NOTES ON FLATNESS AND THE QUOT FUNCTOR ON RINGS

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Abstract. For flat modules M over a ring A we study the similarities between the three statements, $\dim_{\kappa(P)}(\kappa(P)\otimes_A M) = d$ for all prime ideals P of A, the A_P-module M_P is free of rank d for all prime ideals P of A, and M is a locally free A-module of rank d. We have particularly emphasized the case when there is an A-algebra B, essentially of finite type, and M is a finitely generated B-module.

Introduction. The Quot functor $\mathcal{Q}uot_{\mathcal{F}/X/S}$ for a morphism of schemes $f\colon X \to S$, and an $\mathcal{O}_X$-module $\mathcal{F}$ associates to a scheme T over S the $\mathcal{O}_{X_T}$-module quotients $\mathcal{F}_T \to \mathcal{G}$ of the pull-back $\mathcal{F}_T$ of $\mathcal{F}$ to $X_T = X \times_S T$, such that $\mathcal{G}$ is flat over S, and where two quotients are equivalent if they have the same kernel. When X is projective over S it is natural to study the open and closed subfunctor $\mathcal{Q}uot^P_{\mathcal{F}/X/S}$ of quotients such that the restriction of $\mathcal{G}$ to the fiber $f^{-1}(t)$ has Hilbert polynomial P for all points t in T. A. Grothendieck [G] proved that when X is projective over S, and we consider locally noetherian schemes, then $\mathcal{Q}uot^P_{\mathcal{F}/X/S}$ is represented by a scheme which is projective over S.

In many situations it is natural to study $\mathcal{Q}uot_{\mathcal{F}/X/S}$ when $X \to S$ is a morphism of affine schemes associated to an essentially finite A-algebra B, and when $\mathcal{F}$ corresponds to a B-module M. In this situation we do not, in general, have a Hilbert polynomial for the restrictions of the quotients to the fibers. When the dimension $\dim_{\kappa(P)}(\kappa(P)\otimes_B M)$ is finite for all primes P of A, where we have written $\kappa(P) = A_P/PA_P$, it is natural to use this dimension as a substitute for the Hilbert polynomial. In our work on Hilbert schemes [L-S] and [S] we noticed that there are several natural choices for the definition of the subfunctor of $\mathcal{Q}uot_{M/B/A} = \mathcal{Q}uot_{\widetilde{M}/\operatorname{Spec} B/\operatorname{Spec} A}$ corresponding to an integer d. Even in the case of the Hilbert functor $\mathcal{H}ilb_{B/A} = \mathcal{Q}uot_{\mathcal{O}_{\operatorname{Spec} B}/\operatorname{Spec} B/\operatorname{Spec} A}$, which is the case mostly considered in the literature, there are ambiguities. One reason for the confusion is that most authors are not interested in the functor $\mathcal{H}ilb_{B/A}$, but only in its rational points (see [I1] and [I2]).

In this note we try to sort out the relations between the three statements

* M is a locally free A-module of rank d. That is, for every prime ideal P of A there is an element s in A not in P such that M_s is a free A_s-module of rank d.
* M_P is a free A_P-module of rank d for all primes P of A.
* The A-module M is flat and $\dim_{\kappa(P)}(\kappa(P)\otimes_A M) = d$ for all primes P of A.

It is clear that the first assertion implies the second and the second the third. When M is finitely generated and A is noetherian it is well known that these concepts coincide. When A is noetherian and M is not finitely generated, or when A is arbitrary and M is finitely generated they can differ considerably, and even in the noetherian case there are differences between the local and the non-local rings.

We thank Daniel Ferrand for several useful comments concerning the material at the end of Section (2). Thanks to Ferrand Lemma (1.2) also became *coordinate free*, and Theorem (2.4) and its proof got a more attractive form.

The situation that appears most frequently in geometry is when we are given an A-algebra B which is finitely generated, or more generally essentially of finite type, and when M is a finitely generated B-module. In Section (3) we shall focus on this situation and show that under these conditions some of the desirable relations between properties of M and properties of the fibers $\kappa(P) \otimes_A M$ for all prime ideals P of A, still hold.

