathcal{U}$

(*iii*) If $\{U_i\}_{i=0}^{r}$ is a collection of sets from $\mathcal{U}$, then $\bigcap_{i=0}^{r} U_i \in \mathcal{U}$

The sets in $\mathcal{U}$ are called *open* and their complements will be called *closed*.
A space X equipped with a topology $\mathcal{U}$ is called a *topological space*.
If Y is a subset of a topological space X with topology $\mathcal{U}$ then the *induced topology* on Y is the topology defined by $\tau_Y = \{Y \cup U : U \in \mathcal{U}\}$.

It is also useful to know the following property

Definition 1.68. A subset U of a topological space X is *dense* if every nonempty open set in X contains a element of U

Now we can define the Zariski topology on $\mathbb{K}^n$ and on $\mathbb{P}_{\mathbb{K}}^{n-1}$. We have the notation $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$

Definition 1.69. The *Zariski topology* on $\mathbb{K}^n$ is the topology with open sets equal to the the complements of affine varieties, and the closed sets equal to the affine varities.
Similarly the *Zariski topology* on $\mathbb{P}_{\mathbb{K}}^{n-1}$ is the topology with open sets equal to the the complements of projective varieties, and the closed sets equal to the projective varities.

For a proof of why this actually defines a topology on $\mathbb{K}^n$ see [**17**, Proposition 5.5.20]. The case of $\mathbb{P}_{\mathbb{K}}^{n-1}$ is treated the same way.
Earlier in Definition 1.60 we defined what a irreducible variety was. This definition is actually a special case of a more general definition of irreducibility, which we state here.

Definition 1.70. Let Y be a nonempty subset of a topological space X. Then Y is *irreducible* if it cannot be written as the nontrivial union of two closed subsets of Y.

Finally we're ready to define a keyword needed to understand Green's hyperplane restriction theorem.

Definition 1.71. Let $\mathcal{P}$ be a property of elements in $\mathbb{K}^n$ (or $\mathbb{P}^{n-1}_{\mathbb{K}}$). We say that $\mathcal{P}$ hold *generically* in $\mathbb{K}^n$ (or $\mathbb{P}^{n-1}_{\mathbb{K}}$), or *for a generic element* of $\mathbb{K}^n$ (or $\mathbb{P}^{n-1}_{\mathbb{K}}$), if there exists a non-empty Zariski open subset U of $\mathbb{K}^n$ (or $\mathbb{P}^{n-1}_{\mathbb{K}}$) such that $\mathcal{P}$ holds for all elements of U.
Especially we have that a property $\mathcal{P}$ holds for a *generic linear form* $h \in R_1$, if there exists non-empty Zariski open subset U of $\mathbb{K}^n$ such that $\mathcal{P}$ holds for all elements $a_1x_1 + \cdots + a_nx_n \in R_1$ with coefficients $(a_1, \ldots, a_n) \in U$.

The reason for the use of the word generic is because every Zariski open subset happens to be dense. To prove this we need first to prove the following.

Proposition 1.72. *Every nonempty open subset of an irreducible topological space is dense.*

Proof. Let X be an irreducible topological space. Now assume the proposition is false. Then we can find a nonempty open subset U_1 that is not dense. Since U_1 is not dense there also exists another nonempty open subset U_2 such that $U_1 \cap U_2 = \emptyset$. Denote the complement of U_1 as U_1^C and the complement of U_2 as U_2^C. But then we have that $U_1^C \cup U_2^C = X$ and both U_1^C and U_2^C are closed proper subsets of X, since $U_1 \subset U_2^C$ and vice versa. Thus X cannot be irreducible, which is a contradiction. Thus the proposition is true. □

Proposition 1.73. *$\mathbb{K}^n$ and $\mathbb{P}^{n-1}_{\mathbb{K}}$ are both irreducible topological spaces and thus every nonempty Zariski open subset U of $\mathbb{K}^n$ or $\mathbb{P}^{n-1}_{\mathbb{K}}$ is dense.*

Proof. See [**15**, Example 1.4.1 & Excercise 2.4(c)]. □

Chapter 2

Green's theorem

We are ready for any unforeseen event that may or may not occur.

DAN QUAYLE
44th Vice President of the United States

Mathematics follows the path of maximal irony

MARK GREEN

This chapter is dedicated to present the main subject of this thesis and also the research available on the subject that could be found. The research material is mostly gathered from the article [**1**] and its very insightful theorems concerning the subject.
Just as in our first chapter we will have that, unless stated otherwise, $\mathbb{K}$ is an arbitrary infinite field and $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ is standard graded.

2.1. Hilbert functions

A lot of the motivation of why one would like to study Green's hyperplane restriction theorem comes from its applications to Hilbert functions. Thus we get aquinted with these functions in this section.

Definition 2.1. Let I be a homogeneous ideal of R. Define

$$(R/I)_d = \{ \sum_{\mathbf{a}\in\mathbb{N}^n} c_{\mathbf{a}}\mathbf{x}^{\mathbf{a}} \in R/I : \deg(\mathbf{x}^{\mathbf{a}}) = d \text{ whenever } c_{\mathbf{a}} \neq 0\}$$

Then the numerical function $H(R/I, -) : \mathbb{N} \to \mathbb{N}$ defined for $d \in \mathbb{N}$ as

$$H(R/I, d) = \dim_{\mathbb{K}}(R/I)_d$$

is called the *Hilbert function of* R/I.

The Hilbert functions of quotient rings will be our main subject in this chapter since they make the formualtion of Green's hyperplane restriction theorem so much simpler, which we will see in the following section. But before that we state a proposition about Hilbert functions of initial ideals that will come in handy in the next chapter

Proposition 2.2. *Let* I *be a homogeneous ideal of* R *and* $\leq_P$ *a monomial ordering. Then for any* $d \in \mathbb{N}$ *we have*

$$H(R/I, d) = H(R/(\text{in}_P(I)), d)$$

Thus the ideal and it's initial ideal define the same corresponding Hilbert function.

Proof. See [**13**, Proposition 1.11]. □

2.2. Green's hyperplane restriction theorem

We're now ready to formulate the main theorem of this whole thesis. The theorem was first proved by Mark Green in his paper [**12**] from 1988 and is therefore named after him.

Theorem 2.3. (Green's hyperplane restriction theorem)
Let $d \in \mathbb{N}$ *and let* $V \subseteq R_d$ *be a* $\mathbb{K}$*-vector subspace. Further let* $\ell \in R_1$ *be a generic linear form and denote* V_ℓ *the image of* V *in* $R/\langle \ell \rangle$*. Then we have*

$$\text{codim}_{\mathbb{K}}(V_\ell) \leq \text{codim}_{\mathbb{K}}(V)_{<d>}$$

and equality holds if V *is a Lex-segment space.*

Proof. See [**17**, Theorem 5.5.25], [**7**, Theorem 4.2.12] or [**13**, Theorem 3.4] for some modern approaches to the Theorem. Alternatively see the original proof in [**12**, Theorem 1] □

For simplicity we will often refer to this theorem as *Green's theorem* in the thesis since it's understood that we don't mean the famous theorem from integral calculus.
A closely related theorem is that of Macaulay.

Theorem 2.4. (Macaulay's theorem)
Let $d \in \mathbb{N}$ *and let* $V \subseteq R_d$ *be a* $\mathbb{K}$*-vector subspace. Then we have*

$$\text{codim}_{\mathbb{K}}(R_1 \cdot V) \leq \text{codim}_{\mathbb{K}}(V)^{<d>}$$

Here equality holds if V *is a Lex-segment space.*

Proof. See [**17**, Theorem 5.5.27] or [**7**, Theorem 4.2.10]. Note that in Macaulay's theorem one does not need the assumption that $\mathbb{K}$ is infinite. □

Both these theorems translates easily into the languages of Hilbert functions. We get

Corollary 2.5. *Let I be a homogeneous ideal of R. For a generic linear form $\ell \in R_1$ and $d \in \mathbb{N}$ and we have*

(a) $H(R/(I + \langle \ell \rangle), d) \leq H(R/I, d)_{<d>}$ (Green)

(b) $H(R/I, d + 1) \leq H(R/I, d)^{<d>}$ (Macaulay)

Here equality holds if I_d is a Lex-segment space, with the added condition that we have $I_{d+1} = R_1 \cdot I_d$ for equality in Macaulay's theorem.

See [**17**, Corollary 5.5.26] and [**17**, Corollary 5.5.28] to convince you of why these are the natural translation of the aforementioned theorems.

2.3. Persistent extremal behavior for large degrees

With Green's hyperplane restriction theorem finally formualted we can now indulge into the investigations of which this thesis is supposed to cover, namely when this theorem achieves equality.
We start by using our result from Chapter 1 to prove a fact about the persistent behavior of Hilbert-functions.

Proposition 2.6. *Let I be a homogeneous ideal of R. Then there exists an integer d such that for all $j \geq d$ we have*

$$H(R/I, j + 1) = H(R/I, j)^{<j>}$$

Proof. If $I = R$ then $H(R/I, n) = 0$ for all $n \geq 0$. Thus assume $I \subsetneq R$. Then we have that $I_0 = \emptyset$ and thus $H(R/I, 0) = \dim_{\mathbb{K}} \mathbb{K} = 1$. Combined with Macaulay's theorem this enables us to use Proposition 1.31 and the proposition is proven. □

Remark 2.7. The conventional way of proving Proposition 2.6 is by using an argument based on the fact that $\mathbb{K}[x_1, x_2, \ldots, x_n]$ is Noetherian. See e.g. the proof of [**17**, Corollary 5.5.34] or of [**7**, Corollary 4.2.14]. With the use of Proposition 1.31 we managed to reduce this to a combinatorical problem instead.

Proposition 2.6 deals with equality in Macaulay's theorem, but we're mainly interested in equality in Green's theorem. Fortunatly there is a close connection between the two of them

Lemma 2.8. *Let I be a homogeneous ideal in R, $\ell \in R_1$ be a generic linear form and j be an integer such that $j \geq sat(I)$. Further let*

$$M_{j+1}(H(R/I, j+1)) = (\Delta_{j+1}, \Delta_j, \dots, \Delta_\delta)$$

be the $(j+1)^{th}$ Macaulay difference tuple of $H(R/I, j+1)$. Then the following statements are equivalent:

(i) $H(R/I, j+1) = H(R/I, j)^{<j>}$

(ii) $\delta > 1$ *and* $H(R/(I + \langle \ell \rangle), j+1) = H(R/I, j+1)_{<j+1>}$

Proof. See [**2**, Theorem 3.3]. □

Equipped with this lemma we can prove a statement about equality in Green's theorem equivalent to that of Proposition 2.6.

Proposition 2.9. *Let I be a homogeneous ideal of R and let $\ell \in R_1$ be a generic linear form. Then there exists an integer d such that for all $j \geq d$ we have*

$$H(R/(I + \langle \ell \rangle), j) = H(R/I, j)_{<j>}$$

Proof. The following proof is a just a rigorous restatement of [**1**, Remark 2.11] and other arguments spread out in the same article.
Let d_0 be the integer from Proposition 2.6 such that $H(R/I, j+1) = H(R/I, j)^{<j>}$ for all $j \geq d_0$. Now define $d = 2 + \max\{d_0, \mathrm{sat}(I)\}$. Then by Proposition 2.6 we have that $H(R/I, j) = H(R/I, j-1)^{<j-1>}$ for all $j \geq d-1$. Now write

$$M_{d-1}(H(R/I, d-1)) = (\Delta_{d-1}, \Delta_{d-2}, \dots, \Delta_\delta)$$

Now for any $j \geq d$ we have from Remark 1.30 that

$$M_j(H(R/I, j)) = (\Delta_j, \Delta_{j-1}, \dots, \Delta_{\delta+j-d+1})$$

And since $j > d-1$ we have $j - d + 1 > 0$ and thus $\delta + j - d + 1 > 1$ so by Lemma 2.8 we now have that this implies that $H(R/(I + \langle \ell \rangle), j) = H(R/I, j)_{<j>}$ for all $j \geq d$ and we're done. □

Now we want to turn our eyes to the possible geometric interpretations of equality in Green's and Macualays theorems, so from here on we will have that, unless stated otherwise, $\mathbb{V}$ is an arbitrary projective variety (reduced closed subscheme), and $\mathbb{I}(\mathbb{V})$ will denote its homogeneous vanishing ideal. Thus we have that $R[\mathbb{V}] = R/\mathbb{I}(\mathbb{V})$ is the homogeneous coordinate ring of $\mathbb{V}$. Note that for any linear form $\ell \in R_1$ we have that $R/(\mathbb{I}(\mathbb{V}) + \langle \ell \rangle) = R[\mathbb{V}]/\ell R[\mathbb{V}]$.
Motivated by Proposition 2.6 and Proposition 2.9 it makes sense to make the following definitions

Definition 2.10. We define $M(\mathbb{V})$ to be the the least integer d such that the Hilbert function of $R[\mathbb{V}]$ achieves its extremal value in Macaulay's theorem for all degrees $\geq d$. Thus

$$M(\mathbb{V}) = \min\{d \in \mathbb{N} : H(R[\mathbb{V}], j+1) = H(R[\mathbb{V}], j)^{<j>}, \forall j \geq d\}$$

In the same way define $G(\mathbb{V})$ to be the least integer d such that the Hilbert function of $R[\mathbb{V}]$ achieves its extremal value in Green's theorem for all degrees $\geq d$. Thus

$$G(\mathbb{V}) = \min\{d \in \mathbb{N} : H(R[\mathbb{V}]/\ell R[\mathbb{V}], j) = H(R[\mathbb{V}], j)_{<j>}, \forall j \geq d\}$$

Remark 2.11. In Defintion 2.10 we have switched the naming convention of the two functions from how they appear in the article [**1**], where $M(\mathbb{V})$ is defined as the least integer for which Green's theorem achieves it's extremal value and $G(\mathbb{V})$ is the corresponding integer for Macualay's theorem. This is because we want to stress the connection of the two with their corresponding theorems. The reason for the confusing name-convention in [**1**] is probably because $G(\mathbb{V})$ seems to be named after Gotzmann which studied these types of numbers a lot and $M(\mathbb{V})$ seems to be named after Macaulay.

