AN ELEMENTARY, EXPLICIT, PROOF OF THE EXISTENCE OF HILBERT SCHEMES OF POINTS

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Abstract.

There are many beautiful constructions of the Hilbert scheme of closed subschemes of a projective schemes. In many cases where Hilbert schemes are involved, it suffices to know they exist. On the other hand, there are many situations when it is crucial to have an explicit description of the Hilbert schemes. In this article we give a simple construction of Hilbert schemes of points, under general circumstances, that provides such a description. In addition to being useful for computations the construction is short, elementary, and explicit.

Our methods rely on simple algebraic constructions. It gives explicit expressions of members of an affine covering of the Hilbert schemes, involving few variables satisfying natural equations. In particular, it explains the connections with commuting schemes of matrices. We also give some examples showing how our method can be used.

Introduction

There are many beautiful constructions of the Hilbert scheme of closed subschemes of a projective schemes. In many cases where Hilbert schemes are involved, it suffices to know they exist. On the other hand, there are many situations when it is crucial to have an explicit description of the Hilbert schemes. In this article we give a simple construction of Hilbert schemes of points, under general circumstances, that provides such a description. In addition to being useful for computations the construction is short, elementary, and explicit.

The main novelty of this article is the construction of the Hilbert scheme of n points of an affine scheme $\operatorname{Spec}(R)$ over an affine base scheme $\operatorname{Spec}(A)$, where R is any A-algebra. As a consequence of this construction we obtain a description, and easy proof of the existence, of the Hilbert scheme of n points of a family $X \to S$ over an arbitrary base scheme S, where $\mathcal{R}$ is any quasi-coherent graded $\mathcal{O}_S$-algebra. We give explicit expressions for members of an affine covering of the Hilbert schemes, involving few variables satisfying natural equations. This provides a powerful tool for studying the geometric properties of Hilbert schemes of points. In particular we obtain a natural description of an open subset of the *generic* component of the Hilbert schemes, corresponding to n distinct points.

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One of the main features of our construction is that it gives explicit equations defining the Hilbert schemes of n points of $\operatorname{Spec}(R)$ as a closed subscheme of the scheme of commuting $n \times n$-matrices. To illustrate the flexibility of our methods we compute the Hilbert scheme of the scheme $\operatorname{Spec}(S^{-1}A[X])$ over $\operatorname{Spec}(A)$, where S is a multiplicatively closed subscheme of the polynomial ring $A[X]$ in the variable X over A (see [LS], [LST], and [S]).

When S is locally noetherian and $\mathcal{R}$ is locally finitely generated by elements of degree one, our existence result follows from the more general results of Grothendieck [G]. Apparently the first detailed published proof of Grothendiecks result was the one by Altman and Kleiman [AK] valid under general conditions where S is not assumed to be locally noetherian. There are however many other proofs, several of them following the method introduced by Mumford [M] (see [HS], [SE], [St]). The method is made constructive by Haiman and Sturmfels [HS]. Other proofs of the existence of Hilbert schemes, valid under various conditions, can be found in [A], [H], or [N]. However, our methods rely only on simple algebraic constructions and one of the explanations why it is so adapted to computations is that it avoids embeddings into high dimensional grassmannians via Castelnuovo-Mumford regularity.

A proof closer to our point of view is the one by Mark Huibregtse [Hu]. He gives an elementary construction of the Hilbert scheme via open coverings. However, he uses explicit generators of ideals defining the points on the Hilbert scheme, and certain syzygies of these generators to guarantee that the residue ring is free, with a given monomial basis, as a module. Our method completely avoids the choice of particular generators of ideals and the complicated combinatorics of relations between these generators. This is achieved by using fixed free residue modules, and imposing algebra structures on these by using the correspondence between the action of a ring R on an A-module F and the A-algebra homomorphisms from R to the A-module endomorphisms of F.

Our construction is based upon two simple ideas. The first, already mentioned, is to use the well-known description of the R-module structure on an A-module F in terms of A-algebra homomorphisms from R to $\operatorname{End}_A(F)$. This approach has the advantage of removing the annoying ambiguity coming from the different descriptions of an R-module structure on a free A-module F in different bases of F. The second is the observation that the A-module homomorphisms from an A-module M to $\operatorname{End}_A(F)$ correspond to A-algebra homomorphisms from the symmetric algebra $\operatorname{Sym}_A(M \otimes_A \operatorname{End}_A(F)\check{})$ to A. From the latter observation we obtain that the A-algebra homomorphisms from $\operatorname{Sym}_A(M)$ to $\operatorname{End}_A(F)$ correspond to the algebra homomorphisms $H \to A$, where H is the residue of the algebra $\operatorname{Sym}_A(M \otimes_A \operatorname{End}_A(F)\check{})$ by the ideal corresponding to commuting $n \times n$-matrices, and with coefficients in $\operatorname{Sym}_A(M \otimes_A \operatorname{End}_A(F)\check{})$.

1. The basic functors

In this section we define the basic functors used in the remaining part of the article, and we give an explicit relation between A-module homomorphisms $R \to F$ from an A-algebra R to an A-module F, and the A-algebra homomorphisms $R \to \operatorname{End}_A(F)$ to the endomorphisms of F.

1.1 Notation. Let A be a commutative ring with unit, and let F be a free A-module of finite rank with a distinguished element e that is part of a basis of F.

Furthermore we let R be an A-algebra. We write

$$\mathrm{End}_A(F) \xrightarrow{\mathrm{ev}_e} F$$

for the *evaluation map* defined by $\mathrm{ev}_e(u) = u(e)$.

We shall be using several functors from algebras to sets that we shall define next.

1.2 Functors of homomorphisms of modules. For every pair of A-modules M and N we denote by $\mathcal{H}om(M, N)$ the functor from A-algebras to A-modules that on an A-algebra B takes the value $\mathcal{H}om_B(M, N) = \mathrm{Hom}_B(B \otimes_A M, B \otimes_A N)$. When $M = N$ we write $\mathcal{E}nd(M) = \mathcal{H}om(M, M)$. For every A-algebra homomorphism $\varphi : B \to C$ we denote the natural homomorphism by

$$\mathcal{H}om_\varphi : \mathcal{H}om_B(M, N) \to \mathcal{H}om_C(M, N).$$

1.3 Functors of homomorphisms of algebras. Denote by $\mathcal{H}om_{\mathrm{alg}}(R, \mathrm{End}(N))$ the functor from A-algebras to sets that on an A-algebra B takes the value

$$\mathcal{H}om_{B\text{-alg}}(R, \mathrm{End}(N)) = \mathrm{Hom}_{B\text{-alg}}(B \otimes_A R, B \otimes_A \mathrm{End}(N)).$$

Moreover, denote by $\mathcal{H}_R$ the functor from A-algebras to sets that on the A-algebra B takes the value $\mathcal{H}_R(B) = \mathrm{Hom}_{A\text{-alg}}(R, B)$. From the canonical isomorphism $\mathrm{Hom}_{A\text{-alg}}(R, A) \to \mathrm{Hom}_{B\text{-alg}}(B \otimes_A R, B \otimes_A A)$ we obtain a natural isomorphism of functors

$$\mathcal{H}_R \to \mathcal{H}om_{\mathrm{alg}}(R, A).$$

1.4 Subfunctors defined by a unit. Recall that we have a distinguished element e of our free A-module F. For any A-algebra B we let $\mathcal{H}om^e(R, F)$ consist of surjective B-module homomorphisms $u : B \otimes_A R \to B \otimes_A F$ such that the kernel is an ideal in $B \otimes_A R$ and where $u(1_B \otimes 1_R) = 1_B \otimes e$. We have that $\mathcal{H}om^e(R, F)$ is a subfunctor of $\mathcal{H}om(R, F)$.

1.5 Subfunctors defined by sections. For every A-module homomorphism

$$\beta : F \to R \quad \text{such that} \quad \beta(e) = 1_R$$

we denote by $\mathcal{H}om^\beta(R, F)$ the subfunctor of $\mathcal{H}om^e(R, F)$ whose value on an A-algebra B that consists of B-module homomorphisms $u : B \otimes_A R \to B \otimes_A F$ such that the composite homomorphism

$$B \otimes_A F \xrightarrow{\mathrm{id}_B \otimes \beta} B \otimes_A R \xrightarrow{u} B \otimes_A F$$

is the identity.

Denote by $\mathcal{H}om^\beta_{\mathrm{alg}}(R, \mathrm{End}(F))$ the subfunctor of $\mathcal{H}om_{\mathrm{alg}}(R, \mathrm{End}(F))$ that on the A-algebra B consists of the homomorphisms lying in $\mathcal{H}om^\beta_B(R, \mathrm{End}(F))$.

1.6 Remark. We note that for an A-module N there is a natural structure as a left $\mathrm{End}_A(N)$-module on N, and that there is a natural correspondence between R-module structures on N and homomorphisms of A-algebras $R \to \mathrm{End}_A(N)$.