Throughout we have tried to make the presentation self-contained, in some cases presenting proofs of known results.

1. Finitely generated flat modules.

1.1 Notation. Let A be a ring. For each prime ideal P in A we write $\kappa(P) = A_P/PA_P$.

The following result is one way of formulating the criterion for flatness by equations (see e.g. [M], Theorem 7.6, p. 49). We shall use this result instead of Lazard's Theorem ([La1], Theorem 1.2, p. 84) asserting that every flat module is the filtering limit of finitely generated free modules. As was observed by Lazard the results are indeed equivalent.

1.2 Lemma. *Let A be a ring and M an A-module. The following assertions are equivalent:*

(1) *The module M is flat over A.*
(2) *For any finitely presented module N, that is there is an exact sequence $A^m \to A^n \to N \to 0$ of A-modules, the map*

$$\mathrm{Hom}_A(N, A) \otimes_A M \to \mathrm{Hom}_A(N, M) \tag{1.2.1}$$

that sends $u \otimes x$ to the A-linear map sending y to $u(y)x$ is bijective.
(3) *Any A-linear map $N \to M$ from a finitely presented A-module N factors through a finitely generated free A-module.*
(4) *For every A-module homomorphism $u: F \to M$ from a finitely generated free A-module F, and for every element e in the kernel of u, there is a factorization $u = vf$ of u via an A-module homomorphism $f: F \to G$ into a finitely generated free A-module G such that $f(e) = 0$, and an A-module homomorphism $v: G \to M$.*

Proof. For any A-module M the functors $\mathrm{Hom}_A(N, A) \otimes_A M$ and $\mathrm{Hom}_A(N, M)$ are additive and contravariant in N. Since the map (1.2.1) is bijective for $N = A$ it follows that it is bijective for $N = A^n$.

Assume that M is flat over A. Then the two functors are left exact. It follows that the map (1.2.1) is an isomorphism for every finitely presented A-module N. Hence the first assertion implies the second.

Assume that the second assertion holds. Let $u: N \to M$ be an A-linear map from a finitely presented A-module N. Then u is the image by (1.2.1) of an element $\sum_{i=1}^n u_i \otimes x_i$ of $\mathrm{Hom}_A(N, A) \otimes_A M$. Hence u is the composite of the map $N \to A^n$ sending y to $(u_1(y), \ldots, u_n(y))$, and the map $A^n \to M$ sending $(a_1, \ldots, a_n)$ to $\sum_{i=1}^n a_i x_i$. Hence the third assertion follows from the second.

The fourth assertion follows from the third since F/Ae is finitely presented.

Finally we prove that the last assertion implies the first. We shall show that M is flat over A by showing that the map $I \otimes_A M \to M$ is injective for all ideals I of A. Assume that there is an element $x = \sum_{i=1}^m a_i \otimes x_i$ with $a_i \in I$ and $x_i \in M$ in $I \otimes_A M$ that maps to zero in M. Let $u: F \to M$ be the A-linear homomorphism from the free A-module F with basis $f_1, \ldots, f_m$ defined by $u(f_i) = x_i$, and let $y = \sum_{i=1}^m a_i \otimes f_i$. Then $u(y) = x$, and the image e of y by the map $i: I \otimes_A F \to F$ maps to zero by u. Hence the last assertion of the Lemma implies that $u: F \to M$ factors via A-module homomorphisms $f: F \to G$ and $v: G \to M$, where G is a free A-module of finite rank, and where $f(e) = 0$. The map $j: I \otimes_A G \to G$ is injective since G is flat over A. We have that $0 = f(e) = fi(y) = j(\mathrm{id}_I \otimes f)(y)$ and consequently that $(\mathrm{id}_I \otimes f)(y) = 0$. Hence we have that $x = (\mathrm{id}_I \otimes u)(y) = (\mathrm{id}_I \otimes v)(\mathrm{id}_I \otimes f)(y) = 0$.

The following two results are well known (see e.g. Matsumura [M], Theorem 7.10, p. 51). We include proofs to show how Lemma (1.2) can be used in this situation instead of the criterion for flatness by equations.