Now since projective varieties have saturated defining ideals (see [**15**, Exercise 5.10(c)]) we have that $\text{sat}(I(\mathbb{V})) = 0$ and thus by the proof of 2.9 we get that $G(\mathbb{V}) \leq M(\mathbb{V}) + 2$. Note that the article [**1**, p. 12] incorrectly writes this conclusion as $G(\mathbb{V}) \leq M(\mathbb{V})+1$, since they don't take into account their own remark 2.11. However this inequality can be greatly improved.

Proposition 2.12. *Let* $\mathbb{V}$ *be a projective variety. Then either* $G(\mathbb{V}) = M(\mathbb{V})$ *or* $G(\mathbb{V}) = 1$.

Proof. See [**1**, Proposition 3.1] and [**1**, Proposition 3.6].
Note: The original formulation of these propositions states that $\mathbb{V}$ is a closed subscheme and need not be reduced. □

The immediate question one asks after seeing this result is if there is some geometric interpretation to the two different cases of possible $G(\mathbb{V})$. Fortunatly for us there is one whenever $\mathbb{V}$ is equidimensional

Theorem 2.13. *Let* $\mathbb{V}$ *be a equidimensional projective variety. Then either* $M(\mathbb{V}) = \deg \mathbb{V}$ *or* $G(\mathbb{V}) = M(\mathbb{V})$.
Further $G(\mathbb{V}) = 1$ *if and only if* $\mathbb{V}$ *is a hypersurface in a linear subspace of* $\mathbb{P}^{n-1}_{\mathbb{K}}$.

Proof. See [**1**, Corollary 4.5] and [**1**, Corollary 4.6]. □

By Theorem 2.13 we thus get a complete characterization of the equality $G(\mathbb{V}) = 1$ for equidimensional projective varities, showing the geometric use of looking at the persistent equality of Green's hyperplane restriction theorem.

2.4. Equality of Green's theorem for general degrees

We've seen that Green's hyperplane restriction theorem eventually has equality for all degrees large enough. But Green's theorem may of course achieve equality before this happends, and in this section we will see in what cases we can interpret what this means.
In the section when we say that a projective variety $\mathbb{V}$ *achieves equality in Green's theorem in degree* d we will mean that we have the following for a generic linear form, $\ell \in R_1$

$$H(R[\mathbb{V}]/\ell R[\mathbb{V}], d) = H(R[\mathbb{V}], d)_{<d>}$$

In his original article [**12**], where the hyperplane restriction theorem originated, Mark Green included two theorems of what could be said when the theorem achieved equality, [**12**, Theorem 3] and [**12**, Theorem 4]. In the article [**1**, Theorem 3.10] a very beautiful generalisation of these two theorems is proved, which we will state here.

Theorem 2.14. *Let d be an integer and $\mathbb{V}$ be a projective variety that achieves equality in Green's theorem in degree d. If one can find integers l and r such that $1 \leq l \leq d$, $r \geq 1$ and*

$$M_d(H(R[\mathbb{V}], d)) = (r, r, \dots, r)$$

where the d^{th} Macaulay difference tuple has positive length l then

$$I(\mathbb{V})_d = \langle F, L_1, \dots, L_{n-(r+1)} \rangle_d$$

where F is a homogeneous form of degree l and $L_1, \dots, L_{n-(r+1)}$ are linear forms. Thus the ideal of $\mathbb{V}$ has the same structure in degree d as the ideal of a hypersurface of degree l in some $(r+1)$-dimensional linear subspace of $\mathbb{P}^{n-1}_{\mathbb{K}}$

Proof. See [**1**, Theorem 3.10]
Note: The original formulation of the theorem states that $\mathbb{V}$ is a closed subscheme and need not be reduced. □

This is the most general theorem of what happends when we achieve equality in Green's theorem. But if we define some functions operating on the Macaulay difference tuple of a given degree we can formulate some less general theorems revolving around the equality.

Definition 2.15. Let I be a homogeneous ideal in R and x an arbitrary integer. The positive length of the d^{th} Macaulay difference tuple of $H(R/I, d)$ will be denoted $L(R/I, d)$.

Considering $M_d(H(R/I,d))$ as a multiset, the number of elements equal to x in $M_d(H(R/I,d))$ will be deonted $C_x(R/I,d)$.

Example 2.16. If we have the situation described in Theorem 2.14 we have that $L(R[\mathbb{V}],d) = C_r(R[\mathbb{V}],d) = l$ and for any positive integer $x \neq r$ $C_x(R[\mathbb{V}],d) = 0$.

With these tools we can formulate the final theorems concerning equality in Green's theorem

Theorem 2.17. *Let d be an integer and $\mathbb{V}$ be a projective variety that achieves equality in Green's theorem in degree d. Further let*

$$M_d(H(R[\mathbb{V}],d)) = (\Delta_d, \Delta_{d-1}, \dots, \Delta_\delta)$$

be the d^{th} Macaulay difference tuple of $H(R/I,d)$ and assume that $C_0(R[\mathbb{V}],d) = 0$.
Then if $L(R[\mathbb{V}],d) > C_{\Delta_d}(R[\mathbb{V}],d)$ we have that

$$I(\mathbb{V})_{\Delta_d} = \langle L_1, \dots, L_{n-(\Delta_d+1)}\rangle_{\Delta_d}$$

Furthermore if $\delta > 1$, there is a homogeneous polynomial, F, of degree $C_{\Delta_d}(R[\mathbb{V}],d)$ such that

$$I(\mathbb{V})_j \subset \langle F, L_1, \dots, L_{n-(\Delta_d+1)}\rangle_j$$

for every $j \geq d$

Proof. See [**1**, Theorem 4.2]
Note: The original formulation of the theorem states that $\mathbb{V}$ is a closed subscheme and need not be reduced. □

Theorem 2.17 is quite messy on its own, but it has a very nice corollary for equidimensional projective varieties.

Corollary 2.18. *Let d be an integer and $\mathbb{V}$ be a equidimensional projective variety of dimension r that achieves equality in Green's theorem in degree d. Further let*

$$M_d(H(R[\mathbb{V}],d)) = (\Delta_d, \Delta_{d-1}, \dots, \Delta_\delta)$$

be the d^{th} Macaulay difference tuple of $H(R/I,d)$ and assume that $\Delta_d = r$ and $C_0(R[\mathbb{V}],d) = 0$. Then $\mathbb{V}$ is a hypersurface in a linear subspace of $\mathbb{P}^{n-1}_{\mathbb{K}}$.

Proof. See [**1**, Corollary 4.4] □

Remark 2.19. The alert reader may have noticed the similarites of Corollary 2.18 and Theorem 2.13, and the fact is that Corollary 2.18 is used to prove Theorem 2.13 in the article [**1**]. Both are stated here since they are both intersting by themself, even though they are very much connected.

Chapter 3

Reduction of Green's theorem in characteristic zero

Det är stabilt!

Dr. Alban

It's not X you should worry about – it's me!!!

Zero
Rockman X

One very important construction when dealing with the algebra and combinatoris of an ideal is the intial ideal which we encountered in Definition 1.50. Unfortunately the initial ideal depend on the choice of coordintates for our polynomial ring. The solution to this problem is the generic intial ideal of an ideal. Mark Green himself recognised it's usefulness in [**13**].
In this Chapter we will see how Green's theorem can reduced using the generic initial ideals with respect to reverse lexicographical order. The reduction will result in a case where we will only need to consider strongly stable subset and to check if we have equality in Green's theorem one only needs to count the number of elements that are divisible by x_n. In Chapter 4 we will make use of the results from this chapter, so the results in this chapter will be used a lot later on.
The reduction is only valid when $\mathbb{K}$ has characteristic zero, so throughout this chapter we will have unless stated otherwise that $\mathbb{K}$ is a field with characteristic zero and $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ is standard graded. We also have

that $\mathfrak{M}$ denotes the set of all monomials in R as before. We will also have that if I is an ideal of R *that achieves equality in Green's theorem in degree* d we will mean that we have the equality $H(R/(I+\langle\ell\rangle),d) = H(R/I,d)_{<d>}$ for a generic linear form $\ell \in R_1$.
The definitions of this chapter are mostly gathered from [**13**], [**10**] and [**18**]. The careful reader may note that in [**13**] the defintions and theorems are constructed for the case where we have that our field $\mathbb{K} = \mathbb{C}$. One can check that despite this the results from [**13**] are valid for any field $\mathbb{K}$ with characteristic zero.

3.1. Generic initial ideals

The main idea by the generic initial ideals compared to intial ideals is that one wishes to get rid of the dependence on the coordinates $x_1, x_2, \ldots$ in the polynomial ring. To to do this we need to be able to "change the coordinates" of an ideal in R. We let the general linear group act on our polynomials to achieve this. First we remind ourselves of what the general linear group is.

Definition 3.1. The *general linear group* of degree n over $\mathbb{K}$ is the group consisting of invertible $n \times n$ matrices with entries in $\mathbb{K}$ and with matrix multiplication as its binary operation. It is denoted by $\mathbf{GL}_n(\mathbb{K})$.

Now we can define the action of $\mathbf{GL}_n(\mathbb{K})$, and hence all its subgroups, on R in the following way

Definition 3.2. Let $g = (g_{ij}) \in \mathbf{GL}_n(\mathbb{K})$ be a arbitrary matrix and let $f \in R$ be a arbitrary polynomial. We will denote by $g \cdot f$ the standard action of $\mathbf{GL}_n(\mathbb{K})$ on R which is defined by

$$g \cdot f = f(gx_1, gx_2, \ldots, gx_n), \qquad \text{where} \qquad gx_i = \sum_{j=1}^{n} g_{ij}x_j$$

If I is an ideal of R then we define the action $g \cdot I$ by

$$g \cdot I = \{g \cdot f : f \in I\}$$

We're interested in a particular subgroup of the general linear group and its action on ideals. The subgroup is called the Borel subgroup

Definition 3.3. The *Borel subgroup* of $\mathbf{GL}_n(\mathbb{K})$ is the defined as the subgroup consisting of all lower triangular matrices and will be denoted $B_n(\mathbb{K})$. More explicitly we have

$$B_n(\mathbb{K}) = \{(g_{ij}) \in \mathbf{GL}_n(\mathbb{K}) : g_{ij} = 0 \text{ for } j > i\}$$

If I is an ideal of R and we have $b \cdot I = I$ for all $b \in B_n(\mathbb{K})$ then we say that I is *Borel-fixed*.

Remark 3.4. There seems to be some confusion as to whether the Borel subgroup should consist of the lower triangular matrices or the upper triangular matrices in the literature. In [**13**, Definition 1.22] the Borel subgroup is defined as the lower triangular matrices and in [**10**, p. 352] it's defined as the upper triangular matrices. Both books however comes to correct conclusions since they both define the action of $\mathbf{GL}_n(\mathbb{K})$ differently. The action of $\mathbf{GL}_n(\mathbb{K})$ in [**13**] is consistent with our defintion but in [**10**] the action is instead defined by the substitution $gx_j = \sum_{i=1}^{n} g_{ij}x_i$.
However not all literature is as lucky as these two. The book [**18**, pp.21-22] makes the unfortunate mistake to mix the defintions and thus have the action of $\mathbf{GL}_n(\mathbb{K})$ defined as in [**13**] but the Borel subgroup defined as in [**10**]. This mistake becomes even more unfortunate when one takes in consideration that the whole chapter where the defintions are made is about Borel-fixed monomial ideals. To [**18**]'s defense the proofs in the chapter are all correct, since they all seem to ignore the incorrect definition.
The thesis [**11**] has borrowed the faulty definitions from [**18**] and thus makes the same mistake.