1.7 Proposition. *Let $\mathcal{HE}^e_R$ be the functor from A-algebras to sets whose value $\mathcal{HE}^e_R(B)$ at the A-algebra B consists of the B-algebra homomorphisms $\varphi : R \to \operatorname{End}_B(B \otimes_A F)$ such that the composition of the homomorphisms $B \otimes_A R \xrightarrow{\varphi} \operatorname{End}_B(B \otimes_A F) \xrightarrow{\operatorname{ev}_{1_B \otimes e}} B \otimes_A F$ is surjective.*

There is a natural isomorphism of functors

$$\mathcal{HE}^e_R \to \mathcal{H}om^e(R, F) \tag{1.7.1}$$

that, for every A-algebra B, maps a B-algebra homomorphism $\varphi : B \otimes_A R \to \operatorname{End}(B \otimes_A F)$ to $\operatorname{ev}_{1_B \otimes e} \varphi : B \otimes_A R \to B \otimes_A F$. The inverse of the latter homomorphism maps a B-module homomorphism $u : B \otimes_A R \to B \otimes_A F$ to the B-algebra homomorphism $\varphi : B \otimes_A R \to \operatorname{End}_B(B \otimes_A F)$ that makes $B \otimes_A F$ into the unique B-algebra such that u becomes a B-algebra homomorphism.

The isomorphism (1.7.1) *induces a natural isomorphism of functors*

$$\mathcal{HE}^\beta_R \to \mathcal{H}om^\beta(R, F),$$

where $\mathcal{HE}^\beta_R$ is the subfunctor of $\mathcal{HE}^e_R$ consisting of the elements $\varphi \in \mathcal{HE}_R(B)$ such that the composite $B \otimes_A F \xrightarrow{\operatorname{id}_B \otimes_A \beta} B \otimes_A R \xrightarrow{\varphi} \operatorname{End}_B(B \otimes_A F) \xrightarrow{\operatorname{ev}_{1_B \otimes e}} B \otimes_A F$ is the identity.

Proof. The map (1.7.1) described in the proposition is clearly functorial.

To prove the first part of the proposition it thererfore clearly suffices to show the proposition in the case $A = B$. Let $\varphi : R \to \operatorname{End}_A(F)$ be a homomorphism in $\mathcal{HE}^e_R(A)$ and let $u = \operatorname{ev}_e \varphi : R \to F$. Then u is surjective by definition and $u(1_R) = \operatorname{ev}_e \varphi(1_R) = \operatorname{ev}_e(\operatorname{id}_{\operatorname{End}_A(F)}) = \operatorname{id}_{\operatorname{End}_A(F)}(e) = e$. Moreover, the kernel of ev_e is a left ideal in $\operatorname{End}_A(F)$ and consequently the kernel of u is an ideal in R. We have thus constructed a map from $\mathcal{HE}^e_R(A)$ to $\mathcal{H}om^e_A(R, F)$.

Conversely, let $u : R \to F$ be in $\mathcal{H}om^e_A(R, F)$. Then F has a unique A-algebra structure such that u is a homomorphism of algebras. Let

$$\varphi : R \to \operatorname{End}_A(F)$$

be the homomorphism such that the image φ_f of f is defined by $\varphi_f(x) = u(f)x$, where the product on the right side is multiplication in F with the given algebra structure. It is clear that φ is an A-algebra homomorphism. We have $\operatorname{ev}_e \varphi_f = \varphi_f(e) = u(f)e = u(f)$. Hence $u = \operatorname{ev}_e \varphi$. In particular $\operatorname{ev}_e \varphi$ is surjective so that φ is in $\mathcal{H}^e_A(R, \operatorname{End}(F))$. We have thus constructed a map from $\mathcal{H}om^e_A(R, F)$ to $\mathcal{HE}^e_R(A)$.

It is clear that the two maps that we have constructed are inverses.

The last part of the proposition is also clear.

The following result is often useful to compute representants of functors.

1.8 Proposition. *Let $\varphi : R \to \operatorname{End}_A(F)$ be an A-algebra homomorphism such that $\operatorname{ev}_e \varphi$ is surjective, and give F the corresponding structure as an A-algebra such that $\operatorname{ev}_e \varphi$ is an A-algebra homomorphism. Then an endomorphism $v \in \operatorname{End}_A(F)$ commutes with all elements in $\varphi(R)$ if and only if*

$$v(x) = v(e)x$$

for all $x \in F$. In other words, if

$$\chi_\varphi : F \to \operatorname{End}_A(F)$$

is the A-algebra homomorphism defined by $\chi_\varphi(y)(x) = yx$ for all x and y in F, then $\chi_\varphi(F)$ is the subset of $\operatorname{End}_A(F)$ of elements that commute with all elements in $\varphi(R)$.

Proof. Since φ and $\operatorname{ev}_e \varphi$ both are A-algebra homomorphism we obtain, for all r and r' in R, that $\varphi(rr') = \varphi(r)\varphi(r')$, respectively $\varphi(rr')(e) = \varphi(r)(e)\varphi(r')(e)$, such that

$$\varphi(r)\varphi(r')(e) = \varphi(r)(e)\varphi(r')(e). \tag{1.8.1}$$

Moreover we have, since $\operatorname{ev}_e \varphi$ is surjective by assumption, that for $x \in F$ and $v \in \operatorname{End}_A(F)$ there are elements r_v and r_x in R such that $x = \operatorname{ev}_e \varphi(r_x) = \varphi(r_x)(e)$, respectively that $v(e) = \operatorname{ev}_e \varphi(r_v) = \varphi(r_v)(e)$.

Assume first that $v\varphi(r) = \varphi(r)v$ for all $r \in R$. Then $v(x) = v(\varphi(r_x)(e)) = \varphi(r_x)v(e) = \varphi(r_x)\varphi(r_v)(e)$. It follows from (1.8.1) that $v(x) = \varphi(r_x)(e)\varphi(r_v)(e) = xv(e)$, that we wanted to show.

Assume next that $v(x) = v(e)x$ for all $x \in F$. For all $r \in R$ we obtain $v\varphi(r)(x) = v(e)\varphi(r)(x) = \varphi(r_v)(e)\varphi(r)(\varphi(r_x)(e)) = \varphi(r_v)(e)\varphi(rr_x)(e)$. It follows from (1.8.1) that $v\varphi(r)(x) = \varphi(r_v)\varphi(rr_x)(e) = \varphi(r_v r r_x)(e)$. On the other hand we have $\varphi(r)v(x) = \varphi(r)v(e)x = \varphi(r)(\varphi(r_e)(e)\varphi(r_x)(e))$. It follows from (1.8.1) that $\varphi(r)v(x) = \varphi(r)(\varphi(r_e)\varphi(r_x)(e)) = \varphi(rr_v r_x)(e)$. We have thus shown that $v\varphi(r) = \varphi(r)v$.

The last part of the proposition follows since e is part of a basis for F and thus all the elements in F are on the form $v(e)$ for some $v \in \operatorname{End}_A(F)$.

1.9 Corollary. *Let $R[Z]$ be the polynomial ring in the variable Z over R, and let $\beta : F \to R$ be as in* 1.5*. With the notation of Proposition* 1.7 *we have a natural isomorphism of functors*

$$\mathcal{HE}^\beta_{R[Z]} \to \mathcal{HE}^\beta_R \times \mathcal{H}_{\operatorname{Sym}_A(F\check{})} \tag{1.9.1}$$

that for each A-algebra B maps $\psi : B \otimes_A R[Z] \to \operatorname{End}_B(B \otimes_A F)$ to the pair (φ, χ) where $\varphi = \psi | B \otimes_A R$ and where χ is determined as follows:

Let $z \in B \otimes_A F$ be the element determined by $\psi(Z) = \chi_\varphi(z)$, where χ_φ is the injective B-algebra homomorphism of Proposition 1.8 *for the algebra B. Moreover let $u_z : B \to B \otimes_A F$ be the B-module homomorphism determined by $u_z(1) = z$, and let $u_z\check{} : B \otimes_A F\check{} \to B$ be its dual. Then χ is determined by $\chi(w) = u_z\check{}(1_B \otimes w)$ for all $w \in F\check{}$.*

In particular, if H^β_R represents the functor $\mathcal{HE}^\beta_R$ then $H^\beta_R \otimes_A \operatorname{Sym}_A(F\check{})$ represents the functor $\mathcal{HE}^\beta_{R[Z]}$, and the universal families over H^β_R and $H^\beta_R \otimes_A \operatorname{Sym}_A(F\check{})$ are related via the correspondence (1.9.1)*.*

Proof. It follows from Proposition 1.8 that $\psi : B \otimes_A R[Z] \to \operatorname{End}_B(B \otimes_A F)$ belongs to $\mathcal{HE}^\beta_{R[Z]}(B)$ if and only if $\varphi = \psi | B \otimes_A R$ belongs to $\mathcal{HE}^\beta_R(B)$ and $\psi(Z)$ is contained in the image of the B-algebra homomorphism $\chi_\varphi : B \otimes_A F \to \operatorname{End}_B(B \otimes_A F)$ of Proposition 1.8. The latter condition is the same as saying that $\psi(Z)$ is determined by an element in $B \otimes_A F$. Such an element is defined uniquely by a B-module homomorphism $B \to B \otimes_A F$, or its dual $B \otimes_A F\check{} \to B$ as in the corollary, and the dual corresponds to an A-module homomorphism $F\check{} \to B$. Finally such an A-module homomorphism corresponds to an A-algebra homomorphism $\operatorname{Sym}_A(F\check{}) \to B$.