1.3 Lemma. *Let A be a local ring with maximal ideal P and M a flat A-module. Moreover, let F be a free A-module and $u: F \to M$ an A-linear map. If the residue map $u(P): F/PF \to M/PM$ is injective, then the map u is injective.*

Proof. Let e in F be such that $u(e) = 0$. We first prove the Lemma when F is of finite rank. Since M is a flat A-module it follows from Proposition (1.2) that we have a factorization $F \xrightarrow{f} G \xrightarrow{v} M$ of u into A-linear maps, where G is a free A-module of finite rank, and where we have that $f(e) = 0$. Then $u(P)$ factors via $\kappa(P) \otimes_A F \xrightarrow{f(P)} \kappa(P) \otimes_A G \xrightarrow{v(P)} \kappa(P) \otimes_A M$. Since $u(P)$ is injective by assumption, it follows that $f(P)$ is injective. Our claim follows if we show that $F \xrightarrow{f} G$ is injective.

We fix a basis for F and G and let the map f be represented by a matrix. Let n be the rank of F. Since the induced map $f(P)$ is injective, there exist a $(n \times n)$-minor $N(P)$ of the matrix $f(P)$ which is invertible. It follows that the determinant of the corresponding square matrix N of f is invertible since $\det(N) \otimes_A \kappa(P) = \det(N(P))$. Then there exist a matrix N' such that $N'N$ is the identity matrix, and we may construct a map $f' : G \to F$ such that $f'f$ is the identity map. Hence f is injective.

Assume that F has infinite rank. Then the element e is contained in a free A-submodule F' of F of finite rank, which is a direct summand of F. Let $i: F' \to F$ be the inclusion. Then $i(P)$ is injective and thus $u(P)i(P) = ui(P)$ is injective. It follows from the first part of the proof that the map $ui: F' \to M$ is injective. Hence $ui(e) = u(e) = 0$ implies that $e = 0$ and we have proved the Lemma.

1.4 Proposition. *Let M be a finitely generated flat A-module. Then M_P is a free A_P-module for all prime ideals P of A.*

Proof. Let P be a prime ideal of A. Then M_P is a flat A_P-module. Since M is finitely generated it follows from Nakayama's Lemma that we can choose a surjection $u: A_P^n \to M_P$ such that the residue map $u(P): \kappa(P)^n \to \kappa(P) \otimes_{A_P} M_P$ is an isomorphism of $\kappa(P)$-vector spaces. If follows from Lemma (1.3) that $A_P^n \to M_P$ is injective and hence an isomorphism. Thus M_P is a free A_P-module for all prime ideals P in A.

1.5 Corollary. *Let M be a finitely generated flat A-module. If there is an integer d such that*

$$d = \dim_{\kappa(P)}(\kappa(P) \otimes_A M) \tag{1.5.1}$$

for all prime ideals P of A we have that M is locally free.

Proof. Let P be a prime ideal of A. Let $m_1, \dots, m_n$ be a generator set for the A-module M, and let F be a free A-module with basis $f_1, \dots, f_d$. Since M is flat it follows from the Proposition that M_Q is a free A_Q-module for all prime ideals Q of A. It follows from (1.5.1) that M_Q is of rank d. In particular there is an isomorphism $u: F_P \to M_P$ of A_P modules. Choose elements $g_j = \sum_{i=1}^d a_{j,i} f_i$ in F_P such that $u(g_j) = \frac{m_j}{1}$ for $j = 1, \dots, n$. Let t be a common denominator of the elements $u(f_i)$, and of the coefficients $a_{j,i}$ for $i = 1, \dots, d$ and $j = 1, \dots, n$. Then there is a surjective map $v: F_t \to M_t$ of A_t-modules such that the localization of v at P is equal to u. Denote by K the kernel of v. For each prime Q of A we obtain an exact sequence $0 \to K_Q \to F_Q \to M_Q \to 0$ of A_Q-modules. Since F_Q is free of rank d it follows that $K_Q = 0$ for all primes Q of A_t. Consequently we have that $K = 0$. We thus have that M_t is a free A_t-module.