Viewing the matrices in $\mathbf{GL}_n(\mathbb{K})$ as vectors with n^2 coordinates, $\mathbf{GL}_n(\mathbb{K})$ can be seen as a subset of $\mathbb{K}^{n^2}$. Thus we can use the Zariski topology from $\mathbb{K}^{n^2}$ in the obvious way on $\mathbf{GL}_n(\mathbb{K})$, so it makes sense to talk about Zariski open and closed sets in $\mathbf{GL}_n(\mathbb{K})$.
This allows us to formulate the following beautiful result which will motivate us to examine the Borel-fixed ideals.

Theorem 3.5. (Galligo's Theorem)
Let $\leq_P$ be any monomial ordering on $\mathfrak{M}$ and let I be any homogeneous ideal of R. Then there exists a Zariski open subset $U \subseteq \mathbf{GL}_n(\mathbb{K})$ such that $\mathrm{in}_P(g \cdot I)$ is constant and Borel-fixed for all $g \in U$.

Proof. See [**13**, Theorem 1.27] or [**10**, Theorem 15.18 & Theorem 15.20]. □

With Galligo's Theorem formulated we now can define the generic intial ideal

Definition 3.6. Let $\leq_P$ be any monomial ordering on $\mathfrak{M}$ and let I be any homogeneous ideal of R. Then the intial ideal which is constant and Borel-fixed on a Zariski open subset of $\mathbf{GL}_n(\mathbb{K})$, as given by Galligo's Theorem, is called the *generic initial ideal* of I with respect to $\leq_P$ and will be denoted $\mathrm{gin}_P(I)$.

Now Borel-fixed ideals are closely related to the strongly stable monomial ideals as seen by the following result

Proposition 3.7. *Let I be a monomial ideal of R. Then I is Borel-fixed if and only if I is strongly stable.*

Proof. See [**5**, Proposition 2.7] □

Thus Galligo's Theorem together with Proposition 3.7 now gives us the pleasant result that all generic intitial ideals are strongly stable.
The most interesting monomial ordering when dealing with restrictions to hyperplanes turns out to be the reverse lexicographical order. To give a motivation to this claim we present the following result.

Proposition 3.8. *Let I be a homogeneous ideal of R. Further let $\ell \in R_1$ be a generic linear form and denote I_ℓ the image of I in $R/\langle \ell \rangle$. Then we have that*

$$\operatorname{gin}_{rlex}(I_\ell) = (\operatorname{gin}_{rlex}(I))_{x_n}$$

where $(\operatorname{gin}_{rlex}(I))_{x_n}$ denotes the image of $\operatorname{gin}_{rlex}(I)$ in $R/\langle x_n \rangle$.

Proof. See [**13**, Corollary 2.15]. □

Thus we see that a generic hyperplane restriction can via the generic initial ideals with respect to the reverse lexicographical order be made in to a restriction made simply by the hyperplane defined by x_n, which is a great improvement.

3.2. The reduction

We saw that the Hilbert function corresponding to any intial ideal is equal to the ideal defining the initial ideal by Proposition 2.2. This is especially true for the generic intial ideal. Combine this with Proposition 3.8 and the following reduction of Green's theorem should come as no surprise.

Theorem 3.9. *Let d be a positive integer and let B be any strongly stable subset of $\mathfrak{M}_d$. Then the number of monomials in $\mathfrak{M}_d \setminus B$, that are not divisible by x_n, is at most $|\mathfrak{M}_d \setminus B|_{<d>}$.*

Proof. To see a proof that this theorem indeed is a reduction of Green's Theorem for the case of generic intial ideals see [**11**, Theorem 6] or use the argument from the proof of [**13**, Proposition 3.5]. □

We have this nice corollary

Corollary 3.10. *For $d \in \mathbb{N}$ let I be a homogeneous ideal of R with generic initial ideal $\operatorname{gin}_{rlex}(I)$ (with respect to the reverse lexicographical order) and let $B = \operatorname{gin}_{rlex}(I) \cap \mathfrak{M}_d$. Then I achieves equality in Green's theorem in degree d if and only if we have that the number of monomials in $\mathfrak{M}_d \setminus B$ that are not divisible by x_n, is exactly $|\mathfrak{M}_d \setminus B|_{<d>}$.*

Proof. Since Theorem 3.9 is a reduction of Green's theorem to the generic initial ideal-case we have by using the same argument as in the proof of [**11**, Theorem 6] or [**13**, Proposition 3.5] that equality in Green's theorem for a homogeneous ideal implies equality in Theorem 3.9 and vice versa. □

Now we want to examine all possible generic intial ideals that attain equality in Theorem 3.9 and thus the possible generic intial ideals for arbitrary homogeneous ideals that attain equality in Green's theorem. First we need to count the number of monomials divisible by x_n in a monomial ideal.

Definition 3.11. Let d be a positive integer. For any subset $M \subseteq \mathfrak{M}_d$ we define

$$\div_n(M) = |\{m \in M \ : x_n \nmid m\}|$$

Now we can formulate and prove the classification of what possible generic initial ideals (with respect to the reverse lexicographical order) there can exists for ideals that attain equality in Green's theorem for degree d.

Proposition 3.12. *Let d be a positive integer and let B be any strongly stable subset of $\mathfrak{M}_d$. Then B attains equality in Theorem 3.9 if and only if there exists a lex-segment $\Lambda \subseteq \mathfrak{M}_d$, such that $|\Lambda| = |B|$ and $\div_n(B) = \div_n(\Lambda)$,*

Proof. See [**17**, Proposition 5.5.18] or [**11**, Proposition 7] to see why a lex-segment Λ always attains equality in Theorem 3.9.
Now if B attains equality then $\div_n(B) = |B|_{<d>}$. Now if we take the lex-segment Λ with the same size as B, thus $|\Lambda| = |B|$, we get since lex-segments always attains equality in Theorem 3.9 that $\div_n(\Lambda) = |\Lambda|_{<d>} = |B|_{<d>} = \div_n(B)$.
Now assume there exists a lex-segment $\Lambda \subseteq \mathfrak{M}_d$, such that $|\Lambda| = |B|$ and $\div_n(B) = \div_n(\Lambda)$. Then we have, since lex-segments always attains equality in Theorem 3.9 that $\div_n(\Lambda) = |\Lambda|_{<d>} = |B|_{<d>} = \div_n(B)$. □

3.3. Strongly stable subsets

Motivated by the beforementioned reduction of Green's theorem we take some time to study the property of being strongly stable. Remember that we have that $\mathfrak{M}_d$ denotes the set of monomials of degree d in $\mathbb{K}[x_1, \ldots, x_n]$. First we show that there exists an alternate definition for strongly stable subsets apart from Definition 1.45, that hopefully contribute to the understanding of strongly stable subsets (actually this is the conventional definition):

Proposition 3.13. *A non-empty subset B of $\mathbb{K}[x_1, \ldots, x_n]$ is strongly stable if and only if, given any monomial $m \in B$ and indices $i, j \in \{1, \ldots, n\}$, it holds that if $x_j | m$ and $i < j$, then $m\frac{x_i}{x_j} \in B$ also.*

Proof. First let B be a strongly stable set and let $m = \mathbf{x}^{\mathbf{a}} \in B$ be arbitrary. By Definition 1.45 $B \cap \mathfrak{M}$ is an filter of the poset $(\mathfrak{M}, \leq_{\text{str}})$. Now assume we can find indices $i, j \in \{1, \dots, n\}$, such that $x_j | \mathbf{x}^{\mathbf{a}}$ and $i < j$. By Definition 1.13 we have

$$\mathbf{x}^{\mathbf{a}} \frac{x_i}{x_j} = x_1^{a_1} \cdots x_i^{a_i+1} \cdots x_j^{a_j-1} \cdots x_n^{a_n} = \mathbf{x}^{\mu_{i,j}(\mathbf{a})}$$

Now by Definition 1.14 and Proposition 1.43 we thus get that $m\frac{x_i}{x_j} >_{\text{str}} m$, and by the defintion of an filter this implies $m\frac{x_i}{x_j} \in B$.
Now let B be a subset that satisfies the conditions of our proposition, let $\mathbf{x}^{\mathbf{b}} \in B$ and let $\mathbf{x}^{\mathbf{a}}$ be a arbitrary element such that $\mathbf{x}^{\mathbf{a}} <_{\text{str}} \mathbf{x}^{\mathbf{b}}$. By Definition 1.14 we thus have $\mathbf{b} = \mu_{i_1,j_1} \circ \cdots \circ \mu_{i_r,j_r}(\mathbf{a})$, where μ_{i_k,j_k} are strongly stable redistributing functions for $k = 1, \dots, r$. We can assume that all our strongly stable redistributing functions differs from the identity since we can remove all those functions from the composition that don't. But now since B satisfies the conditions from our proposition we must have that $\mathbf{x}^{\mu_{i_r,j_r}(\mathbf{a})} = \frac{x_{i_r}}{x_{j_r}}\mathbf{x}^{\mathbf{a}}$ and thus $\mathbf{x}^{\mu_{i_r,j_r}(\mathbf{a})} \in B$. But then by the same reasoning $\mathbf{x}^{\mu_{i_{r-1},j_{r-1}} \circ \mu_{i_r,j_r}(\mathbf{a})} \in B$ and so on until we conclude $\mathbf{x}^{\mathbf{b}} = \mathbf{x}^{\mu_{i_1,j_1} \circ \cdots \circ \mu_{i_r,j_r}(\mathbf{a})} \in B$. Thus $B \cap \mathfrak{M}$ is an filter of the poset $(\mathfrak{M}, \leq_{\text{str}})$ and we're finished. □

It's a simple consequence of Proposition 3.13 and Definition 1.7 that the following also is true.

Corollary 3.14. *Let B be a non-empty subset of $\mathbb{K}[x_1, \dots, x_n]$. Then $B \cap \mathfrak{M}$ is a order ideal of the poset $(\mathfrak{M}, \leq_{str})$ if and only if, given any monomial $m \in B$ and indices $i, j \in \{1, \dots, n\}$ it holds that if $x_j | m$ and $i > j$ then $m\frac{x_i}{x_j} \in B$ also.*
If $B \cap \mathfrak{M}$ is a order ideal of the poset $(\mathfrak{M}, \leq_{str})$ we say that B is dually strongly stable*.*

Now we can restate some facts using our new terminology.

Proposition 3.15. *The set-theoretical complement of a strongly stable subset is dually strongly stable. Further a dually strongly stable subset becomes strongly stable and vice versa using the index-bijection $i \mapsto n - i + 1$. Thus there exists two natural bijections between strongly stable subsets and dually strongly stable subsets.*

Proof. The first assertion is just a restatment of Proposition 1.17. For the other assertions we note that with the given index-bijection one gets that if $i, j \in \{1, \dots, n\}$ such that $i < j$ then $n - i + 1 > n - j + 1$ and the claim follows from the definition.
To see that both these operations induces bijections, one notes that taking

complements is a bijective operation and so is a bijective change of variables, since the variables uniquely define the monomials. □

3.4. Classifying equality in the monomial case of Greens theorem

Since in the reduced form Green's hyperplane restriction theorem revolves around strongly stable subsets, and we've seen that the lex-segments always achieve equality in Green's theorem it's interesting to investigate which strongly stable subsets share the properties of the lex-segments that make them attain equality, namely the cardinality and the number of monomials not divisible by x_n. Therefore it's natural to make the following definition

Definition 3.16. Let d be a positive integer, let $\mathfrak{M}_d$ denote the set of monomials of degree d in $\mathbb{K}[x_1, \dots, x_n]$, let $\Lambda \subseteq \mathfrak{M}_d$ be a lex-segment and let $\mathbb{B}$ be the set of all strongly stable subsets of $\mathfrak{M}_d$. Then we define

$$\mathbb{B}(\Lambda) = \{B \in \mathbb{B} : |\Lambda| = |B|, \div_n(B) = \div_n(\Lambda)\}$$

Remark 3.17. We see that if $B \in \mathbb{B}(\Lambda)$ then B attains equality in Theorem 3.9 by Proposition 3.12.
Further we also note that $x_1^d \in \Lambda$ since x_1^d is the maximal monomial of $\mathfrak{M}_d$ with respect to the lexigraphical order and $x_1^d \in B$ since if $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is any element in B, then since B is strongly stable we have that $x_1^{a_1+a_2+\cdots+a_n}$ is in B. But $a_1 + a_2 + \cdots + a_n = d$ and thus $x_1^d \in B$.