2. Representing functors of maps to endomorphisms

In this section we show how the A-module homomorphisms $M \to \operatorname{End}_A(F)$ from an A-module M to the endomorphisms of a free A-module F can be described by A-algebra homomorphisms $\operatorname{Sym}_A(M \otimes_A \operatorname{End}_A(F)\check{}) \to A$.

2.1 Notation. Let M be a fixed A-module. We write $N\check{} = \operatorname{Hom}_A(N, A)$ for the dual of an A-module N, and let $\operatorname{Sym}_A(N)$ be the symmetric algebra of N over A. We shall consider N as the submodule of the graded A-algebra $\operatorname{Sym}_A(N)$ consisting of elements of degree one.

2.2 Canonical isomorphisms. Since $\operatorname{End}_A(F)$ is a free A-module of finite rank the *evaluation map*

$$\mathrm{ev} : \operatorname{End}_A(F) \otimes_A \operatorname{End}_A(F)\check{} \to A$$

that maps $u \otimes \varphi$ to $\varphi(u)$ corresponds, by duality, to a natural A-module homomorphism

$$t : A \to \operatorname{End}_A(F)\check{} \otimes_A \operatorname{End}_A(F). \tag{2.2.1}$$

2.3 Lemma. *For every A-algebra B the homomorphism of B-modules*

$$\operatorname{Hom}_A(M \otimes_A \operatorname{End}_A(F)\check{}, B) \to \operatorname{Hom}_A(M, B \otimes_A \operatorname{End}_A(F)) \tag{2.3.1}$$

that maps $u : M \otimes_A \operatorname{End}_A(F)\check{} \to B$ to the composite homomorphism

$$M \xrightarrow{\sim} M \otimes_A A \xrightarrow{\mathrm{id}_M \otimes t} M \otimes_A \operatorname{End}_A(F)\check{} \otimes_A \operatorname{End}_A(F) \xrightarrow{u \otimes \mathrm{id}_{\operatorname{End}_A(F)}} B \otimes_A \operatorname{End}_A(F)$$

is an isomorphism.

Proof. Choosing a basis for $\operatorname{End}(F)$, and the dual basis for $\operatorname{End}(F)\check{}$ it is easy to check that the homomorphism of the lemma has as inverse the isomorphism that maps an A-module homomorphism $v : M \to B \otimes_A \operatorname{End}_A(F)$ to $(\mathrm{id}_B \otimes \mathrm{ev})(v \otimes \mathrm{id}_{\operatorname{End}(F)\check{}})$.

2.4 Universal homomorphism. Recall that we identify $M \otimes_A \operatorname{End}(F)\check{}$ with a submodule of $\operatorname{Sym}_A(M \otimes_A \operatorname{End}_A(F)\check{})$. Hence we obtain from the A-module homomorphism

$$M \otimes_A A \xrightarrow{\mathrm{id}_M \otimes t} M \otimes_A \operatorname{End}_A(F)\check{} \otimes_A \operatorname{End}_A(F). \tag{2.4.1}$$

an A-module homomorphism $M \to \operatorname{Sym}_A(M \otimes_A \operatorname{End}_A(F)\check{}) \otimes_A \operatorname{End}_A(F)$, and consequently a natural homomorphism of $\operatorname{Sym}_A(M \otimes_A \operatorname{End}_A(F)\check{})$-modules

$$\mu : \operatorname{Sym}_A(M \otimes_A \operatorname{End}_A(F)\check{}) \otimes_A M \to \operatorname{Sym}_A(M \otimes_A \operatorname{End}_A(F)\check{}) \otimes_A \operatorname{End}_A(F)$$

uniquely determined by $\mu(1 \otimes x) = x \otimes t(1)$.

2.5 Proposition. *There is a natural isomorphism of functors*

$$\mathcal{H}_{\operatorname{Sym}_A(M \otimes_A \operatorname{End}_A(F)\check{})} \to \mathcal{H}om(M, \operatorname{End}(F)) \tag{2.5.1}$$

such that, for every A-algebra B, a B-algebra homomorphism $\varphi : \operatorname{Sym}_A(M \otimes_A \operatorname{End}_A(F)\check{}) \to B$ is mapped to the B-module homomorphism $\mathcal{H}om_\varphi(\mu) : B \otimes_A M \to B \otimes_A \operatorname{End}_A(F)$.

Proof. We shall show that the homomorphism (2.5.1) is the composite of the following three isomorphisms:

(1) $\mathrm{Hom}_{A\text{-alg}}(\mathrm{Sym}_A(M \otimes_A \mathrm{End}_A(F)\check{}), B) \to \mathrm{Hom}_A(M \otimes_A \mathrm{End}_A(F)\check{}, B)$ that we obtain from the definition of symmetric algebras.
(2) The map (2.3.1).
(3) The canonical isomorphism $\mathrm{Hom}_A(M, B \otimes_A \mathrm{End}_A(F)) \to \mathrm{Hom}_B(B \otimes_A M, B \otimes_A \mathrm{End}_A(F))$.

Let $u = \varphi|(M \otimes_A \mathrm{End}(F)\check{})$. The image of φ by the composite map of the three isomorphisms is the B-module homomorphism determined on the element $1_B \otimes x$ by $(u \otimes \mathrm{id}_{\mathrm{End}_A(F)})(\mathrm{id}_M \otimes t)(x \otimes 1_A)$ for all $x \in M$, and this homomorphism is equal to $\mathcal{H}om_\varphi(\mu)$ since it follows from (2.4.1) that for all $x \in M$ we have $\mathcal{H}om_\varphi(\mu)(1_B \otimes x) = (\varphi \otimes \mathrm{id}_{\mathrm{End}_A(F)})\mu(1_{\mathrm{Sym}_A(M \otimes_A \mathrm{End}_A(F)\check{})} \otimes x) = (\varphi \otimes \mathrm{id}_{\mathrm{End}_A(F)})(x \otimes t(1_A))$.

As the three isomorphisms are functorial in B we have proved the proposition.

3. Representing functors of maps to commuting endomorphisms

In section 2 we gave the connection between A-module homomorphisms $M \to \mathrm{End}_A(F)$ and A-algebra homomorphisms $\mathrm{Sym}_A(M \otimes_A \mathrm{End}_A(F)\check{}) \to A$. Here we show how the A-module homomorphisms $M \to \mathrm{End}_A(F)$ such that the image consists of commuting matrices correspond to A-algebra homomorphisms $H \to A$, where H is a natural residue algebra of $\mathrm{Sym}_A(M \otimes_A \mathrm{End}_A(F)\check{})$. Hence we obtain a correspondence between A-algebra homomorphisms $\mathrm{Sym}_A(M) \to \mathrm{End}_A(F)$ and A-algebra homomorphisms $H \to A$.

3.1 The ideal of zeroes of a homomorphism. Let $u : N \to P$ be a homomorphism of an A-module N into a free A-module P of finite rank. We denote by $\mathfrak{I}_Z(u)$ the ideal in A where u is zero. More precisely, the ideal $\mathfrak{I}_Z(u)$ is the image of the composite homomorphism $N \otimes_A P\check{} \xrightarrow{\mathrm{id}_{P\check{}} \otimes u} P \otimes_A P\check{} \xrightarrow{\mathrm{ev}} A$. Then, for every A-algebra $\varphi : A \to B$, the homomorphism

$$B \otimes_A N \xrightarrow{\mathrm{id}_B \otimes u} B \otimes_A P$$

is zero if and only if $A \xrightarrow{\varphi} B$ factors via the residue map $A \to A/\mathfrak{I}_Z(u)$.