Remark. When A is noetherian and M is a finitely generated A-module we have that if M_P is a free A_P-module, then there is an element t in A not in P such that M_t is a free A_t-module. Indeed, in the proof of Corollary (1.5) we constructed a surjective map $v: F_t \to M_t$ from a free A_t-module of rank equal to the rank of M_P, whose localization at P is an isomorphism. Hence the localization K_P of the kernel K of v at P is zero. Since A is noetherian by assumption, we have that K is finitely generated and thus we can find an element s in A not contained in P such that $K_s = 0$. It follows that $v_s: F_{st} \to M_{st}$ is an isomorphism of A_{st}-modules. In particular it follows from Proposition (1.4) that if M is flat, then M is locally free.

With the following example we will show that when A is not noetherian we can have a finitely generated flat A-module M such that M_P is free for all prime ideals P of A, but where M is not locally free. In particular it follows that condition (1.5.1) is necessary in Corollary (1.5).

1.6 Example. Let $B = k[y_1, y_2, \dots]$ be the polynomial ring in the variables $y_1, y_2, \dots$ over the field k, and let A be the residue ring of B by the ideal generated by the polynomials $y_i(y_i - 1)$ for $i = 1, 2, \dots$. Denote by x_i the class of y_i in A. Let P be a prime ideal of A. Then, for each i, the ideal P contains either x_i or $x_i - 1$. It follows that the prime ideals of A are the ideals $(x_1 - \delta_1, x_2 - \delta_2, \dots)$, for all choices of $\delta_1, \delta_2, \dots$, where δ_i, here and below, will take the values 0 and 1. We obtain in particular that $A/P = \kappa(P) = k$.

We note that the ring A is reduced. Indeed, if a polynomial $f(y_1, \dots, y_n)$ in B maps to a nilpotent element in A we must have that $f(\delta_1, \dots, \delta_n) = 0$ for all choices of $\delta_1, \dots, \delta_n$. It is easy to show, by induction on n, that this implies that $f(y_1, \dots, y_n)$ is in the ideal generated by the elements $y_1(y_1 - 1), \dots, y_n(y_n - 1)$. Hence the class of $f(y_1, \dots, y_n)$ in A is zero.

Let $P = (x_1 - \delta_1, x_2 - \delta_2, \dots)$. Then P is a prime ideal of A. For each i we have that $(x_i - \delta_i)(x_i + \delta_i - 1) = 0$, and clearly $x_i + \delta_i - 1 \notin P$. Consequently we have that the class of $x_i - \delta_i$ is zero in A_P. Hence we have that $A_P = \kappa(P) = k$ for all prime ideals P in A. In particular any module M over A is flat.

Fix a prime P of A. We have that the A-module $M = \kappa(P)$ is generated by one element. Moreover we have that $M_P = A_P \otimes_A M = \kappa(P) \otimes_A M = \kappa(P)$, and that $M_Q = A_Q \otimes_A M = \kappa(Q) \otimes_A M = 0$ for all prime ideals Q of A different from P.

In $\operatorname{Spec} A$ every non-empty open set contains infinitely many points. Indeed, let A_f be the ring of a non-empty principal open set in $\operatorname{Spec} A$, where f is the residue class of the polynomial $f(y_1, \dots, y_n)$ in A. Then $f(\delta_1, \dots, \delta_n) \neq 0$ for some $\delta_1, \dots, \delta_n$. Then, for all choices of $\delta_{n+1}, \delta_{n+2}, \dots$, the prime $(x_1 - \delta_1, \dots, x_n - \delta_n, x_{n+1} - \delta_{n+1}, \dots)$ is in $\operatorname{Spec} A_f$. Since $M = \kappa(P)$ has fiber k at one point and fiber zero at the remaining points, it follows that $M = \kappa(P)$ can not be locally free.

The condition that M is finitely generated is necessary in Corollary (1.5), even when A is noetherian, as shown by the following example communicated to us by C. Walter.