We now classify all such sets belonging to $\mathbb{B}(\Lambda)$ for small values on n.

The case n=1,2.

Proposition 3.18. *Let d be a positive integer, let $\mathfrak{M}_d$ denote the set of monomials of degree d in $\mathbb{K}[x_1, \dots, x_n]$, where n is either 1 or 2. Then a subset is strongly stable if and only if it is a Lex-segment.*
Thus if $\Lambda \subseteq \mathfrak{M}_d$ is a lex-segment we have $|\mathbb{B}(\Lambda)| = 1$. Further we have that $\div_n(\Lambda) = 0$ if $n = 1$ and $\div_n(\Lambda) = 1$ if $n = 2$.

Proof. We have by Proposition 1.16 and Proposition 1.43 that the orders $\leq_{\text{lex}}$ and $\leq_{\text{str}}$ coincide on $\mathfrak{M}_d$. Thus both orders are total and since $\mathfrak{M}_d$ is finite we have by Corollary 1.6 that $|B| = |\Lambda|$ implies $B = \Lambda$.
To see that $\div_n(\Lambda) = 0$ if $n = 1$ one notes that in this case we have $\mathfrak{M}_d = \{x_1^d\}$. To see that $\div_n(\Lambda) = 1$ if $n = 2$ one combines Remark 3.17 with the fact that in $\mathfrak{M}_d$ x_1^d is the only monomial not divisible by x_2. □

It seems like the cases $n = 1, 2$ were very simple ones. This is due to the fact that the lexicographical order and the strongly stable order coincided there. Unfortunatly that is not the case when $n > 2$.

The case n=3. By Example 1.15 we see that that the lexicographical order and the strongly stable order no longer coincide. Thus we need to look closer on how the strongly stable subsets look when $n = 3$.
By Proposition 1.5 we have that a strongly stable subset is completely defined by its minimal elements. It turns out that in $\mathbb{K}[x_1, x_2, x_3]$ we can in a very neat way express what these minimal elements look like

Proposition 3.19. *Let d be a positive integer, let $\mathfrak{M}_d$ denote the set of monomials of degree d in $\mathbb{K}[x_1, x_2, x_3]$ and let B be a strongly stable subset of $\mathfrak{M}_d$. Then there are two sequences of positive integers $\{\alpha_i\}_{i=1}^r$ and $\{\beta_i\}_{i=1}^r$ that satisfies*

(i) $0 \leq \alpha_r < \alpha_{r-1} < \cdots < \alpha_1 \leq \beta_1 < \beta_2 < \cdots < \beta_r \leq d$

(ii) $\div_3(B) = \beta_r + 1$

(iii) $|B| = \div_3(B) + \alpha_1(\beta_1 + \frac{1-\alpha_1}{2}) + \sum_{i=2}^{r} \alpha_i(\beta_i - \beta_{i-1})$

(iv) $\{x_1^{d-\beta_1} x_2^{\beta_1-\alpha_1} x_3^{\alpha_1}, \ldots, x_1^{d-\beta_r} x_2^{\beta_r-\alpha_r} x_3^{\alpha_r}\}$ *are the defining minimal elements of B*

Proof. We start by choosing our α_1 so that it's the maximal exponent of x_3 for any monomial in B. Thus

$$\alpha_1 = \max\{\alpha \in \mathbb{N} : x_3^{\alpha} | m \text{ for some } m \in B\}$$

We choose our β_1 in a corresponding way but with the exponent of x_3 fixed to α_1

$$\beta_1 = \max\{\beta \in \mathbb{N} : x_2^{\beta-\alpha_1} x_3^{\alpha_1} | m \text{ for some } m \in B\}$$

Now we define

$$B_1 = \{m \in B : m \leq_{\text{str}} x_1^{d-\beta_1} x_2^{\beta_1-\alpha_1} x_3^{\alpha_1}\} = \tag{3.1}$$

$$= \{x_1^{d-i} x_2^{i-\alpha_1+j} x_3^{\alpha_1-j} \in B : (\alpha_1 - j) \leq i \leq \beta_1, 0 \leq j \leq \alpha_1,\, i, j \in \mathbb{N}\} \tag{3.2}$$

We now have by construction

$$\div_3 (B_1) = |\{i \in \mathbb{N} : x_1^{d-i} x_2^{i} \in B_1\}| = \beta_1 + 1$$

Keeping j fixed in the definition of B_1 we see that there are $\beta_1 - \alpha_1 + j + 1$ different integer choices for i. Thus we get

$$|B_1| = \sum_{j=0}^{\alpha_1} \beta_1 - \alpha_1 + j + 1 = \alpha_1(\beta_1 - \alpha_1) + \beta_1 + 1 + \sum_{j=0}^{\alpha_1} j =$$

$$= \alpha_1(\beta_1 - \alpha_1) + \beta_1 + 1 + \frac{\alpha_1(1 + \alpha_1)}{2} = \div_3(B_1) + \alpha_1(\beta_1 + \frac{1 - \alpha_1}{2})$$

Now since B is strongly stable we must have by construction $B_1 \subseteq B$. Now if $B_1 = B$, β_1 and α_1 satisfies the properties of the proposition, by the above calculations, and we're done. Otherwise we can find a α_2 and β_2 such that

$$\alpha_2 = \max\{\alpha \in \mathbb{N} : x_3^\alpha | m \text{ for some } m \in B \setminus B_1\}$$

and

$$\beta_2 = \max\{\beta \in \mathbb{N} : x_2^{\beta-\alpha_2} x_3^{\alpha_2} | m \text{ for some } m \in B \setminus B_1\}$$

and we define in the same way as before

$$B_2 = \{m \in B \setminus B_1 : m \leq_{\text{str}} x_1^{d-\beta_2} x_2^{\beta_2-\alpha_2} x_3^{\alpha_2}\} =$$
$$= \{x_1^{d-i} x_2^{i-\alpha_2+j} x_3^{\alpha_2-j} \in B \setminus B_1 : (\alpha_2 - j) \leq i \leq \beta_2, 0 \leq j \leq \alpha_2 \text{ for } i, j \in \mathbb{N}\}$$

But since by definition α_1 was the maximum exponent of x_3 and by the maximality of β_1 we have that B_1 contains all monomials divisible by $x_3^{\alpha_1}$. Thus we must have $\alpha_2 < \alpha_1$ and also $\beta_2 > \beta_1$, since otherwise the monomial $x_1^{d-\beta_2} x_2^{\beta_2-\alpha_2} x_3^{\alpha_2}$ would be in B_1 which is a contradiction. Thus $\alpha_2 - j \leq \alpha_2 < \alpha_1 \leq \beta_1$ and we can write

$$B_2 = \{x_1^{d-i} x_2^{i-\alpha_2+j} x_3^{\alpha_2-j} \in B : \beta_1 \leq i \leq \beta_2, 0 \leq j \leq \alpha_2 \text{ for } i, j \in \mathbb{N}\}$$

Keeping j fixed in the definition of B_2 we see that there are $\beta_2 - \beta_1$ different integer choices for i. Thus we get

$$|B_2| = \sum_{j=0}^{\alpha_2} \beta_2 - \beta_1 = \alpha_2(\beta_2 - \beta_1) + \beta_2 - \beta_1$$

which gives us

$$\div_3(B_1 \cup B_2) = |\{i \in \mathbb{N} : x_1^{d-i} x_2^i \in B_1 \cup B_2\}| = \beta_2 + 1$$

and finally

$$|B_1 \cup B_2| = \alpha_1(\beta_1 - \alpha_1) + \beta_1 + 1 + \frac{\alpha_1(1+\alpha_1)}{2} + \alpha_2(\beta_2 - \beta_1) + \beta_2 - \beta_1 =$$
$$= \alpha_1(\beta_1 - \alpha_1) + \beta_2 + 1 + \frac{\alpha_1(1+\alpha_1)}{2} + \alpha_2(\beta_2 - \beta_1) =$$
$$= \div_3(B_1 \cup B_2) + \alpha_1(\beta_1 + \frac{1-\alpha_1}{2}) + \alpha_2(\beta_2 - \beta_1)$$

Now since B is strongly stable we must have by construction $B_1 \cup B_2 \subseteq B$. Now if $B_1 \cup B_2 = B$, $\{\beta_1, \beta_2\}$ and $\{\alpha_1, \alpha_2\}$ satisfies the properties of the proposition, by above calculations, and we're done. Otherwise we continue in the exact same way as before and define

$$\alpha_i = \max\{\alpha \in \mathbb{N} : x_3^\alpha | m \text{ for some } m \in B \setminus (\bigcup_{j=1}^{i-1} B_j)\}$$

and

$$\beta_i = \max\{\beta \in \mathbb{N} : x_2^{\beta-\alpha_i} x_3^{\alpha_i} | m \text{ for some } m \in B \setminus (\bigcup_{j=1}^{i-1} B_j)\}$$

and

$$(3.3) \quad B_i = \{x_1^{d-l} x_2^{l-\alpha_i+j} x_3^{\alpha_i-j} \in B : \beta_{i-1} \leq l \leq \beta_i, 0 \leq j \leq \alpha_i \text{ for } l, j \in \mathbb{N}\}$$

Since $|B|$ is finite our process must terminate for some r such that $B = \bigcup_{i=1}^{r} B_r$. Using induction and the same reasoning as before we thus get:

$$\div_3(\Lambda) = \div_3(B) = |\{i \in \mathbb{N} : x_1^{d-i} x_2^{i} \in \bigcup_{i=1}^{r} B_i\}| = \beta_r + 1$$

and

$$|\Lambda| = |B| = \div_3(\Lambda) + \alpha_1(\beta_1 + \frac{1-\alpha_1}{2}) + \sum_{i=2}^{r} \alpha_i(\beta_i - \beta_{i-1})$$

And also $0 \leq \alpha_r < \alpha_{r-1} < \cdots < \alpha_1 \leq \beta_1 < \beta_2 < \cdots < \beta_r \leq d$. The sequences $\{\alpha_i\}_{i=1}^r$ and $\{\beta_i\}_{i=1}^r$ are thus seen to satisfy the properties of the proposition.
The elements $\{x_1^{d-\beta_1} x_2^{\beta_1-\alpha_1} x_3^{\alpha_1}, \ldots, x_1^{d-\beta_r} x_2^{\beta_r-\alpha_r} x_3^{\alpha_r}\}$ are seen to be the minimal elements of B with respect to $\leq_{\text{str}}$ by construction. □

Now we have by Proposition 1.46 that every lex-segment is a strongly stable subset. Thus it's interesting how the minimal elements of a lex-segment with respect to $\leq_{\text{str}}$ look like.

Proposition 3.20. *Let B be a strongly stable subset of $\mathbb{K}[x_1, x_2, x_3]$ and let $\{\beta_i\}_{i=1}^r$ and $\{\alpha_i\}_{i=1}^r$ be it's defining sequences from Lemma 3.19. Then B is a lex-segment if and only if we have one of the following cases:*

1) $r = 1$, $\beta_1 = \alpha_1 + 1$

2) $r = 1$, $\beta_1 = \alpha_1$

3) $r = 2$, $\beta_1 = \alpha_1$, $\beta_2 = \beta_1 + 1$

Proof. In case *1)* we get from (3.1) that

$$B = \{x_1^{d-i} x_2^{i-\alpha_1+j} x_3^{\alpha_1-j} \in B : (\alpha_1 - j) \leq i \leq \alpha_1 + 1, 0 \leq j \leq \alpha_1,\ i, j \in \mathbb{N}\}$$

which is equal to all monomials bigger than $x_1^{d-\alpha_1-1} x_2^1 x_3^{\alpha_1}$ in the lexicographic order and is a lex-segment.
In case *2)* we get from (3.1) that

$$B = \{x_1^{d-i} x_2^{i-\alpha_1+j} x_3^{\alpha_1-j} \in B : (\alpha_1 - j) \leq i \leq \alpha_1, 0 \leq j \leq \alpha_1,\ i, j \in \mathbb{N}\}$$

which is equivalent to all monomials bigger than $x_1^{d-\alpha_1} x_3^{\alpha_1}$ in the lexicographic order and is a lex-segment.