3.2 Definition. Let H be the residue algebra of $\mathrm{Sym}_A(M \otimes_A \mathrm{End}_A(F)\check{})$ modulo the smallest ideal containing the ideals

$$\mathfrak{I}_Z(\mu(1 \otimes x)\mu(1 \otimes y) - \mu(1 \otimes y)\mu(1 \otimes x)) \quad \text{for all} \quad x, y \quad \text{in} \quad M,$$

where μ is the homomorphism of 2.4. Moreover, let

$$\rho_H : \mathrm{Sym}_A(M \otimes_A \mathrm{End}_A(F)\check{}) \to H$$

be the residue class homomorphism. Denote by

$$\mu_H : H \otimes_A \mathrm{Sym}_A(M) \to H \otimes_A \mathrm{End}_A(F)$$

the H-algebra homomorphism uniquely defined by

$$\mu_H(1 \otimes x) = \mathcal{H}om_{\rho_H}(\mu)(1 \otimes x) \quad \text{for all} \quad x \in M.$$

3.3 Proposition. *We have a natural isomorphism of functors*

$$\mathcal{H}_H \to \mathcal{H}om_{\mathrm{alg}}(\mathrm{Sym}_A(M), \mathrm{End}(F))$$

that for every A-algebra B maps an A-algebra homomorphism $\varphi : H \to B$ to the B-algebra homomorphism $\mathcal{H}om_\varphi(\mu_H) : B \otimes_A \mathrm{Sym}_A(M) \to B \otimes_A \mathrm{End}_A(F)$.

Proof. It follows from the definition of H and Proposition 2.5 that there is a bijection between the set $\mathrm{Hom}_{A\text{-alg}}(H, B)$ of A-algebra homomorphisms $\varphi : H \to B$ and the set of B-module homomorphisms $u : B \otimes_A M \to B \otimes_A \mathrm{End}(F)$ such that the elements $u(1 \otimes x)$, for all x in M, commute. Under this bijection φ corresponds to the homomorphism u given by $u(1 \otimes x) = (\mathcal{H}om_{\varphi\rho_H}(\mu))(1 \otimes x) = (\mathcal{H}om_\varphi \mathcal{H}om_{\rho_H}(\mu))(1 \otimes x) = (\mathcal{H}om_\varphi(\mu_H))(1 \otimes x)$.

From the definition of symmetric products it follows that the set of B-module homomorphisms $u : B \otimes_A M \to B \otimes_A \mathrm{End}_A(M)$ such that the elements $u(1 \otimes x)$ commute for all x in M corresponds bijectively to the set of B-algebra homomorphisms $\psi : B \otimes_A \mathrm{Sym}_A(M) \to B \otimes_A \mathrm{End}_A(F)$ such that $\psi|(B \otimes_A M) = u$. We have thus proved that the homomorphism $\mathcal{H}_H(B) \to \mathcal{H}om_{B\text{-alg}}(\mathrm{Sym}_A(M), \mathrm{End}(F))$ described in the proposition is an isomorphism. It is clear from the construction of the homomorphism that it is functorial in B.

4. Sections and closed subschemes

In section 3 we described the connection between A-algebra homomorphisms $\mathrm{Sym}_A(M) \to \mathrm{End}_A(F)$ and A-algebra homomorphisms $H \to A$. Here we show how we, for any residue algebra R of $\mathrm{Sym}_A(M)$, can construct a natural residue algebra H_R of H such there is a similar connection between A-algebra homomorphisms $R \to \mathrm{End}_A(F)$ and A-algebra homomorphisms $H_R \to A$. This we refine further so that we, for every A-module homomorphism $\beta : F \to R$, obtain a correspondence between A-algebra homomorphisms $\varphi : R \to \mathrm{End}_A(F)$ such that $\mathrm{ev}_e \varphi\beta = \mathrm{id}_F$ and A-algebra homomorphisms $H_R^\beta \to A$ from a natural residue algebra H_R^β of H_R.

4.1 Definition. Let $\mathfrak{I}$ be an ideal in $\mathrm{Sym}_A(M)$ and let $\iota : \mathfrak{I} \to \mathrm{Sym}_A(M)$ be the homomorphism given by the inclusion of $\mathfrak{I}$ in $\mathrm{Sym}_A(M)$. Write $R = \mathrm{Sym}_A(M)/\mathfrak{I}$ and denote by v the composite of the H-module homomorphisms

$$H \otimes_A \mathfrak{I} \xrightarrow{\mathrm{id}_H \otimes \iota} H \otimes_A \mathrm{Sym}_A(M) \xrightarrow{\mu_H} H \otimes_A \mathrm{End}_A(F),$$

where H is defined in 3.2. Let H_R be the residue ring of H modulo the ideal $\mathfrak{I}_Z(v)$, of zeroes of v, and let

$$\rho_{H_R} : H \to H_R$$

be the residue homomorphism. Then the composite homomorphism

$$H_R \otimes_A \mathfrak{I} \xrightarrow{\mathrm{id}_{H_R} \otimes \iota} H_R \otimes_A \mathrm{Sym}_A(M) \xrightarrow{\mathcal{H}om_{\rho_{H_R}}(\mu_H)} H_R \otimes_A \mathrm{End}_A(F)$$

is zero and thus induces an H_R-algebra homomorphism

$$\mu_{H_R} : H_R \otimes_A R \to H_R \otimes_A \mathrm{End}_A(F).$$

4.2 Proposition. *We have a natural isomorphism of functors*

$$\mathcal{H}_{H_R} \to \mathcal{H}om_{\text{alg}}(R, \text{End}(F))$$

such that for every A-algebra B the image of an A-algebra homomorphism $\varphi : H_R \to B$ is $\mathcal{H}om_\varphi(\mu_{H_R}) : B \otimes_A R \to B \otimes_A \text{End}_A(F)$.

Proof. By the definition of H_R an A-algebra homomorphism $\psi : H \to B$ factors via the residue homomorphism $\rho_{H_R} : H \to H_R$ if and only if the composite homomorphism

$$B \otimes \mathfrak{I} \xrightarrow{\text{id}_B \otimes \iota} B \otimes_A \text{Sym}_A(M) \xrightarrow{\mathcal{H}om_\psi(\mu_H)} B \otimes_A \text{End}_A(F)$$

is zero, that is, if and only if the homomorphism $\mathcal{H}om_\psi(\mu_H) : B \otimes_A \text{Sym}_A(M) \to B \otimes_A \text{End}_A(F)$ factors via the residue map $B \otimes_A \text{Sym}_A(M) \to B \otimes_A (\text{Sym}_A(M)/\mathfrak{I})$. The proposition thus follows from Proposition 3.3.

4.3 Definition. Recall that F is a free A-module of finite rank with a distinguished element e that is part of a basis. We fix an A-module homomorphism

$$\beta : F \to R \quad \text{such that} \quad \beta(e) = 1_R.$$

Let u be the composite of the H_R-module homomorphisms

$$H_R \otimes_A F \xrightarrow{\text{id}_{H_R} \otimes \beta} H_R \otimes_A R \xrightarrow{\mu_{H_R}} H_R \otimes_A \text{End}_A(F) \xrightarrow{\text{id}_{H_R} \otimes \text{ev}_e} H_R \otimes_A F.$$

We denote by H_R^β the residue ring of H_R modulo the ideal $\mathfrak{I}_Z(\text{id}_{H_R \otimes_A F} - u)$ and let

$$\rho_{H_R^\beta} : H_R \to H_R^\beta$$

be the residue homomorphism. Moreover, we write

$$\mu_{H_R^\beta} = \mathcal{H}om_{\rho_{H_R^\beta}}(\mu_{H_R}) : H_R^\beta \otimes_A R \to H_R^\beta \otimes_A \text{End}_A(F).$$

4.4 Proposition. *We have a natural isomorphism of functors*

$$\mathcal{H}_{H_R^\beta} \to \mathcal{H}om^\beta_{\text{alg}}(R, \text{End}_A(F))$$

that for every A-algebra B maps an A-algebra homomorphism $\varphi : H_R^\beta \to B$ to $\mathcal{H}om_\varphi(\mu_{H_R^\beta}) : B \otimes_A R \to B \otimes_A \text{End}_A(F)$.

Proof. By the definition of H_R^β an A-algebra homomorphism $\psi : H_R \to B$ factors via the residue homomorphism $\rho_{H_R^\beta} : H_R \to H_R^\beta$ in a homomorphism $\varphi : H_R^\beta \to B$ if and only if the composite of the B-module homomorphisms

$$B \otimes_A F \xrightarrow{\text{id}_B \otimes \beta} B \otimes_A R \xrightarrow{\mathcal{H}om_\psi(\mu_{H_R})} B \otimes_A \text{End}_A(F) \xrightarrow{1_B \otimes \text{ev}_e} B \otimes_A F$$

is the identity and $\mathcal{H}om_\psi(\mu_{H_R}) = \mathcal{H}om_{\varphi\rho_{H_R^\beta}}(\mu_{H_R}) = \mathcal{H}om_\varphi(\mu_{H_R^\beta})$. The proof thus follows from Proposition 4.2.

We sum up what we have done in the following result.