1.7 Example. Let $A = \mathbf{Z}$ be the ring of integers and let M be the $\mathbf{Z}$-submodule

$$M = \{x \in \mathbf{Q} \colon v_p(x) \geq -1 \text{ for all primes } p \in \mathbf{Z}\}$$

of the rational numbers $\mathbf{Q}$, where $v_p(x) = d$ if $x = \frac{m}{n} p^d$ with m and n prime to p. If P is a maximal ideal of $\mathbf{Z}$ corresponding to a prime integer p, we have that $M_P = \frac{1}{p}\mathbf{Z}_P$. In particular the $\mathbf{Z}$-module M is a flat. Furthermore we have an isomorphism $\mathbf{Q} \to M \otimes_{\mathbf{Z}} \mathbf{Q} = M_{(0)}$. Hence we have that $\dim_{\kappa(P)}(\kappa(P) \otimes_{\mathbf{Z}} M) = 1$ for all prime ideals P in the ring $\mathbf{Z}$. However we obviously have that $M_n = \{\frac{x}{n^m} \colon x \in M, m \in \mathbf{Z}\}$ is not finitely generated $\mathbf{Z}_n$-module for any non-zero integer n. In particular M is not locally free.

1.8 Example. We shall give another, perhaps more typical, example of a ring A, together with a flat A-module M, such that M_P is a free A_P-module of rank 1 for each prime P of A, but such that M is neither a finite, nor a locally free A-module.

Denote by A the product $\prod_{i \in \mathbf{N}} K_i$ of a field $K = K_i$ for $i \in \mathbf{N}$. Let I be the ideal in A consisting of elements $a = (a_i)_{i \in \mathbf{N}}$ with finite support $\operatorname{Supp}(a) = \{i \colon a_i \neq 0\}$. That is, the ideal I is the direct sum $\oplus_{i \in \mathbf{N}} K_i$ of the field $K = K_i$ for $i \in \mathbf{N}$. Let $M = I \oplus A/I$.

We first show that the ring A is *absolutely flat*, that is all A-modules M are flat. Note that there are no inclusions of prime ideals in A. Indeed, let P be a prime ideal and let a be an element in A not in P. If a is not a unit in A we let b be an element in A having support on the complement of $\operatorname{Supp}(a)$. Then $ab = 0$, and consequently we have b in P. The element $a + b$ is congruent to a modulo P. We have that $a + b$ is a unit in A since $\operatorname{Supp}(a + b) = \mathbf{N}$, hence a is a unit in A/P. Thus A/P is a field and all prime ideals are maximal, and minimal.

The ring A is reduced and consequently any fraction ring of A is reduced. In particular the stalks A_P are reduced for all prime ideals P in A. In our ring A all prime ideals P are minimal, thus we get that $A_P = \kappa(P)$. Consequently any module M is flat over A.

Hence, when we localize the exact sequence

$$0 \to I \to A \to A/I \to 0 \qquad (1.8.1)$$

in a prime ideal P of A we see that we either have that I_P is a free A_P-module of rank 1 and $(A/I)_P = 0$, or we have that $I_P = 0$ and $(A/I)_P$ is a free A_P-module of rank 1. In both cases we have that $M_P = A_P \otimes_A M = \kappa(P) \otimes_A M$ is a free $\kappa(P)$-module of rank 1.

We have that I is not a finitely generated A-module, since the elements of I otherwise would have support on a finite subset of $\mathbf{N}$. However I is a quotient of M, so M is not a finitely generated A-module either.

We can tell exactly for which prime ideals P we have that $I_P = 0$. Indeed, it is easily seen that there is an inclusion preserving bijection between ideals in A and filters of $\mathbf{N}$. This correspondence associates to an ideal I of A the ultrafilter consisting of the complement in $\mathbf{N}$ of the support $\operatorname{Supp}(a) = \{i \in \mathbf{N}: a_i \neq 0\}$ of the elements $a = (a_i)_{i\in\mathbf{N}}$ of I. Under this correspondance the prime ideals of A correspond to the ultra filters of $\mathbf{N}$. The trivial ultra filters, that is the ultra filters consisting of the sets containing a fixed integer, correspond to the maximal ideals consisting of elements with one fixed coordinate equal to zero.