In case *3)* we get from (3.1) and (3.3) and case *2)* that B is the union of all monomials bigger than $x_1^{d-\alpha_1}x_3^{\alpha_1}$ and all monomials smaller than $x_1^{d-\alpha_1}x_3^{\alpha_1}$ but bigger than $x_1^{d-\alpha_1+1}x_2^{\alpha_1+1-\alpha_2}x_3^{\alpha_2}$ in the lexicographic order. This is a lex-segment.
Now assume $r \geq 3$. Then we have $\alpha_2 < \beta_2$ so $x_1^{d-\beta_2}x_3^{\beta_2} \notin B$ but $x_1^{d-\beta_3}x_2^{\beta_3} \in B$. Thus B is not a lex-segment since $x_1^{d-\beta_2}x_3^{\beta_2} >_{\text{lex}} x_1^{d-\beta_3}x_2^{\beta_3}$.
Now assume $r = 2$ and that B is a lex-segment. Then $\beta_1 = \alpha_1$ since otherwise $x_1^{d-\beta_1}x_3^{\beta_1} \notin B$ but $x_1^{d-\beta_2}x_2^{\beta_2} \in B$ and $x_1^{d-\beta_1}x_3^{\beta_1} >_{\text{lex}} x_1^{d-\beta_2}x_2^{\beta_2}$. Further $\beta_2 = \beta_1 + 1$ since otherwise $x_1^{d-\beta_1-1}x_3^{\beta_1+1} \notin B$ but $x_1^{d-\beta_2}x_2^{\beta_2} \in B$ and $x_1^{d-\beta_1-1}x_3^{\beta_1+1} >_{\text{lex}} x_1^{d-\beta_2}x_2^{\beta_2}$.
Now assume $r = 1$ and that $\alpha_1 \leq \beta_1 - 2$. Then $x_1^{d-\beta_1+1}x_3^{\beta_1-1} \notin B$ but $x_1^{d-\beta_1}x_2^{\beta_1-\alpha_1}x_3^{\alpha_1} \in B$ so B is not a Lex-segment since $x_1^{d-\beta_1+1}x_3^{\beta_1-1} >_{\text{lex}} x_1^{d-\beta_1}x_2^{\beta_1-\alpha_1}x_3^{\alpha_1}$. □

Now we're almost ready to classify all elements of $\mathbb{B}(\Lambda)$. But first we need a very beautiful result.

Lemma 3.21. *Let d be a positive integer, let $\mathfrak{M}_d$ denote the set of monomials of degree d in $\mathbb{K}[x_1, x_2, x_3]$. Further let $t \leq d$ be an integer and denote $\mathbb{B}(t)$ the set of all strongly stable subset of $\mathfrak{M}_d$ with cardinality t. Then we have $|\mathbb{B}(t)| = \hat{p}(t)$ where $\hat{p}(t)$ denotes the number of partitions of t into distinct parts as defined in Definition 1.19*

Proof. We prove this by creating a bijection. Let $\hat{P}(t)$ denote the set of all partitions of t into distinct parts and let $\{\lambda_i\}_{i=1}^k \in \hat{P}(t)$ be arbitrary. Now we define the bijection $\varphi : \hat{P}(t) \to \mathbb{B}(t)$ in the following way

$$\varphi(\{\lambda_i\}_{i=1}^k) = \bigcup_{i=1}^{k} \{x_1^{d-j}x_2^{j-i+1}x_3^{i-1} \in \mathfrak{M}_d : i-1 \leq j \leq \lambda_i + i - 1\}$$

To see that $\varphi(\{\lambda_i\}_{i=1}^k) \in \mathbb{B}(t)$ we denote $B(\lambda_i) = \{x_1^{d-j}x_2^{j-i+1}x_3^{i-1} \in \mathfrak{M}_d : i-1 \leq j \leq \lambda_i + i - 1\}$ and note that given $l \in \{1, 2, \dots, k\}$ we have

$$\{m \in \mathfrak{M}_d : m \leq_{\text{str}} x_1^{d-\lambda_l+l-1}x_2^{\lambda_l}x_3^{l-1}\} =$$

$$B(\lambda_l) \cup (\bigcup_{i=1}^{l-1} \{x_1^{d-j}x_2^{j-l+1+i}x_3^{l-1-i} \in \mathfrak{M}_d : l-1-i \leq j \leq \lambda_l + l - 1\})$$

But now for $\varphi(\{\lambda_i\}_{i=1}^k)$ to be strongly stable we must have that $\lambda_l + l - 1 \leq \lambda_{l-1} + l - 2 \leq \lambda_{l-2} + l - 3 \leq \cdots \leq \lambda_1$ so we have that

$$\{x_1^{d-j}x_2^{j-l+1+i}x_3^{l-1-i} \in \mathfrak{M}_d : l-1-i \leq j \leq \lambda_l + l - 1\}) \subseteq B(\lambda_{l-i})$$

But since we have $\lambda_l < \lambda_{l-1} < \lambda_{l-2} < \cdots < \lambda_1$ this is the case and thus we get that

$$\{m \in \mathfrak{M}_d : m \leq_{\text{str}} x_1^{d-\lambda_l+l-1} x_2^{\lambda_l} x_3^{l-1}\} \subseteq \bigcup_{i=1}^{l} B(\lambda_i) \subseteq \bigcup_{i=1}^{k} B(\lambda_i) = \varphi(\{\lambda_i\}_{i=1}^k)$$

So $\varphi(\{\lambda_i\}_{i=1}^k)$ is strongly stable. Further

$$|\varphi(\{\lambda_i\}_{i=1}^k)| = \sum_{i=1}^{k} |\{x_1^{d-j} x_2^{j-i+1} x_3^{i-1} \in \mathfrak{M}_d : i-1 \leq j \leq \lambda_i+i-1\}| = \sum_{i=1}^{k} \lambda_i = t$$

so $\varphi(\{\lambda_i\}_{i=1}^k) \in \mathbb{B}(t)$. Now we only need to prove that φ is a bijection. It's easy to see that φ is injective since the sets $\{x_1^{d-j} x_2^{j-i+1} x_3^{i-1} \in \mathfrak{M}_d : i-1 \leq j \leq \lambda_i + i - 1\}$ are unique given λ_i. To see that it's surjective let B be a strongly stable subset. Then we can find a sequence of positive integers, $\{a_i\}_{i=1}^r$ such that

$$B = \bigcup_{i=1}^{r} \{x_1^{d-j} x_2^{j-i+1} x_3^{i-1} \in \mathfrak{M}_d : i-1 \leq j \leq a_i + i - 1\}$$

but since for given $l \in \{1, 2, \ldots, k\}$ we demand that $\{m \in \mathfrak{M}_d : m \leq_{\text{str}} x_1^{d-a_l+l-1} x_2^{a_l} x_3^{l-1}\} \subset B$ we must have that by the same reasoning as above that $a_l + l - 1 \leq a_{l-1} + l - 2 \leq a_{l-2} + l - 3 \leq \cdots \leq a_1$ and thus $a_l < a_{l-1} < a_{l-2} < \cdots < a_1$ and we see that we get $\varphi^{-1}(B) = \{a_i\}_{i=1}^r$ so φ is surjective and thus bijective. □

Remark 3.22. A alternative proof of Lemma 3.21 is given by Snellman in [**21**, Proposition 7.6]. The proof by Snellman was discovered after our proof was written. The proof by Snellman uses a clever trick with recursion and the proof thus follows by the simplicity of strongly stable subset in two variables. Our proof could be considered a more straight forward approach to the problem. The interested reader is encouraged to read the interesting article [**21**].

Proposition 3.23. *Let d be a positive integer, let $\mathfrak{M}_d$ denote the set of monomials of degree d in $\mathbb{K}[x_1, x_2, x_3]$, and let $\Lambda \subseteq \mathfrak{M}_d$ be a lex-segment. Further let $x_1^{d-s} x_2^{s-t} x_3^t$ be the minimal element of Λ. Then $\div_3(\Lambda) = s + 1$ and we have a bijection between $\mathbb{B}(\Lambda)$ and the number of partitions of $s - t$ into distinct parts. Thus $|\mathbb{B}(\Lambda)| = \hat{p}(s - t)$ where $\hat{p}(s - t)$ is defined as in Definition 1.19.*

Proof. If $x_1^{d-s} x_2^{s-t} x_3^t$ is the minimal element of Λ then since Λ is a lex-segment we have that $s = \max\{a \in \mathbb{N} : x_1^{d-a} x_2^a | m \text{ for some } m \in \Lambda\}$, since if $x_1^{d-s-1} x_2^{s+1} | m$ for some m then $x_1^{d-s} x_2^{s-t} x_3^t >_{\text{lex}} m$ and $m \notin \Lambda$. By Proposition 3.19 and Proposition 3.20 we thus get that $\div_3(\Lambda) = s + 1$.

Let B be a arbitrary element in $\mathbb{B}(\Lambda)$ and B^C it's set-theoretical complement. Further let $\mathbb{B}(s-t)$ be defined as in Lemma 3.21.
By Proposition 3.19 and Definition 3.16 we have for our strongly stable subset B we must have $\beta_r = s-1$, where $\{\beta_i\}_{i=0}^r$ is the sequence defined in Proposition 3.19. By the definition of β_r (see proof of Proposition 3.19) we thus get that all elements with the exponent of x_1 being larger than $d-s$ is in B^C. Now by Proposition 3.15 B^C is a dually strongly stable subset. Now we denote $\Lambda_0 = \{x_1^{a_1}x_2^{a_2}x_3^{a_3} \in \mathfrak{M}_d : a_1 > d-s\}$, then we can construct the map $\psi : \mathfrak{M}_d \setminus \Lambda_0 \to \mathfrak{M}_d$

$$\psi(x_1^{a_1}x_2^{a_2}x_3^{a_3}) \mapsto x_1^{a_1-d+s}x_2^{a_2}x_3^{a_3+d-s}$$

Since we demand that $x_1^{a_1}x_2^{a_2}x_3^{a_3} \in \mathfrak{M}_d \setminus \Lambda_0$ we have that $a_1 \leq d-s$. Thus $a_1 - d + s \geq 0$ and the map is well defined.
Now we claim that $\psi(B^C \setminus \Lambda_0)$ is dually strongly stable. Let $\psi(m) \in B^C \setminus \Lambda_0$ and assume we can find indices $i, j \in \{1, \ldots, n\}$, such that $x_j | \psi(m)$, $i > j$. Then $x_j | m$ since $\psi(\frac{m}{x_j}) = \frac{\psi(m)}{x_j}$. But then since B^C is dually strongly stable and since $\psi(m)\frac{x_i}{x_j} = \psi(m\frac{x_i}{x_j})$ we have $m\frac{x_i}{x_j} \in B^C \setminus \Lambda_0$. Thus $\psi(m)\frac{x_i}{x_j} \in \psi(B^C \setminus \Lambda_0)$ and $\psi(B^C \setminus \Lambda_0)$ is dually strongly stable.
We construct the map $\varphi : \mathbb{B}(\Lambda) \to \mathbb{B}(s-t)$

$$\varphi(B) = \{x_1^{a_3+d-s}x_2^{a_2}x_3^{a_1-d+s} \in \mathfrak{M}_d : x_1^{a_1}x_2^{a_2}x_3^{a_3} \in B^C \setminus \Lambda_0\}$$

We want to show that φ is a bijection. Now if we denote γ for the index-bijection of the variables mentioned in Proposition 3.15 then φ can be seen as the composition

$$\varphi(B) = \gamma \circ \psi((B^C) \setminus \Lambda_0)$$

Now Proposition 3.15 combined with the fact that $\psi(B^C \setminus \Lambda_0)$ was dually strongly stable gives us that $\varphi(B)$ is strongly stable.
Next we show that φ is injective. By earlier arguments all elements with the exponent of x_1 being larger than $d-s$ is in B^C, thus if $B_1, B_2 \in \mathbb{B}(\Lambda)$ are two distinct elements then B_1^C and B_2^C must differ outside Λ_0. This combined with Proposition 3.15 gives us that that the mapping $(B^C) \setminus \Lambda_0$ is injective. Now γ is injective by Proposition 3.15 and ψ is injective since we only preform addition in the exponents which is a injective operation. Thus φ is injective.
To see that $\varphi(B) \in \mathbb{B}(s-t)$ we now note that we have that $|B \setminus \Lambda_0| = |\Lambda \setminus \Lambda_0| = |\{x_1^{d-s}x_2^{s-t-1}x_3^{t+1}, x_1^{d-s}x_2^{s-t-2}x_3^{t+2}, \ldots, x_1^{d-s}x_3^{s}\}| = s-t$. By the injectivity of φ we have $|\varphi(B)| = s-t$ and since $\varphi(B)$ was strongly stable we have $\varphi(B) \in \mathbb{B}(s-t)$.
Now all that is left is to show that φ is surjective. Let $B_t \in \mathbb{B}(s-t)$ be an arbitrary element. Then we have $\varphi^{-1}(B_t) = (\psi^{-1} \circ \gamma^{-1}(B_t) \cup \Lambda_0)^C$ which is a well-defined element of $\mathbb{B}(\Lambda)$ by the injectivity of all operations in the composition.