4.5 Theorem. *Let F and M be A-modules with F free of finte rank, with a distinguished element e that is part of a basis. Moreover, let $\mathfrak{I}$ be an ideal in $\mathrm{Sym}_A(M)$ and let $R = \mathrm{Sym}_A(M)/\mathfrak{I}$. For all A-module homomorphism $\beta : F \to R$ such that $\beta(e) = 1_R$ we have a natural isomorphism of functors*

$$\mathcal{H}_{H_R^\beta} \to \mathcal{H}om^\beta(R, F)$$

that is determined by mapping $\mathrm{id}_{H_R^\beta}$ to the homomorphism $H_R^\beta \otimes_A R \to H_R^\beta \otimes_A F$ obtained from $(\mathrm{id}_{H_R^\beta} \otimes \mathrm{ev}_e)\mathcal{H}om_\rho(\mu) : H_R^\beta \otimes M \to H_R^\beta \otimes_A F$, where we let $\rho = \rho_{H_R^\beta}\rho_{H^\beta}\rho_H : \mathrm{Sym}_A(M \otimes_A \mathrm{End}_A(F)\check{}) \to H_R^\beta$ denote the residue homomorphism. That is, the functor $\mathcal{H}om^\beta(R, F)$ is represented by the A-algebra H_R^β and the universal family is is the homomorphism $H_R^\beta \otimes_A R \to H_R^\beta \otimes_A F$ obtained from $(\mathrm{id}_{H_R^\beta} \otimes \mathrm{ev}_e)\mathcal{H}om_\rho(\mu) : H_R^\beta \otimes_A M \to H_R^\beta \otimes_A F$.

Proof. It follows from Proposition 1.7 that we have a natural isomorphism of functors $\mathcal{HE}_R^\beta \to \mathcal{H}om^\beta(R, F)$ and from Proposition 4.4 we have a natural isomorphism of functors $\mathcal{H}_{H_R^\beta} \to \mathcal{H}om_{\mathrm{alg}}^\beta(R, \mathrm{End}(F))$. Consequently the theorem follows from the canonical isomorphism of functors $\mathcal{H}om_{\mathrm{alg}}^\beta(R, \mathrm{End}(F)) \to \mathcal{HE}_R^\beta$ that we obtain from the natural isomorphism of B-algebras $B \otimes_A \mathrm{End}_A(F) \to \mathrm{End}_B(B \otimes_A F)$.

5. The Hilbert Functor

In this section we show how our results for the Hilbert scheme of n points in $\mathrm{Spec}(R)$ can be used to obtain the Hilbert scheme of any projective scheme over an arbitrary base.

5.1 The Hilbert functor in the affine case. Let R be an A-algebra. We let $\mathcal{H}ilb_{R/A}^n$ be the functor from A-algebras to sets, that to an A-algebra B associates the set $\mathcal{H}ilb_{R/A}^n(B)$ of surjective B-algebra homomorphism $\varphi : B \otimes_A R \to Q$, where Q is a locally free B-module of rank n. For every A-module homomorphism

$$\beta : F \to R \quad \text{such that} \quad \beta(e) = 1_R$$

we let $\mathcal{H}ilb_{R/A}^\beta$ be the subfunctor of $\mathcal{H}ilb_{R/A}^n$ that to B associates the set $\mathcal{H}ilb_{R/A}^\beta(B)$ of those φ such that

$$B \otimes_A F \xrightarrow{\mathrm{id}_B \otimes_A \beta} B \otimes_A R \xrightarrow{\varphi} Q$$

is surjective, and thus an isomorphism.

5.2 The existence of sections. For every homomorphism $\varphi : B \otimes_A R \to Q$ in $\mathcal{H}ilb_{R/A}^n(B)$ and every maximal ideal in B we can find an element $f \in B$, not in the maximal ideal, and an A-module homomorphism $\beta_Q : F \to R$ with $\beta_Q(e) = 1$, such that the composite homeomorphism

$$B_f \otimes_A F \xrightarrow{\mathrm{id}_{B_f} \otimes_A \beta_Q} B_f \otimes_A R \xrightarrow{\varphi_f} Q_f$$

of B_f-modules is surjective, and thus an isomorphism. Hence the functors $\mathcal{H}ilb_{R/A}^\beta$, for all choises of β, form an open cover of $\mathcal{H}ilb_{R/A}^n$. We can even choose the β to map a fixed basis of F to a given set of generators of R to obtain an open covering, and when $R = A[X_1, X_2, \dots, X_m]$ we can map the basis of F to monomials in $X_1, \dots, X_m$ of degree strictly less than n.

5.3 Lemma. *We have a natural isomorphism of functors*

$$\mathcal{H}om^{\beta}(R,F) \to \mathcal{H}ilb^{\beta}_{R/A}$$

given, for every A-algebra B, by the canonical homomorphism $\mathcal{H}om^{\beta}_B(R,F) \to \mathcal{H}ilb^{\beta}_R(B)$ such that the image of a B-module homomorphism $u : B \otimes_A R \to B \otimes_A F$ is equal to itself when $B \otimes_A F$ is given the unique B-algebra structure such that u becomes a B-algebra homomorphism.

Proof. To see that the morphism given in the lemma is an isomorphism we construct an inverse. For every B-algebra homomorphism $\varphi : B \otimes_A R \to Q$ in $\mathcal{H}ilb^{\beta}_R(B)$ the composite homomorphism $B \otimes_A F \xrightarrow{\mathrm{id}_B \otimes \beta} B \otimes_A R \xrightarrow{\varphi} Q$ is an isomorphism. Via this isomorphism $B \otimes_A F$ obtains a unique B-algebra structure such that $B \otimes_A R \xrightarrow{(\varphi(\mathrm{id}_B \otimes \beta))^{-1}\varphi} B \otimes_A F$ is in $\mathcal{H}om^{\beta}_B(R,F)$. It is clear that this defines an inverse to the map of the lemma.

5.4 Theorem. *Let R be an A-algebra. The functor $\mathcal{H}ilb^{n}_{R/A}$ is representable. More precisely, the functor $\mathcal{H}ilb^{n}_{R/A}$ is covered by the open subfunctors $\mathcal{H}ilb^{\beta}_{R/A}$ for all A-module homomorphism $\beta : F \to R$, and for each β we have a natural isomorphism of functors*

$$\mathcal{H}_{H^{\beta}_R} \to \mathcal{H}ilb^{\beta}_{R/A}$$

that maps $\mathrm{id}_{H^{\beta}_R}$ to $\mu_{H^{\beta}_R} : H^{\beta}_R \otimes_A R \to H^{\beta}_R \otimes_A F$. That is, the functor $\mathcal{H}ilb^{\beta}_{R/A}$ is represented by the A-algebra H^{β}_R and the universal homomorphism is $\mu_{H^{\beta}_R}$.

Proof. The first part of the theorem we observed in Section 5.2, and the isomorphism of the theorem is the composite of the natural isomorphism of functors of Theorem 4.5 and Lemma 5.3.

5.5 The Hilbert functor. Let X be a scheme over a base scheme S and let $p : X \to S$ be the structure homomorphism. For a morphism $T \to S$ from a scheme T we let $p_T : T \times_S X \to T$ denote the projection. The T-points $\mathcal{H}ilb^{n}_{X/S}(T)$ of the *Hilbert functor* $\mathcal{H}ilb^{n}_{X/S}$ of n points of X over S consists of the closed subschemes Z of $T \times_S X$ such that Z is finite over T and $(p_T)_*(\mathcal{O}_Z)$ is a locally free $\mathcal{O}_T$-module of rank n (see [LST]).

5.6 Lemma. *Let R be a graded A-algebra. For every prime ideal $\mathfrak{p}$ of A write $\boldsymbol{\kappa}(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$.*

Let Z be a closed subscheme of $\mathrm{Proj}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R)$ that is finite over $\mathrm{Spec}(\boldsymbol{\kappa}(\mathfrak{p}))$. Then there is an element $a \in A$ not in $\mathfrak{p}$ and an element $f \in R_a$ such that Z is contained in the open subscheme $\mathrm{Spec}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_{A_a} (R_a)_{(f)}) = \mathrm{Spec}(\boldsymbol{\kappa}(\mathfrak{p})) \times_{\mathrm{Spec}(A_a)} \mathrm{Spec}((R_a)_{(f)})$ of $\mathrm{Proj}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_{A_a} R_a) = \mathrm{Spec}(\boldsymbol{\kappa}(\mathfrak{p})) \times_{\mathrm{Spec}(A_a)} \mathrm{Proj}(R_a)$.