We have that $I_P = 0$ exactly when P corresponds to a non-trivial ultra filter. Indeed, let $a = (a_i)_{i\in\mathbf{N}}$ be an element of I. Then $ab = 0$ for all elements b in A whose support is in the complement of the support of a. Such an element b has cofinite support, that is, the complement of the support is finite. However, it is easily seen that an ultra filter is non-trivial if and only if it contains the filter of all cofinite sets.

We have that if P is a prime ideal corresponding to a trivial ultra filter, then there exist a f not in P such that $M_f = I_f = A_f$. The module M is however not locally free, that is there exist prime ideals P in A such that M_f is not free for any f not in P. Indeed if there for each prime ideal P exists f_P not in P such that M_{f_P} is free, then there exist prime ideals $P_1, \dots, P_m$ such that $\sum_{i=1}^m a_i f_{P_i} = 1$, with $a_1, \dots, a_m$ in A. We have that $M_P = A_P$ for all prime ideals P in A. It follows that M_{f_P} is a finitely generated A_{f_P}-module. Let $x_1, \dots, x_n$ be elements in M such that the classes of $x_1, \dots, x_n$ generate $M_{f_{P_i}}$ as an $A_{f_{P_i}}$-module for $i = 1, \dots, m$. Then $x_1, \dots, x_n$ generate M as an A-module. In particular we would have that M is finitely generated, which we have seen is not the case. Thus M is not locally free.

2. Flat modules over local rings.

In this section we shall consider the case when the ring A is local. Our main objective is to investigate under which conditions a flat A-module M is free when the fiber dimension $\dim_{\kappa(P)}(\kappa(P) \otimes_A M)$ is constant for all primes P of M.

Remark. Assume that A is reduced and that Q is a minimal prime ideal of A. Then:

(1) For all A-modules M we have that $\kappa(Q) \otimes_A M = M_Q$.

(2) If M is flat we have that $M = QM$ if and only if $M_Q = 0$.

The first assertions holds because when A is reduced and Q is minimal then A_Q is the field $\kappa(Q)$.

To prove the second assertion we observe that since $Q_Q = 0$ we have that localization gives an injective homomorphism $A/Q \to A_Q$. When M is flat over A we obtain an injection $A/Q \otimes_A M \to A_Q \otimes_A M$, that is, an injection $M/QM \to M_Q$. Hence $M_Q = 0$ implies that $M = QM$, and $M/QM = 0$ implies that $0 = (M/QM)_Q = M_Q/QM_Q = M_Q/Q_QM_Q = M_Q$.

First we prove a result of which many variations are known.

2.1 Proposition. *Let A be a ring, F a free A-module, and M a flat A-module. Moreover $u: F \to M$ be an A-linear map. If the residue map*

$$u(P): \kappa(P) \otimes_A F \to \kappa(P) \otimes_A M$$

is injective for all maximal ideals P of A we have that u is injective and that the cokernel of u is a flat A-module.

Proof. The kernel of u is zero if and only if its localization at all maximal ideals of A is zero. Moreover the cokernel of u is a flat A-module if and only if its localization is flat at all maximal ideals. Hence we can assume that A is local.

The first part of Proposition (2.1) follows from Lemma (1.3).

To prove the second part let N be the cokernel of u. To prove that N is flat over A we shall verify that the map $I \otimes_A N \to N$ is injective for all ideals I of A. We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & I\otimes_A F & \longrightarrow & I\otimes_A M & \longrightarrow & I\otimes_A N & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \xrightarrow{u} & M & \longrightarrow & N & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A/I\otimes_A F & \xrightarrow{\mathrm{id}_{A/I}\otimes u} & A/I\otimes_A M & \longrightarrow & A/I\otimes_A N. & &
\end{array}$$

It is clear that the right and left vertical sequences, as well as the top horizontal sequence, are exact. The middle vertical sequence is exact because M is flat, and the middle horizontal sequence is exact by the first part of the Proposition. We have that $\kappa(P) \otimes_A A/I = \kappa(P)$. Hence tensoring the source and target of $\mathrm{id}_{A/I} \otimes_A u$ by $\kappa(P)$ over A we have that $\mathrm{id}_{A/I} \otimes_A u$ induces $u(P)$ which is injective by assumption. Hence it follows from the first part of the Proposition that the bottom horizontal sequence is exact. It follows from the diagram that $I \otimes_A N \to N$ is injective.