Finally since φ was a bijection we get from Lemma 3.21 that $|\mathbb{B}(\Lambda)| = |\mathbb{B}(s-t)| = \hat{p}(s-t)$. □

3.5. Summary

We take a moment to summarize the work in the chapter. We have proven the following

Theorem 3.24. *Let I be a homogeneous ideal of $\mathbb{K}[x_1, \ldots, x_n]$ where n is either 1 or 2. Then I achieves equality in Green's theorem in all degrees.*

Proof. By Proposition 3.18, Galligo's Theorem and Proposition 3.8 the generic initial ideal of I is a lex-segment in all degrees. By Proposition 3.12 and Corollary 3.10 this implies that I achieves equality in Green's theorem in all degrees. □

And also the following

Theorem 3.25. *Let I be a homogeneous ideal of $\mathbb{K}[x_1, x_2, x_3]$, $d \in \mathbb{N}$ and $B = \mathrm{gin}_{rlex}(I) \cap \mathfrak{M}_d$. Further let $\Lambda \subseteq \mathfrak{M}_d$ be the lex-segment such that $|\Lambda| = |B|$ and let $x_1^{a_1} x_2^{a_2} x_3^{a_3}$ be the minimal element of Λ.*
Then I achieves equality in Green's theorem in degree d if and only if we can find a partition, $\{\lambda_i\}_{i=1}^k$, of a_2 into distinct parts such that $\varphi^{-1}(\{\lambda_i\}_{i=1}^k) = B$, where φ is the bijection from the proof of Proposition 3.23

Proof. By Proposition 3.23, Galligo's Theorem and Proposition 3.8 the only possible generic initial ideal that achieves equality in Theorem 3.9 in degree d are those in the image of φ^{-1}.
By Proposition 3.12 and Corollary 3.10 the theorem thus follows. □

Chapter 4

Computational Plücker embeddings

Welcome, welcome. The Grass Man is no longer popular, mm.. [...] Don't you wanna watch the Grass Man being eaten by a demon? The cost is only 100 coin.

Man at Carnival-desk
Breath of Fire II, GBA

In Chapter 3 we characterized what possible generic initial ideals there are for a homogeneous ideal that achieves equality in Green's theorem for a given degree. In this chapter we instead try to find the space of all possible homogeneous ideals for a given generic intial ideal that achieves equality in Green's theorem. We will do this by computational means, by computing the Plücker embeddings of the Grassmanian of such spaces in the computer algebra system named CoCoA[**8**].
As earlier we will have that, unless stated otherwise, $\mathbb{K}$ is an arbitrary infinite field and $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ is standard graded. We also have that $\mathfrak{M}$ denotes the set of all monomials in R, and thus $\mathfrak{M}_d$ will denote all monomials of degree d for any $d \in \mathbb{N}$. $\mathbf{GL}_k(\mathbb{K})$ will denote the general linear group of degree k over $\mathbb{K}$ and we also have that a *pivot element of a matrix* is a non-zero entry of the matrix such that all entries in the same column are zero.

4.1. The Grassmannian

We first need to define what the Grassmannian and the Plücker embedding is. The definitions from this section are versions of definitions found in the books [**13**], [**20**] and [**14**].

Definition 4.1. Let V be a vector space. The *Grassmannian* $\mathbf{Gr}(k, V)$ is the set of all k-dimensional subspaces of the vector space V.

Example 4.2. By Definition 1.52 we have that $\mathbf{Gr}(1, \mathbb{K}^{n+1}) = \mathbb{P}^n_{\mathbb{K}}$.

To be able to extract the geometric information from the Grassmannian one uses what is called the *Plücker embedding*.

Theorem 4.3. (The Plücker embedding)
Let V be a vector space over the field $\mathbb{K}$ with dimension d. Then $\mathbf{Gr}(k, V)$ can then be embedded in $\mathbb{P}^{\binom{d}{k}-1}_{\mathbb{K}}$.

Proof. Since the Plücker embedding is vital to the chapter, we're going to outline its construction but leave the details to the interested reader. For the proofs of the theorem see [**20**, pp. 72-73] or [**14**, pp. 209-211] (these proofs are written for the case $\mathbb{K} = \mathbb{C}$, but are vaild for a arbitrary infinite field aswell). Our construction follows the outline of the proof in [**20**].
Let $W \in \mathbf{Gr}(k, V)$ be a k-dimensional subspace, and choose a ordered basis $\{e_j\}_{j=1}^d$ for V. Since W is a k-dimensional vectorsubspace we can find a ordered basis $\{w_i\}_{i=1}^k$ for W. Being a subspace of V we can find coefficients w_{ij} such that $w_i = \sum_{j=1}^d w_{ij} e_j$. Now the resulting matrix (w_{ij}), which will be called the *Plücker matrix of* W, has full rank since its rows are linearly independent by construction. Further we know from linear algebra that two matrices of full rank (w_{ij}) and (v_{ij}) span the same subspace if and only if there exists a matrix $g \in \mathbf{GL}_k(\mathbb{K})$ such that $(w_{ij}) = g \cdot (v_{ij})$. Thus we can identify our subspace W with the equivalence class

$$[(w_{ij})] = \{g \cdot (w_{ij}) : g \in \mathbf{GL}_k(\mathbb{K})\}$$

Since the matrix representation of our subspace must be invariant under matrix-multiplication with $\mathbf{GL}_k(\mathbb{K})$ by above arguments.
Now we denote the $k \times k$ subdeterminant of (w_{ij}) formed by the columns $1 \leq j_1 < \cdots < j_k \leq d$ by $\Delta_{(j_1,\ldots,j_k)}$. These determinants will be called the *Plücker coordinates*. Then the Plücker embedding is given by the map

$$[(w_{ij})] \mapsto [\Delta_{(1,\ldots,k)} : \cdots : \Delta_{(j_1,\ldots,j_k)} : \cdots : \Delta_{(d-k+1,\ldots,d)}] \in \mathbb{P}^{\binom{d}{k}-1}_{\mathbb{K}}$$

This map is well-defined since we have for any $g \in \mathbf{GL}_k(\mathbb{K})$

$$[(w_{ij})] = [g \cdot (w_{ij})] \mapsto$$
$$\mapsto [\det g \cdot \Delta_{(1,\dots,k)} : \dots : \det g \cdot \Delta_{(j_1,\dots,j_k)} : \dots : \det g \cdot \Delta_{(d-k+1,\dots,d)}] =$$
$$= [\Delta_{(1,\dots,k)} : \dots : \Delta_{(j_1,\dots,j_k)} : \dots : \Delta_{(d-k+1,\dots,d)}]$$

Thus since W could be identified with the equivalence class $[(w_{ij})]$ we have that W can be imbedded in $\mathbb{P}_{\mathbb{K}}^{\binom{d}{k}-1}$ via the Plücker embedding. And since W was arbitrary the whole of $\mathbf{Gr}(k, V)$ can be embedded. □

But one can prove an even stronger result than this namely that $\mathbf{Gr}(k, V)$ is a projective variety.

Theorem 4.4. *Let V be a vector space over the field $\mathbb{K}$ with dimension d. Then $\mathbf{Gr}(k, V) \subset \mathbb{P}_{\mathbb{K}}^{\binom{d}{k}-1}$ is a projective variety. The homogeneous polynomials defining $\mathbf{Gr}(k, V)$ as a projective variety are called the* Plücker relations

Proof. See [**14**, pp. 209-211] (This proof is written for the case $\mathbb{K} = \mathbb{C}$, but is vaild for a arbitrary infinite field aswell). □

Remark 4.5. A rigorous treatment of the Plücker relations would be too lengthy for this paper, but since we will be working with computational Plücker embeddings we can compute the relations in CoCoA. The complete code can be seen in Appendix A, but the part that will compute the Plücker relations is the following part.

```
Wmat := MakeMatByRows(SubDimm,Dimm,b[1,1]..b[SubDimm,Dimm]);
ChoiceOfCol := Subsets(1..Dimm,SubDimm);
PluckGen := [];
DetGen :=[];
For J := 1 To PluckDimm Do
  Append(DetGen,Det(Submat(Wmat,1..SubDimm,ChoiceOfCol[J])));
  Append(PluckGen,p[J] - Det(Submat(Wmat,1..SubDimm,ChoiceOfCol[J])))↘
  →;
EndFor;

PluckerRel := Gens(Elim5(b[1,1]..b[SubDimm,Dimm],Ideal(PluckGen)));
```

Basically what the code does is that it constructs all possible subdeterminants and identify each of them with a Plücker coordinate, `p[J]`. Then it tells the computer to compute the relations between the Plücker coordinates without any reference to the subdeterminants. The result is the Plücker relations.

We summarize the terminology from the Plücker embedding

Definition 4.6. If $W \subseteq V$ is a vectorsubspace, $\{e_j\}_{j=1}^d$ is an ordered basis for V, $\{w_i\}_{i=1}^k$ is a ordered basis for W and $w_i = \sum_{j=1}^d w_{ij} e_j$ then the

$k \times d$ matrix $P_W = (w_{ij})$ is called the *Plücker matrix of W with respect to the basis* $\{w_i\}_{i=1}^k$. The determinant of the $k \times k$ minor of P_W taking the columns indexed by $1 \le j_1 < \cdots < j_k \le d$ is denoted $\Delta_{(j_1,\ldots,j_k)}(P_W)$ and called a *Plücker coordinate of W*. When its clear which Plücker matrix we use for our Plücker coordinates we will simply write $\Delta_{(j_1,\ldots,j_k)}$.

4.2. Generic initial ideal space

Now when we are aquainted with the Grassmannian and the Plücker embedding we can use it to create a representation of the space of our interest, namely the space of all possible homogeneous ideals with a given generic initial ideal. We will use the reverse lexicographical order when computing generic initial ideals, since the generic initial ideals with respect to the reverse lexicographical order are well behaved as we saw in Chapter 3. Thus when we write $\text{gin}(I)$ for a homogeneous ideal $I \subset R$ it will be understood that we mean the generic initial ideal with respect to the reverse lexicographical order, $\text{gin}_{\text{rlex}}(I)$.
Before continuing let's define our object of interest for this chapter

Definition 4.7. Let d be a positive integer and let $\mathfrak{g}$ be a strongly stable subset of $\mathfrak{M}_d$. Then we define

$$\star(\mathfrak{g}) = \{I \subset R : I \text{ is a homogeneous ideal and } \text{gin}(I_d) \cap \mathfrak{M}_d = \mathfrak{g}\}$$

Now if we choose $\mathfrak{g}$ such that there exist a lex-segment $\Lambda \subset \mathfrak{M}_d$ such that $\mathfrak{g} \in \mathbb{B}(\Lambda)$, where $\mathbb{B}(\Lambda)$ is defined as in Definition 3.16, then we know by Proposition 3.12 and Corollary 3.10 that for any $I \in \star(\mathfrak{g})$ we have that I achieves equality in Green's theorem in degree d. And this is exactly what we are looking for, especially when we have already studied how $\mathbb{B}(\Lambda)$ looks like in Chapter 3.
We want a way to represent our space $\star(\mathfrak{g})$ and we will use the Plücker embedding for this purpose. First we examine how the initial ideal affects the Plücker embedding.

Proposition 4.8. *Let d be a positive integer, $\mathfrak{g}$ be a subset of $\mathfrak{M}_d = \{e_1, \ldots, e_{d_0}\}$ where we have $d_0 = |\mathfrak{M}_d| = \binom{n+d-1}{d}$ and $e_1 >_{rlex} \cdots >_{rlex} e_{d_0}$. Further let I be a homogeneous ideal of R.*
Then if we have $\text{in}_{rlex}(I_d) \cap \mathfrak{M}_d = \mathfrak{g}$ there must exist a Plücker matrix of I_d (in reduced row echelon form) with pivot elements in the columns given by the representation of $\mathfrak{g}$ in the basis $\{e_i\}_{i=1}^{d_0}$.

Proof. Let $\{w_i\}_{i=1}^k$ be a basis for the $\mathbb{K}$ vector space I_d. We have by construction that $\{e_1, \ldots, e_{d_0}\}$ is a basis for the $\mathbb{K}$ vectorspace R_d.
Let $\text{in}_{\text{rlex}}(I_d) \cap \mathfrak{M}_d = \mathfrak{g}$. Then for our Plücker matrix of I_d with respect to the basis $\{w_i\}_{i=1}^k$ we will have for row i that there exists a minimal j_0

such that $e_{j_0} = s \cdot w_{ij_0}$ for some $s \in \mathbb{K}$. Then we will have that $w_{ij} = 0$ for all $j < j_0$. Using Gauss-Jordan elimination one gets a pivot in the column j_0. Continuing this way for every basis element with a unique initial term will give a pivot in the columns given by the initials terms of the basis $\{w_i\}_{i=1}^k$. These terms obviously belongs to $\mathrm{in}_{\mathrm{rlex}}(I_d)$ and thus must belong to $\mathfrak{g}$. Now if $|\{\mathrm{in}_{\mathrm{rlex}}(w_1), \dots, \mathrm{in}_{\mathrm{rlex}}(w_k)\}| \neq |\mathfrak{g}|$ then since the initial ideal and the ideal share the same dimension in every degree by Proposition 2.2 we must have some monomial in $\mathfrak{g}$ that is acquired as the initial term of a polynomial created via addition and subtraction of basis elements. Thus via Gauss-Jordan elimination we can acquire the pivot representing this monomial, since Gauss-Jordan eliminiation is just addition and subtraction of the rows representing the basis polynomials. Continuing with the Gauss-Jordan elimination we finally get a pivot for every element of $\mathfrak{g}$.