Proof. Since Z is finite over $\mathrm{Spec}(\boldsymbol{\kappa}(\mathfrak{p}))$ the fiber of the induced morhpism $Z \to \mathrm{Spec}(\boldsymbol{\kappa}(\mathfrak{p}))$ consists of a finite number of points, corresponding to homogeneous prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_k$ in $\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R$ that do not contain the irrelevant ideal. Their union consequently do not contain the irrelevant ideal. Hence we can find a homogeneous element $g \in \boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R$ of positive degree that is not contained in any of the ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_k$. Thus Z is contained in the open subscheme $\mathrm{Spec}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R)_{(g)})$ of $\mathrm{Proj}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R)$.

Clearly we can find an element $a \in A$ not in $\mathfrak{p}$ and an element $f \in R_a$ such that $1_{\boldsymbol{\kappa}(\mathfrak{p})} \otimes f$ is the image of g by the natural isomorphism $\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R \to \boldsymbol{\kappa}(\mathfrak{p}) \otimes_{A_a} R_a$. However, then $\operatorname{Spec}((\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R)_{(g)}) = \operatorname{Spec}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_{A_a} (R_a)_{(f)})$, and we have proved the lemma.

5.7 Theorem. *Let S be a scheme and $\mathcal{R}$ a quasi-coherent graded $\mathcal{O}_S$-algebra. Then the functor $\mathcal{H}ilb^n_{\operatorname{Proj}(\mathcal{R})/S}$ is representable.*

More precisely, the functor $\mathcal{H}ilb^n_{\operatorname{Proj}(\mathcal{R})/S}$ is covered by the representable open subfunctors $\mathcal{H}ilb^n_{\operatorname{Spec}(R_{(f)})/\operatorname{Spec}(A)}$, where $\operatorname{Spec}(A)$ is an open affine subscheme of S, where $R = \Gamma(\operatorname{Spec}(A), \mathcal{R}|\operatorname{Spec}(A))$, and where f is a homogeneous element of R.

Proof. For every affine open subset U of S we have that $\mathcal{H}ilb^n_{p^{-1}(U)/U}$ is an open subfunctor of $\mathcal{H}ilb^n_{X/S}$ and these subfunctors, for all U in an open covering of S, cover $\mathcal{H}ilb^n_{X/S}$. In order to represent $\mathcal{H}ilb^n_{X/S}$ we can thus assume that $S = \operatorname{Spec}(A)$ is affine. Write $R = \Gamma(\operatorname{Spec}(A), \mathcal{R})$ such that $X = \operatorname{Proj}(R)$. For every $a \in A$ and every $f \in R_a$ the functor $\mathcal{H}ilb^n_{\operatorname{Spec}((R_a)_{(f)})/\operatorname{Spec}(A_a)}$ is an open subfunctor of $\mathcal{H}ilb^n_{X/S}$. It follows from Theorem 5.4 that in order to prove the theorem it suffices to prove that these subfunctors cover the functor $\mathcal{H}ilb^n_{X/S}$. In order to show this it suffices to show that for every prime ideal $\mathfrak{p}$ in A and every closed subscheme Z of $\operatorname{Proj}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R)$ that is finite over $\operatorname{Spec}(\boldsymbol{\kappa}(\mathfrak{p}))$, and with $\Gamma(Z, \mathcal{O}_Z)$ of dimension n over $\boldsymbol{\kappa}(\mathfrak{p})$, there is an $a \in A$, not in $\mathfrak{p}$ and an $f \in R_a$ such that Z is contained in the open subscheme $\operatorname{Spec}(\boldsymbol{\kappa}(\mathfrak{p})) \times_{\operatorname{Spec}(A_a)} \operatorname{Spec}((R_a)_{(f)}) = \operatorname{Spec}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_{A_a} (R_a)_{(f)})$ of $\operatorname{Proj}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_{A_a} R_a)$. It consequently follows from Lemma 5.6 that the functors $\mathcal{H}ilb^n_{\operatorname{Spec}(R_a)_{(f)}/\operatorname{Spec}(A_a)}$ cover $\mathcal{H}ilb^n_{X/S}$, and we have proved the theorem.

6. The Hilbert scheme in coordinates

In this section we express our construction of the Hilbert scheme of n points in $\operatorname{Spec}(R)$ in local coordinates. We obtain explicit expression of an open affine covering of the Hilbert scheme in terms of variables and relations.

6.1 Notation. We choose a basis $e = T_1, T_2, \dots, T_n$ of the A-module F, and let $T_1\check{}, \dots, T_n\check{}$ be the dual basis of $F\check{}$. Correspondingly we obtain a basis T_{ij} for $i, j = 1, \dots, n$ of $\operatorname{End}_A(F)$, and a dual basis $T_{ij}\check{}$ for $\operatorname{End}_A(F)\check{}$. In particular $\mathrm{id}_{\operatorname{End}_A(F)} = \sum_{i=1}^n T_{ii}$ and $\mathrm{id}_{\operatorname{End}_A(F)\check{}} = \sum_{i=1}^n T_{ii}\check{}$. Then

$$T_{ij}(T_k) = \delta_{jk} T_i \quad \text{and} \quad T_{ij}\check{}(T_k\check{}) = \delta_{ik} T_j\check{}.$$

In these bases we have

$$t(1_A) = \sum_{i,j=1}^n T_{ij}\check{} \otimes T_{ij},$$

where t is defined in Section (2.2.1). For every A-algebra B we consider T_{ij} as a basis for the B-module $\operatorname{End}_B(B \otimes_A F)$ and thus identify $B \otimes_A \operatorname{End}_A(F)$ with $\operatorname{End}_B(B \otimes_A F)$ via the homomorphism that maps $1 \otimes T_{ij}$ to T_{ij}. Then we have for all b_{ij} in B that

$$\mathrm{ev}_{1_B \otimes T_1}\Big(\sum_{i,j=1}^n b_{ij} T_{ij}\Big) = (\mathrm{id}_B \otimes \mathrm{ev}_{T_1})\Big(\sum_{i,j=1}^n b_{ij} \otimes T_{ij}\Big) = \sum_{i=1}^n b_{i1}.$$

Assume that M is a free A-module with basis $\{Y_s\}_{s\in S}$ for some index set S. Then $\mathrm{Sym}_A(M\otimes_A \mathrm{End}_A(F)\check{})$ is the polynomial ring over A in the independent variables $Y_s\otimes T_{ij}\check{}$, for $i,j=1,\dots,n$ and $s\in S$. We write $U_{ij}^s = Y_s\otimes T_{ij}\check{}$, and $A[U] = \mathrm{Sym}_A(M\otimes_A \mathrm{End}_A(F)\check{})$. As above we identify $M\otimes_A \mathrm{End}_A(F)\check{}$ with the degree 1 part of $\mathrm{Sym}_A(M\otimes_A \mathrm{End}_A(F)\check{})$, and we shall identify M with a submodule of $M\otimes_A \mathrm{End}_A(F)\check{}$ via the map that takes x to $x\otimes \mathrm{id}_{\mathrm{End}_A(F)\check{}}$. In particular $Y_s = Y_s\otimes \mathrm{id}_{\mathrm{End}_A(F)\check{}} = Y_s\otimes\sum_{i=1}^n T_{ii}\check{} = \sum_{i=1}^n U_{ii}^s$, and $\mathrm{Sym}_A(M) = A[Y]$ is the A-algebra in $A[U]$ generated by the elements Y_s for $s\in S$. We write

$$(U_{ij}^s) = \sum_{i,j=1}^n U_{ij}^s\otimes T_{ij} = \begin{pmatrix} U_{11}^s & \cdots & U_{1n}^s \\ \vdots & \ddots & \vdots \\ U_{n1}^s & \cdots & U_{nn}^s \end{pmatrix}.$$

For every polynomial $f(Y)$ in $A[Y]$ we write $f((U_{ij}^s))$ for the element in $A[U]\otimes_A \mathrm{End}_A(F)$ obtained by substituting the matrix (U_{ij}^s) for the variable Y_s. The $A[U]$-module homomorphism $\mu : A[U]\otimes_A M \to A[U]\otimes_A \mathrm{End}_A(F)$ is determined by

$$\mu(1_{A[U]}\otimes Y_s) = \sum_{i,j=1}^n Y_s\otimes T_{ij}\check{}\otimes T_{ij} = \sum_{i,j=1}^n U_{ij}^s\otimes T_{ij} = (U_{ij}^s).$$

6.2 Coordinates of the representing ring.

(1) Let $\mathfrak{I}_1$ be the ideal in $A[U]$ generated by the coordinates of the matrices

$$(U_{ij}^s)(U_{ij}^t) - (U_{ij}^t)(U_{ij}^s)$$

for all s,t in S. Then $H = A[U]/\mathfrak{I}_1$, and $\mu_H : H\otimes_A A[Y] \to H\otimes_A \mathrm{End}_A(F)$ is determined by

$$\mu_H(1_H\otimes f(Y)) = f((U_{ij}^s))$$

for all $f(Y)$ in $A[Y]$.