The following Theorem is due to D. Lazard ([La2] Théorème 1, p. 65). Our proof uses Lemma (1.2). We are thankful to D. Ferrand for making us aware of Theorem (2.2) and pointing out that it implies Corollary (2.3).

2.2 Theorem. *Let A be a local ring and M a flat A-module. Moreover, let m be an element of M such that* $\mathrm{ann}(m)$ *is non-zero, and such that the maximal ideal is the only prime ideal of A that contains* $\mathrm{ann}(m)$. *Then m is in $\sqrt{0}M$.*

Proof. Let $u: A \to M$ be the A-linear map that sends 1 to m. It follows from Lemma (1.2) that for every element a of $\mathrm{ann}(m)$ there is a factorization $u = v_a f_a$, where $f_a: A \to F_a$ and $v_a: F_a \to M$, and F_a is a finitely generated free A-module. Then we have that $v_a(f_a(1)) = m$ and consequently that $a \in \mathrm{ann}(f_a(1)) \subseteq \mathrm{ann}(m)$.

Fix a non-zero element b in $\mathrm{ann}(m)$ and let $f_1, \dots, f_p$ be a basis of $F = F_b$. Write $x = f_b(1) = \sum_{i=1}^{p} a_i f_i$ and $v = v_b$.

We shall consider finitely generated free A-modules G and factoriations $F \to G \to M$ of $v: F \to M$. Note that if x maps to y in G then $\mathrm{ann}(x) \subseteq \mathrm{ann}(y) \subseteq \mathrm{ann}(m)$.

Let $F' \to G_1 \to M$ and $F' \to G_2 \to M$ be two factorizations of $v': F' \to M$ and let y_1 and y_2 be the images in G_1 respectively G_2 og an element x' in F'. Then we can find a factorization $F' \to G \to M$, with maps $G_1 \to G$ and $G_2 \to G$ such that y_1 and y_2 map to the same element in G. Indeed, let $s: G_1 \oplus G_2 \to M$ be the sum of the maps $G_1 \to M$ and $G_2 \to M$. Then the element $(y_1, -y_2)$ in $G_1 \oplus G_2$ maps to zero in M. It follows from Lemma (1.2) that there is a factorization of s in maps $f: G_1 \oplus G_2 \to G$ and $G \to M$ such that $f(y_1, 0) - f(0, y_2) = f(y_1, -y_2) = 0$. Let $G_1 \to G$ and $G_2 \to G$ be the maps induced by the canonical maps of G_1 respectively G_2 in G. Then we clearly have that y_1 and y_2 have the same image y in G. In particular we have that the ideals $\operatorname{ann}(y_1)$ and $\operatorname{ann}(y_2)$ are contained in $\operatorname{ann}(y)$.

Denote by I the union of the ideals $\operatorname{ann}(y)$ for all the elements y obtained in this way. Then $I = \operatorname{ann}(m)$. Indeed, we observed that $I \subseteq \operatorname{ann}(m)$. On the other hand, for each $a \in \operatorname{ann}(m)$ we observed that there is a factorization of $u: A \to M$ in maps $f_a: A \to F_a$ and $v_a: F_a \to M$, where F_a is free of finite rank, such that $a \in \operatorname{ann}(f_a(1))$. Using the above with $F' = A$ and with the maps $A \to F_a$ and $A \to F$ we obtain a free A-module G of finite rank toghether with maps $F_a \to G$ and $F \to G$ such that if y in G is the image of x then $a \in \operatorname{ann}(f_a(1)) \subseteq \operatorname{ann}(y)$, and $F \to G \to M$ is an extension of v.