□

Example 4.9. Let I be a homogeneous ideal of $R = \mathbb{C}[x_1, x_2, x_3]$ with $I_2 = \langle 3x_1^2 + x_1x_2, x_1x_2 + 9x_3^2, 2x_1x_3 + 14x_2x_3 \rangle$. Our Plücker matrix with respect to this basis becomes

$$P_{I_2} = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 2 & 14 & 0 \end{pmatrix}$$

Notice how the columns are ordered by the reverse lexicographical order $x_1^2 >_{\mathrm{rlex}} x_1x_2 >_{\mathrm{rlex}} x_2^2 >_{\mathrm{rlex}} x_1x_3 >_{\mathrm{rlex}} x_2x_3 >_{\mathrm{rlex}} x_3^2$.
We have that $\mathrm{in}_{\mathrm{rlex}}(I_2) \cap \mathfrak{M}_d = \langle x_1^2, x_1x_2, x_1x_3 \rangle \cap \mathfrak{M}_d = \{x_1^2, x_1x_2, x_1x_3\}$. Proposition 4.8 gives us

$$g \cdot P_{I_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 7 & 0 \end{pmatrix}$$

where we have

$$g = \begin{pmatrix} 1/3 & -1/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

From Proposition 4.8 we see that the initial ideal clearly affects the form of the Plücker matrix. In a similiar way the form of a Plücker matrix is affected by the generic initial ideal. For a generic initial ideal we have by Galligo's Theorem that, when using the standard action of matrices (see Definition 3.2), the initial ideal must be constant on a open subset of $\mathbf{GL}_n(\mathbb{K})$. Proposition 4.8 thus suggest that the pivot elements should be constant on the same open subset. Fortunatly we only need to check invariance under a smaller subset of $\mathbf{GL}_n(\mathbb{K})$ namely the *unipotent subgroup*

Definition 4.10. The *unipotent subgroup* of the Borel subgroup $B_n(\mathbb{K})$ is the defined as the subgroup consisting of all lower triangular matrices with ones on the diagonal and will be denoted $\mathcal{U}_n(\mathbb{K})$. More explicitly we have that

$$\mathcal{U}_n(\mathbb{K}) = \{(g_{ij}) \in \mathbf{GL}_n(\mathbb{K}) : g_{ij} = 0 \text{ for } j > i \text{ and } g_{ii} = 1 \text{ for all } i\}$$

The unipotent subgroup behaves nicely as we will se in the following proposition

Proposition 4.11. *$\mathcal{U}_n(\mathbb{K})$ is a irreducible topological space under the (induced) Zariski topology.*

Proof. Use the same arguments as in [**6**, Proposition 5-2.2] combined with the fact that $\mathcal{U}_n(\mathbb{K})$ is generated by the lower elementary matrices. □

We will from here on use the induced (Zariski) topology on $\mathcal{U}_n(\mathbb{K})$ because of this.
Now we ready to prove that it is enough to check invariance under the unipotent subgroup when constructing the generic initial ideal, which will be very useful when computing generic initial ideal later on.

Proposition 4.12. *Let $\leq_P$ be any monomial ordering on $\mathfrak{M}$ and let I be any homogeneous ideal of R. Then there exists an dense open subset of $U \subset \mathcal{U}_n(\mathbb{K})$ where $\text{in}_P(g \cdot I) = \text{gin}_P(I)$ for all $g \in U$.*

Proof. By Galligo's Theorem we have that there exists a Zariski open subset U_0 such that $\text{in}_P(g \cdot I) = \text{gin}_P(I)$ for all $g \in U_0$. We get from [**10**, Theorem 15.18] that this subset meets $\mathcal{U}_n(\mathbb{K})$ nontrivially, thus $U_0 \cap \mathcal{U}_n(\mathbb{K}) \neq \emptyset$. We denote $U = U_0 \cap \mathcal{U}_n(\mathbb{K})$. Now we have that $\text{in}_P(g \cdot I) = \text{gin}_P(I)$ for all $g \in U$. Further U is open in $\mathcal{U}_n(\mathbb{K})$ by definition of the induced topology and since it is nonempty we get from Proposition 1.72 and Proposition 4.11 that it is dense in $\mathcal{U}_n(\mathbb{K})$. □

Now we're ready to see how the generic initial ideal will affect our Plücker matrices. The generic initial ideal has a much more complicated affect on the Plücker matrix than the initial ideal. Thus we will only examine a special case of the generic initial ideals and see the affect on the Plücker matrices in this case. More complex cases can be constructed in almost the same way, but they will left for the curious reader to construct.

Proposition 4.13. *Let d be a positive integer and let $\{e_1, \ldots, e_{d_0}\} = \mathfrak{M}_d$ where we have $d_0 = |\mathfrak{M}_d| = \binom{n+d-1}{d}$ and $e_1 >_{rlex} \cdots >_{rlex} e_{d_0}$. Further let I be a arbitrary homogeneous ideal of R and let $\dim \text{gin}(I_d) = k$.*
Then the condition that $\text{gin}(I_d) \cap \mathfrak{M}_d \neq \{e_1, \ldots, e_k\}$ can be formulated as

the vanishing of linear polynomials in the Plücker coordinates of the Plücker embedding of I_d.

Proof. Since I is supposed to be a arbitrary homogeneous ideal of R we formulate this symbolically by having $I_d = \langle b_{11}e_1 + \cdots + b_{1d_0}e_{d_0}, b_{21}e_1 + \cdots + b_{2d_0}e_{d_0},\ \ldots\ , b_{k1}e_1 + \cdots + b_{kd_0}e_{d_0}\rangle$. We can view the b_{ij}'s as indeterminates, since we later on want to find linear polynomials in the Plücker coordinates. Thus let (b_{ij}) denote our Plücker matrix filled with the indeterminates b_{ij} for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, d_0$.
Now by Proposition 4.12 we have for a "generic" choice of a matrix $g \in \mathfrak{U}_n(\mathbb{K})$ that $\mathrm{in}_{\mathrm{rlex}}(g \cdot I_d) = \mathrm{gin}_{\mathrm{rlex}}(I_d)$. We put

$$g = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ g_{2\,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_{n\,1} & g_{n\,2} & \cdots & 1 \end{pmatrix}$$

for this purpose, and we treat the $g_{i\,j}$'s as indeterminates, since we want g to act as "generic" as possible. The resulting polynomials, $f \in g \cdot I_d$, will be polynomials of the ring $\mathbb{K}[x_1, x_2, \ldots, x_n, g_{2\,1}, g_{3\,1}, \ldots, g_{n\,n-1}]$ (or even more precise of the ring $\mathbb{K}[x_1, x_2, \ldots, x_n, g_{2\,1}, g_{3\,1}, \ldots, g_{n\,n-1}, b_{11}, b_{12}, \ldots, b_{kd_0}]$), but we treat them as polynomials of the ring $\mathbb{K}[x_1, x_2, \ldots, x_n]$, viewing the indeterminates $g_{i\,j}$ (and $b_{i\,j}$) as scalars in $\mathbb{K}$. We denote the resulting basis polynomials by $\{u_i\}_{i=1}^k$.
Constructing the Plücker matrix, P_{gI_d} for $g \cdot I_d$ with respect to the basis $\{u_i\}_{i=1}^k$ we get by Proposition 4.8 that the pivot elements should be given by the representation of $\mathrm{gin}(I_d) \cap \mathfrak{M}_d$ in the basis $\{e_i\}_{i=1}^{d_0}$. Now since we have $\mathrm{gin}(I_d) \cap \mathfrak{M}_d \neq \{e_1, \ldots, e_k\}$ this means that $\Delta_{(1,\ldots,k)}(P_{gI_d}) = 0$. We denote this first determinant of P_{gI_d} by $w = \Delta_{(1,\ldots,k)}(P_{gI_d})$.
Since $w = 0$ for an open, hence dense, subset of $\mathfrak{U}_n(\mathbb{K})$ we know that w must be zero no matter the value of the indeterminants $g_{i\,j}$. Thus the coefficients for monomials constructed by the indeterminants $g_{i\,j}$ must all be zero. But these coefficients are themselves polynomials in the indeterminates $b_{11}, b_{12}, \ldots, b_{kd_0}$ and even further they are homogeneous linear polynomials in the Plücker coordinates. So by setting these coefficients to zero we have formulated the condition that a arbitrary homogeneous ideal I of R will satisfy $\mathrm{gin}(I_d) \cap \mathfrak{M}_d \neq \{e_1, \ldots, e_k\}$ as the vanishing of linear polynomials in the Plücker coordinates of the Plücker embedding of I_d. □

We now have this nice corollary.

Corollary 4.14. *Let d be a positive integer and let $\mathfrak{g}$ be a strongly stable subset of $\mathfrak{M}_d = \{e_1, \dots, e_{d_0}\}$ with $e_1 >_{rlex} \cdots >_{rlex} e_{d_0}$. Then if $\mathfrak{g}$ is the only strongly stable subset of $\mathfrak{M}_d$ that satisfies $|\mathfrak{g}| = k$ and $\mathfrak{g} \neq \{e_1, \dots, e_k\}$ then $\star(\mathfrak{g})$ can be embedded as a projective variety of $\mathbb{P}_{\mathbb{K}}^{\binom{d_0}{k}-1}$.*

Proof. Let I be a arbitrary homogeneous ideal of R with $\dim \operatorname{gin}(I_d) = k$. Since $\operatorname{gin}(I_d) \cap \mathfrak{M}_d \neq \{e_1, \dots, e_k\}$ implies $\operatorname{gin}(I_d) \cap \mathfrak{M}_d = \mathfrak{g}$ we have that the linear polynomials from Proposition 4.13 only vanishes for those ideals I which satisfy $\operatorname{gin}(I_d) \cap \mathfrak{M}_d = \mathfrak{g}$. This is exactly the set $\star(\mathfrak{g})$. Together with the Plücker relations from Theorem 4.4 these linear polynomials makes $\star(\mathfrak{g})$ into a projective variety of $\mathbb{P}_{\mathbb{K}}^{\binom{d_0}{k}-1}$. □

4.3. A worked example

In Corollary 4.14 we showed how one turn a special case of $\star(\mathfrak{g})$ for $\mathfrak{g} \subset \mathfrak{M}_d$ into a projective variety. Since the construction is a bit messy and the proof barely describes the process we take some time to go through a worked example of the construction.