(2) Let $\mathfrak{I}$ be an ideal of $A[Y]$ and let $\mathfrak{I}_2$ be the ideal in $A[U]$ generated by the coordinates of the matrices

$$f((U_{ij}^s)) \quad \text{for} \quad f\in\mathfrak{I}.$$

Then $H_{A[Y]} = A[U]/(\mathfrak{I}_1,\mathfrak{I}_2)$.

(3) Let $\beta : F\to A[Y]$ be an A-module homomorphism and write $f_k(Y) = \beta(T_k)$ for $k=1,\dots,n$. Denote by $\mathfrak{I}_3$ the ideal in $A[U]$ generated by the coefficients of $T_1,\dots,T_n$ in

$$\mathrm{ev}_{1_{A[U]}\otimes T_1}(f_k((U_{ij}^s))) - T_k \quad \text{for} \quad k=1,\dots,n.$$

Then $H_{A[Y]}^\beta = A[U]/(\mathfrak{I}_1,\mathfrak{I}_2,\mathfrak{I}_3)$.

7. The generic open subset of the Hilbert scheme of affine space

We use the explicit coordinate description in section 6 of an open covering of the Hilbert scheme to give a simple expression of an open subset of the *generic* component of the Hilbert scheme of n points in $\mathrm{Spec}(A[Y])$, where Y_s for $s \in S$ is a set of independent variables over A.

7.1 Notation. We shall keep the notation of Section 6, and assume in addition that S has a distinguished element s_1. Denote by

$$\beta_1 : F \to A[Y]$$

the A-module homomorphism defined by

$$\beta_1(T_i) = Y_1^{i-1} \quad \text{for} \quad i = 1, \dots, n.$$

We write

$$C_1 = C_1^{s_1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & a_1^{s_1} \\ 1 & 0 & 0 & \cdots & 0 & 0 & a_2^{s_1} \\ 0 & 1 & 0 & \cdots & 0 & 0 & a_3^{s_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & a_n^{s_1} \end{pmatrix},$$

in the basis T_{ij} of $\mathrm{End}_A(F)$. Then

$$C_1^n = a_1^{s_1} I_n + a_2^{s_1} C_1 + \cdots + a_n^{s_1} C_1^{n-1}.$$

Let

$$\varphi_1 : A[Y_1] \to \mathrm{End}_A(F)$$

be the A-algebra homomorphism determined by $\varphi_1(Y_1) = C_1$. Then the composite homomorphism

$$F \xrightarrow{\beta_1} A[Y_1] \xrightarrow{\varphi_1} \mathrm{End}_A(F) \xrightarrow{\mathrm{ev}_e} F$$

is the identity on F.

7.2 Lemma. *Let $\varphi : A[Y] \to \mathrm{End}_A(F)$ be an A-algebra homomorphism. Then the composite homomorphism*

$$F \xrightarrow{\beta_1} A[Y] \xrightarrow{\varphi} \mathrm{End}_A(F) \xrightarrow{\mathrm{ev}_e} F \tag{7.2.1}$$

is the identity on F if and only if φ is defined by

$$\varphi(Y_1) = C_1 \quad \textit{and} \quad \varphi(Y_s) = a_1^s I_n + a_2^s C_1 + \cdots + a_n^s C_1^{n-1}, \tag{7.2.2}$$

for all $s \in S \setminus \{s_1\}$, with $a_1^s, \dots, a_n^s$ in A.

Proof. Assume that φ is of the form (7.2.2). Since $\varphi(Y_1) = C_1$ the composite of the homomorphisms of (7.2.1) is the same as the composite of the homomorphisms of (7.1.1), and thus, as we observed in Section 7.1, equal to the identity of F.

Conversely, assume that the composite of the maps (7.2.1) is the identity on F. Since $\varphi(Y_s)$ commute with $\varphi(Y_1) = C_1$ for all $s \in S$, it follows from Proposition 1.8 that $\varphi(Y_s)$ is determined by $\varphi(Y_s)(T_j) = \varphi(Y_s)(T_1)T_j$ for $j = 1, \dots, n$, where F has the A-algebra structure determined by the surjection $\mathrm{ev}_e \varphi_1 = \mathrm{ev}_e(\varphi | A[Y_1])$. Since φ and $\mathrm{ev}_e \varphi$ are A-algebra homomorphisms we obtain, as in (1.8.1), that $T_i T_j = \varphi(Y_1^{i-1})(T_1)\varphi(Y_1^{j-1})(T_1) = \varphi(Y_1^{i-1})\varphi(Y_1^{j-1})(T_1) = C_1^{i-1}C_1^{j-1}(T_1) = C_1^{i-1}(T_j)$ for $i, j = 1, \dots, n$. Write $\varphi(Y_s)(T_1) = a_1^s T_1 + \cdots + a_1^s T_n$. Then $\varphi(Y_s)(T_1)T_j = (a_1^s T_1 + \cdots + a_n^s T_n)T_j = (a_a^s I_n + a_2^s C_1 + \cdots + a_n^s C_1^{n-1})(T_j)$ for $j = 1, \dots, n$. Hence $\varphi(Y_s) = a_1^s I_n + a_2^s C_1 + \cdots + a_n^s C_1^{n-1}$ as we wanted to prove.

7.3 Proposition. *We have that $H^{\beta_1}_{A[Y]}$ is the polynomial ring over A in the independent variables $U^{s_1}_{1n}, \dots, U^{s_1}_{nn}$ and $U^{s}_{11}, \dots, U^{s}_{n1}$ for all s in S different from s_1. The universal family is given by the ideal in $A[Y] \otimes_A H^{\beta_1}_{A[Y]}$ generated by the elements $Y_1^n - U^{s_1}_{1n} - U^{s_1}_{2n} Y_1 - \dots - U^{s_1}_{nn} Y_1^{n-1}$ and $Y_s - U^{s}_{1n} - U^{s}_{2n} Y_1 - \dots - U^{s}_{nn} Y_1^{n-1}$ for $s \in S \setminus \{s_1\}$.*

Proof. The ideal $\mathfrak{I}_3$ of 6.2(3) is generated by the coefficients of T_k in the polynomials $\mathrm{ev}_{1_{A[U]} \otimes_A T_1}((U^s_{ij})^{k-1}) - T_k$ for $k = 1, \dots, n$. It follows from Lemma 7.2 that, modulo the ideal $\mathfrak{I}_3$, we have that $(U^{s_1}_{ij})$ is congruent to the *companion* matrix

$$C_U = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & U^{s_1}_{1n} \\ 1 & 0 & 0 & \cdots & 0 & 0 & U^{s_1}_{2n} \\ 0 & 1 & 0 & \cdots & 0 & 0 & U^{s_1}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & U^{s_1}_{nn} \end{pmatrix}$$

and that

$$U^s_{ij} \equiv U^s_{11} I_n + U^s_{21} C_U + \dots + U^s_{n1} C_U^{n-1} \pmod{\mathfrak{I}_3}.$$

In particular the matrices (U^s_{ij}) commute modulo $\mathfrak{I}_3$, that is $\mathfrak{I}_1 \subseteq \mathfrak{I}_3$, and since $\mathfrak{I}_2 = 0$ in this case, it follows that $H^{\beta_1}_{A[Y]} = A[U]/\mathfrak{I}_3$.

To prove the last part of the proposition we note that since $I_n, C_U, \dots, C_U^{n-1}$ clearly are linearly independent over $H^{\beta_1}_{A[Y]}$ the $H^{\beta_1}_{A[Y]}$-algebra homomorphism $\varphi_U : A[Y] \otimes_A H^{\beta_1}_{A[Y]} \to \mathrm{End}_{H_{A[Y]}}(H^{\beta_1}_{A[Y]} \otimes_A F)$ defined by $\varphi_U(Y_1) = C_U$ and $\varphi_U(Y_s) = C^s_U = U^s_{1n} I_n + U^s_{2n} C_U + \dots + U^s_{nn} C_U^{n-1}$ for $s \in S \setminus \{s_1\}$ has kernel generated by the elements in the last part of the proposition.

8. The Hilbert scheme of points on a line

To illustrate how easily certain question of Hilbert schemes can be handled by our explicit expressions of open coverings of Hilbsert schemes we describe the Hilbert scheme of n points in $\mathrm{Spec}(S^{-1}A[X])$, where S is a multiplicative set in the polynomial ring $A[X]$ in the variable X over A.

8.1 The open covering. Let $A[X]$ be the polynomial ring in the variable X over A and let S be a multiplicatively closed subset of $A[X]$ containing 1. Moreover, let $\{Y_s\}_{s \in S}$ be a collection of independent variables with $Y_1 = X$. We denote by M the free A-module generated by the elements Y_s for $s \in S$ and, as in 6.2 we write $A[Y] = \mathrm{Sym}_A(M)$. Then there is a surjective A-algebra homomorphism

$$\varphi : A[Y] \to S^{-1}A[X]$$

defined by $\varphi(Y_1) = X$ and $\varphi(Y_s) = 1/s(X)$ for all other s in S. The kernel of φ is generated by the elements

$$s(X)Y_s - 1 \quad \text{for all } s \text{ in } S \text{ different from } 1.$$

Let B be an A-algebra and let

$$\psi : B \otimes_A S^{-1}A[X] \to C$$

be a surjective homomorphism of B-algebras, where C is a free B-module of rank n and write $\psi(1\otimes X)=f$. It follows from the Cayley-Hamilton Theorem that there is a relation $f^n+b_1f^{n-1}+\cdots+b_n=0$ in C, with $b_1,\dots,b_n$ in B. Consequently the image of $B\otimes_A A[X]$ by ψ is in the B-module generated by $1,f,\dots,f^{n-1}$. Let s be in S and write $\psi(1\otimes s)=g$. Then g is invertible in C. Write $\psi(1\otimes s^{-1})=g^{-1}$. Since $\psi(1\otimes s^{-1})$ also satisfies a relations $(g^{-1})^n+a_1(g^{-1})^{n-1}+\cdots+a_n=0$ in C, with $a_1,\dots,a_n$ in B, we obtain that $\psi(1\otimes s^{-1})=g^{-1}=-a_1-\cdots-a_ng^{n-1}$, and thus the image of $1/s(X)$ lies in the B-module generated by $1,f,\dots,f^{n-1}$. When F is the free A-module with basis $e=T_1,\dots,T_n$ and

$$\beta : F\to A[X]$$

is defined by $\beta(T_i)=X^{i-1}$ for $i=1,\dots,n$, then $\mathcal{H}ilb^n_{S^{-1}A[X]/A}=\mathcal{H}ilb^\beta_{S^{-1}A[X]/A}$.

8.2 Description of the coordinate ring of the Hilbert scheme. In this example the ideal $\mathfrak{I}_3$ in 6.2(3) is generated by the coefficients of $T_1,\dots,T_n$ in

$$\mathrm{ev}_{1_{A[U]}\otimes T_1}((U^s_{ij})^{k-1})-T_k \quad \text{for} \quad k=1,\dots,n.$$

The relation $\mathrm{ev}_{1_{A[U]}\otimes T_1}((U^1_{ij})^{k-1})=T_k$ expressed that the first column in $(U^1_{ij})^{k-1}$ is equal to the column vector with 1 in the k'th coordinate and 0 elsewhere. It follows from Lemma 7.2 the matrix (U^1_{ij}) is congruent, modulo the ideal $\mathfrak{I}_3$, to the *companion* matrix

$$C_U=\begin{pmatrix} 0&0&0&\cdots&0&0&U^1_{1n}\\ 1&0&0&\cdots&0&0&U^1_{2n}\\ 0&1&0&\cdots&0&0&U^1_{3n}\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\ 0&0&0&\cdots&0&1&U^1_{nn}\end{pmatrix}.$$

In this example the elements given in (6.2)(2) are

$$s((U^1_{ij}))(U^s_{ij})-1 \quad \text{for} \quad s\in S.$$

Then we have $\det s((U^1_{ij}))\det(U^s_{ij})\equiv 1 \pmod{\mathfrak{I}_2}$. In particular, the elements $d_s=\det(s((U^1_{ij}))$ are invertible modulo $\mathfrak{I}_2$, and $(U^s_{ij})\equiv d_s^{-1}V_s \pmod{\mathfrak{I}_2}$, where V_s is the *adjoint* matrix of $s((U^1_{ij}))$, and consequently has coordinates that are polynomials in U^1_{ij} for $i,j=1,\dots,n$. We see that $A[U]/\mathfrak{I}_2=D^{-1}A[U^1_{11},\dots,U^1_{ij},\dots,U^1_{nn}]$ where D is the multiplicatively closed subset consisting of all products of the d_s for $s\in S$. Moreover we have seen that $A[U]/(\mathfrak{I}_2,\mathfrak{I}_3)=E^{-1}A[U^1_{1n},\dots,U^1_{nn}]$, where E consists of the elements obtined from the elements of D by the specialization that takes (U^1_{ij}) to the companion matrix C_U. Moreover, since the matrices $s((U^1_{ij}))$ commute for $s\in S$, the matrices (U^s_{ij}) commute modulo $\mathfrak{I}_2$. Consequently $\mathfrak{I}_1$ of 6.2(2) is contained in $\mathfrak{I}_2$ and $H^\beta_R=A[U]/(\mathfrak{I}_2,\mathfrak{I}_3)$.

In order to get a more attractive presentation of $E^{-1}A[U^1_{1n},\dots,U^1_{nn}]$ we factor the characteristic polyomial of C_U as

$$p_{C_U}(T)=T^n-U^1_{nn}T^{n-1}-\cdots-U^1_{1n}=\prod_{i=1}^n(T-Z_i)$$

where $Z_1, \dots, Z_n$ are independent variables over A. That is, we have $U^1_{n-i+1\,n} = (-1)^{i+1} c_i(Z)$ for $i = 1, \dots, n$, where $c_i(Z)$ is the i'the elementary symmetric function in $Z_1, \dots, Z_n$. It follows from the *Spectral Mapping Theorem* ([EL],[LSvT], [L] Chapter XIV, §3, Theorem 3.10, p. 566) that

$$\det(s(C_U)) = \prod_{i=1}^{n} s(Z_i).$$

It follows that the functor $\mathcal{H}ilb^n_{S^{-1}A[X]/A}$ is represented by the polynomial ring $A[c_1(Z), \dots, c_n(Z)]$ localized in the elements $\prod_{i=1}^n s(Z_i)$ for all $s \in S$.

9. A degenerate open subset of the Hilbert scheme of affine space

In this section we describe an open subset of a component of the Hilbert scheme containing many subschemes of n points with support at a fixed point. One of the important features of this set is that it can be used (see [I]) to show that the Hilbert scheme of n points in $\operatorname{Spec}(A[Y_1, \dots, Y_m])$ is reducible when $m \geq 3$ and n is large.

9.1 Notation. We shall keep the notation of Section 6, and assume in addition that $S = \{1, \dots, m\}$ such that $R = A[Y_1, \dots, Y_m]$. Determine the integer d by the inequalities

$$\binom{d+m-1}{m} < n \leq \binom{d+m}{m} \tag{9.1.1}$$

and let $s = \binom{d+m}{m} - n$. We order the monomials in the variables $Y_1, \dots, Y_m$ lexicographically and let $m_1, m_2, \dots$ be the monomials in this ordering. Hence the equalities (9.1.1) are equivalent with the condition that the n'th monomial is of degree d.

9.2 The section defining the open subset. Let

$$\beta : F \to A[Y]$$

be the A-module homomorphism defined by $\beta(T_i) = m_i$ for $i = 1, \dots, n$. For each $s \times (\binom{d-1+m}{m-1} - s)$-matrix $a = (a_{ij})$ with entries from A we define an A-module homomorphism

$$u_a : A[Y] \to F$$

by

$$u_a(m_i) = \begin{cases} T_i \quad \text{for} \quad i = 1, \dots, n \\ \sum_{j=\binom{d+m-1}{m}+1}^{n} a_{ij} T_j \quad \text{for} \quad i = n+1, \dots, \binom{d+m}{m} \\ 0 \quad \text{for} \quad i = \binom{d+m}{m} + 1, \dots \end{cases}.$$

It is clear that the kernel of u_a is the ideal generated by the homogeneous monomials of degree strictly greater than d and by the polynomials

$$m_i - \sum_{j=\binom{d-1+m}{m}+1}^{n} a_{ij} m_j \quad \text{for} \quad i = n+1, \dots, \binom{d+m}{m}.$$

Moreover, it is clear that different matrices $a = (a_{ij})$ give different ideals. Consequently we obtain for each $s \times (\binom{d+m-1}{m-1} - s)$-matrix $a = (a_{ij})$ a unique A-algebra homomorphism

$$\varphi_a : A[Y] \to \mathrm{End}_A(F)$$

such that $\mathrm{ev}_{T_1} \varphi_a = u_a$. In particular we have that φ_a corresponds to an element in $\mathcal{H}ilb^{\beta}_{A[Y]}(A)$. We see that the dimension of $\mathcal{H}ilb^{\beta}_{A[Y]}$ is at least equal to $s(\binom{d+m-1}{m-1} - s)$. In order to get the dimension of $\mathcal{H}ilb^{\beta}_{A[Y]}$ as big as possible we must choose $s = \lfloor \binom{d+m-1}{m-1}/2 \rfloor$. Easy computations (see [I], § 3) show that the dimension of $\mathcal{H}ilb^{\beta}_{A[Y]}$ is at least equal to $n^{2-(2/m)}(m!/2)^{-(2/m)}(m^2/16)$ when $d \geq 2m^2$.

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