The example we are going to check is the case $\mathfrak{g} = \{x_1^2, x_1x_2, x_1x_3\} \subset \mathfrak{M}_2$ in the ring $\mathbb{K}[x_1, x_2, x_3]$. Using our notation from Corollary 4.14 we have $e_1 = x_1^2$, $e_2 = x_1x_2$, $e_3 = x_2^2$, $e_4 = x_1x_3$, $e_5 = x_2x_3$ and $e_6 = x_3^2$. Thus we see that $\mathfrak{g} \neq \{e_1, e_2, e_3\} = \{x_1^2, x_1x_2, x_2^2\}$. Further $\mathfrak{g}$ is the only strongly stable subset satisfying $|\mathfrak{g}| = 3$ and $\mathfrak{g} \neq \{e_1, e_2, e_3\}$, since the only other strongly stable subset in $\mathfrak{M}_2$ of order 3 is $\{e_1, e_2, e_3\}$. Thus we can use Corollary 4.14.
Now if $I_d = \langle b_{1\,1}e_1 + \cdots + b_{1\,6}e_6, b_{2\,1}e_1 + \cdots + b_{2\,6}e_6, b_{3\,1}e_1 + \cdots + b_{3\,6}e_6\rangle$ we get the Plücker matrix

$$P_{I_d} = \begin{pmatrix} b_{1\,1} & b_{1\,2} & b_{1\,3} & b_{1\,4} & b_{1\,5} & b_{1\,6} \\ b_{2\,1} & b_{2\,2} & b_{2\,3} & b_{2\,4} & b_{2\,5} & b_{2\,6} \\ b_{3\,1} & b_{3\,2} & b_{3\,3} & b_{3\,4} & b_{3\,5} & b_{3\,6} \end{pmatrix}$$

Our "generic" matrix $g \in \mathfrak{U}_3(\mathbb{K})$ becomes

$$g = \begin{pmatrix} 1 & 0 & 0 \\ r & 1 & 0 \\ s & t & 1 \end{pmatrix}$$

We're now interested in the first 3×3 minor of the Plücker matrix $P_{gI_d} = (a_{i\,j})$. This minor will have the following elements
$a_{1\,1} = b_{1\,3}r^2 + b_{1\,5}rs + b_{1\,6}s^2 + b_{1\,2}r + b_{1\,4}s + b_{1\,1}$
$a_{1\,2} = b_{1\,5}rt + 2b_{1\,6}st + 2b_{1\,3}r + b_{1\,5}s + b_{1\,4}t + b_{1\,2}$
$a_{1\,3} = b_{1\,6}t^2 + b_{1\,5}t + b_{1\,3}$
$a_{2\,1} = b_{2\,3}r^2 + b_{2\,5}rs + b_{2\,6}s^2 + b_{2\,2}r + b_{2\,4}s + b_{2\,1}$

$a_{2\,2} = b_{2\,5}rt + 2b_{2\,6}st + 2b_{2\,3}r + b_{2\,5}s + b_{2\,4}t + b_{2\,2}$
$a_{2\,3} = b_{2\,6}t^2 + b_{2\,5}t + b_{2\,3}$
$a_{3\,1} = b_{3\,3}r^2 + b_{3\,5}rs + b_{3\,6}s^2 + b_{3\,2}r + b_{3\,4}s + b_{3\,1}$
$a_{3\,2} = b_{3\,5}rt + 2b_{3\,6}st + 2b_{3\,3}r + b_{3\,5}s + b_{3\,4}t + b_{3\,2}$
$a_{3\,3} = b_{3\,6}t^2 + b_{3\,5}t + b_{3\,3}$
Taking the determinant of this minor results in an enormous polynomial which we only write out the beginning of

$$w = \Delta_{(1,2,3)}(P_{gI_d}) = r^3t^3(-b_{1\,6}b_{2\,5}b_{3\,3} + b_{1\,5}b_{2\,6}b_{3\,3} + b_{1\,6}b_{2\,3}b_{3\,5} - \\ - b_{1\,3}b_{2\,6}b_{3\,5} - b_{1\,5}b_{2\,3}b_{3\,6} + b_{1\,3}b_{2\,5}b_{3\,6}) + r^2st^2(3b_{1\,6}b_{2\,5}b_{3\,3} + \cdots$$

By identifying Plücker coordinates with their respective determinants from the Plücker matrix P_{I_d}, e.g. $\Delta_{(3,5,6)}(P_{I_d}) = \Delta_{(3,5,6)} = -b_{1\,6}b_{2\,5}b_{3\,3} + b_{1\,5}b_{2\,6}b_{3\,3} + b_{1\,6}b_{2\,3}b_{3\,5} - b_{1\,3}b_{2\,6}b_{3\,5} - b_{1\,5}b_{2\,3}b_{3\,6} + b_{1\,3}b_{2\,5}b_{3\,6}$, we can simplify our polynomial

$$w = r^3t^3\Delta_{(3,5,6)} - r^2st^2\Delta_{(3,5,6)} + r^2t^3(\Delta_{(2,5,6)} + \Delta_{(3,4,6)}) + 3rs^2t\Delta_{(3,5,6)} + \\ + r^2t^2(\Delta_{(2,3,6)} + \Delta_{(3,4,5)}) - 2rst^2(\Delta_{(2,5,6)} + \Delta_{(3,4,6)}) + rt^3(\Delta_{(1,5,6)} + \Delta_{(2,4,6)}) - \\ - s^3\Delta_{(3,5,6)} - 2rst(\Delta_{(2,3,6)} + \Delta_{(3,4,5)}) + s^2t(\Delta_{(2,5,6)} + \Delta_{(3,4,6)}) + \\ + rt^2(2\Delta_{(1,3,6)} + \Delta_{(2,4,5)}) - st^2(\Delta_{(1,5,6)} + \Delta_{(2,4,6)}) + t^3\Delta_{(1,4,6)} + \\ + s^2(\Delta_{(2,3,6)} + \Delta_{(3,4,5)}) + rt(\Delta_{(1,3,5)} - \Delta_{(2,3,4)}) - st(2\Delta_{(1,3,6)} + \Delta_{(2,4,5)}) + \\ + t^2(\Delta_{(1,2,6)} + \Delta_{(1,4,5)}) + s(\Delta_{(2,3,4)} - \Delta_{(1,3,5)}) + t(\Delta_{(1,2,5)} - \Delta_{(1,3,4)}) + \Delta_{(1,2,3)}$$

Now since $w = 0$ regardless of the values of r, s and t we get set all the coefficients of monomials in $\mathbb{K}[r, s, t]$ to zero. Removing copies we get the following relations

$$\begin{aligned} &\Delta_{(2,5,6)} + \Delta_{(3,4,6)} = 0 && \Delta_{(2,3,6)} + \Delta_{(3,4,5)} = 0 \\ &\Delta_{(1,5,6)} + \Delta_{(2,4,6)} = 0 && 2\Delta_{(1,3,6)} + \Delta_{(2,4,5)} = 0 \\ &\Delta_{(1,3,5)} - \Delta_{(2,3,4)} = 0 && \Delta_{(1,2,6)} + \Delta_{(1,4,5)} = 0 \\ &\Delta_{(1,2,5)} - \Delta_{(1,3,4)} = 0 && \Delta_{(3,5,6)} = 0 \\ &\Delta_{(1,4,6)} = 0 && \Delta_{(1,2,3)} = 0 \end{aligned}$$

Now these are our homogeneous linear polynomials in the Plücker coordinates that must vanish whenever we have $\text{gin}(I_d) \cap \mathfrak{M}_d = \mathfrak{g}$ for a homogeneous ideal I of $\mathbb{K}[x_1, x_2, x_3]$.
Together with the Plücker relations, which can be computed by the process described in Remark 4.5, these homogeneous polynomials make $\star(\mathfrak{g})$ into a projective variety.

Now we can verify our result in this case by comparing with what we know about this case from Chapter 2. Let $I \in \star(\mathfrak{g})$ be a homogeneous ideal. Since we have that $H(R/I, d) = 6-3 = \binom{3}{2}$ we get that $M_d(H(R/I, d)) = (1)$

Table 1. Computation time for diffrent sizes of Plücker matrices

Plücker matrix dimension	Computation time
3×6	24 sec
3×10	≈ 45 min
4×10	- (stopped manually after 12 hours)

so we can use Theorem 2.14 to conclude that $I_d = \langle L_1, L_2 \rangle_d$ for some pair of linear forms L_1 and L_2.
Now by using our computed projective variety for $\star(\mathfrak{g})$ we
MATS SUPER FÖRKLARING HÄR...

4.4. Computation time

We have seen that we're able to compute a Plücker embedding of our spaces $\star(\mathfrak{g})$ in CoCoA so now we should be able to use the tools in CoCoA to evaluate some new cases of equality in Green's theorem that wasn't explored fully in Chapter 2. Unfortunately for us there was a major drawback with the approach developed in this chapter, namely computation time.

The hope was that the approach would open up for some new interesting results concerning equality in Green's theorem, but already at a testing stage of the computations there we're big issues with the computation time. A lot of smaller problems was possible to fix with clever coding and some help from the people of the CoCoATeam, but in the end there was still big issues with computation time as can be seen in Table 1.

By Table 1 we can see that we were not able to compute the embeddings for Plücker matrices of size larger than 3×10 (the case 3×10 corresponds to the case of $\mathfrak{g} = \{x_1^3, x_1^2 x_2, x_1^2 x_3\} \subset \mathfrak{M}_3 \subset \mathbb{K}[x_1, x_2, x_3]$). The interesting unexplored cases of equality in Green's theorem require Plücker matrix sizes of at least 5×15, thus making our approach computationally impractical.

The explanation to why the computation time grows seemingly uncontrollably is because of the number of Plücker relations which proved to grow extremely quick. This is illustrated in Table 2.
The conclusion thus is that the approach developed in this chapter holds some theoretical merit but is unfortunately computationally impractical when it comes to produce new insights into the subject of when Green's theorem achieves equality.

Table 2. Number of Plücker relations for diffrent sizes of Plücker matrices

Plücker matrix dimension	Number of Plücker relations
2×4	1
3×6	35
3×10	2310
4×10	-

Chapter 5

Future work

The future will be better tomorrow

Dan Quayle

44th Vice President of the United States

The natural questions at this point are: How do we go from here? What can be done to continue the work in the thesis? What could've been done differently?
These are the questions we will answer in this chapter.

5.1. Combinatorics

bajs

5.2. Ideals

Hej

Appendix A

Algorithm code

With computers you can waste time a lot faster
Darryl Mc Cullough

Following is the code used in CoCoA for the Plücker embedding of ideals.

A.1. The code

```
D:=2;
SubDimm:=3; --- Must be smaller than Bin(3+D-1, D)

Dimm:=Bin(3+D-1, D);

Use R1 ::= QQ[x,y,z,c[1..Dimm],b[1..SubDimm,1..Dimm],r,s,t], ↘
→DegRevLex;

KonvertToSupport := [];
For JJ := 1 To SubDimm Do
        For J:= 1 To Dimm Do
                Append(KonvertToSupport,[b[JJ,J],1]);
        EndFor;
EndFor;

Set Indentation;
Use R0 ::= QQ[x,y,z], DegRevLex;

Polka := DensePoly(D);
Md0 := Ideal(Support(Polka));

Use R1;
```

```
Naturlig := RMap(x, y, z);
Md := Image(Md0, Naturlig);
Li := Support(BringIn(Polka));

For J:= 1 To Len(Li) Do
        Insert(Li,J,Li[J]*c[J]);
        Remove(Li,J+1);
EndFor;
Polka2 := Sum(Li);

Eqq := GenRepr(Subst(Polka2, [[x,x], [y,r*x+y], [z,s*x+t*y+z]]), Md);

MatrixList := [];
Radkonvert := [];
For JJ := 1 To SubDimm Do
        Radkonvert := [];
        For J:= 1 To Dimm Do
                Append(Radkonvert,[c[J],b[JJ,J]]);
        EndFor;
        For J:= 1 To Dimm Do
                Append(MatrixList,Subst(Eqq[J], Radkonvert));
        EndFor;
EndFor;

M0 := MakeMatByRows(SubDimm, Dimm, b[1,1]..b[SubDimm,Dimm]);
M := MakeMatByRows(SubDimm, Dimm, MatrixList);
FirstSubMatrix := Submat(M,1..SubDimm,1..SubDimm);
GrassDet := Det(FirstSubMatrix);

CoeffGrass:=[];
DummyR:= Coefficients(GrassDet, r);
For J := 1 To Len(DummyR) Do
        DummyS:= Coefficients(DummyR[J], s);
        For JJ := 1 To Len(DummyS) Do
                DummyT := Coefficients(DummyS[JJ], t);
                For JJJ := 1 To Len(DummyT) Do
                        If DummyT[JJJ] <> 0 Then
                                Append(CoeffGrass,DummyT[JJJ]);
                        EndIf;
                EndFor;
        EndFor;
EndFor;
MinCoeffGrass := Gens(Minimalized(Ideal(CoeffGrass)));
Len(MinCoeffGrass);

PluckDimm := Bin(Dimm, SubDimm);

W := Concat([1,1,1],NewList(PluckDimm,SubDimm),NewList(SubDimm*Dimm↘
→,1));
Use R2 ::= QQ[x,y,z,p[1..PluckDimm],b[1..SubDimm,1..Dimm]],Weights(W)↘
→;
```

```
Wmat := MakeMatByRows(SubDimm,Dimm,b[1,1]..b[SubDimm,Dimm]);
ChoiceOfCol := Subsets(1..Dimm,SubDimm);
PluckGen := [];
DetGen :=[];
For J := 1 To PluckDimm Do
  Append(DetGen,Det(Submat(Wmat,1..SubDimm,ChoiceOfCol[J])));
  Append(PluckGen,p[J] - Det(Submat(Wmat,1..SubDimm,ChoiceOfCol[J]))↘
  →;
EndFor;

CoeffGrass2 := BringIn(MinCoeffGrass);

GrassGenKoord := [];
For J := 1 To Len(CoeffGrass2) Do
  Append(GrassGenKoord,GenRepr(CoeffGrass2[J],Ideal(DetGen)));
EndFor;

GrassGen := Flatten(List(Mat(GrassGenKoord)*Transposed(Mat([ p[1]..p[↘
→PluckDimm] ]))));
PluckerRel := Gens(Elim5(b[1,1]..b[SubDimm,Dimm],Ideal(PluckGen)));

Green := Ideal(ConcatLists([GrassGen,PluckerRel]));
GreenGen := Gens(Green);
GreenGen;
Len(GreenGen);
Len(PluckerRel);

--Slut!
```

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