We have that $b \in \operatorname{ann}(x)$. In particular $\operatorname{ann}(x) \neq 0$. Hence $\operatorname{ann}(x)$ is contained in the maximal ideal of A. It follows that the elements $a_1, \dots, a_p$ are contained in the maximal ideal of A. Since, by hypothesis, the maximal prime ideal of A is the only prime ideal containing $I = \operatorname{ann}(m)$, and thus the radical of I, we have that $I_{a_i} = A_{a_i}$ for $i = 1, \dots, p$. Consequently there are elements $c_1, \dots, c_p$ in I such that the image of c_i in A_{a_i} is invertible. As we saw above we can find a factorization $F \to G \to M$ such that, if y in G is the image of x, then the elements $c_1, \dots, c_p$ are in $\operatorname{ann}(y)$. In particular we have that $(\operatorname{ann}(y))_{a_i} = A_{a_i}$ for $i = 1, \dots, p$.

Write $y = \sum_{i=1}^{q} b_i g_i$, where $g_1, \dots, g_q$ is a basis for G. Let P be a prime ideal of A. If P does not contain $\operatorname{ann}(y) = \operatorname{ann}(b_1, \dots, b_q)$ it is clear that P contains $b_1, \dots, b_q$. On the other hand if P contains $\operatorname{ann}(y)$, then $P_{a_i} = A_{a_i}$ for $i = 1, \dots, p$ since we have seen that $(\operatorname{ann}(y))_{a_i} = A_{a_i}$. In particular P contains the elements $a_1, \dots, a_p$. We have that x maps to y by the A-linear map $F \to G$ of free A modules with bases $f_1, \dots, f_p$ respectively $g_1, \dots, g_q$. It follows that each element b_i is a linear combination of of the elements $a_1, \dots, a_p$. Hence the elements $b_1, \dots, b_q$ are contained in P. We have proved that the elements $b_1, \dots, b_q$ are in all the prime ideals of A and consequently in the radical of A. Hence we have proved the Theorem.

2.3 Corollary. *Let A be a ring and M a flat A-module such that $M_P = 0$ for all minimal prime ideals P of A. We then have that $\sqrt{0}M = M$.*

Proof. Replacing A by $A/\sqrt{0}$ and M by $M/\sqrt{0}M$ we may assume that A is reduced. The Corollary the asserts that if $M_P = 0$ for all minimal prime ideals P of A then $M = 0$.

Assume that M is not zero, and let m be a non-zero element of M. The image of m in M_P is zero for all minimal prime ideals P of A. Hence we have that $\operatorname{ann}(m)$ is not contained in any minimal prime of A. Let Q be a prime ideal which is minimal among the prime ideals in A that contain $\operatorname{ann}(m)$. Then QA_Q is the only prime ideal in A_Q containing $\operatorname{ann}(m)A_Q$. It follows from the Theorem that the image of m in M_Q is contained in $\sqrt{0}M_Q$. Hence there is an element $s \in A$ not in Q such that sm is in $\sqrt{0}M$. Since A is reduced we must have that $sm = 0$, that is s is in $\operatorname{ann}(m)$. However, this is impossible since we assumed that s was not in Q and that Q contains $\operatorname{ann}(m)$. This contradicts the assumption that M is not zero and we have proved the Corollary.

2.4 Theorem. *Let A be a local ring with a nilpotent radical, and let M be a flat A-module. Denote the maximal ideal of A by P. Assume that $\dim_{\kappa(P)}(\kappa(P)\otimes_A M)$ is finite and that*

$$\dim_{\kappa(P)}(\kappa(P)\otimes_A M)=\dim_{\kappa(Q)}(\kappa(Q)\otimes_A M)$$

for all minimal prime ideals Q of A.

Then M is a free A-module of rank $\dim_{\kappa(P)}(\kappa(P)\otimes_A M)$.

In particular the result holds when A is noetherian or when A is reduced.

Proof. Let F be a free A-module of rank $d=\dim_{\kappa(P)}(\kappa(P)\otimes_A M)$. We can find an A-linear map $u\colon F\to M$ such that $u(P)\colon \kappa(P)\otimes_A F\to\kappa(P)\otimes_A M$ is an isomorphism. It follows from Proposition (2.1) that u is injective and that the cokernel N of u is flat.

Since N is flat we have an exact sequence $0\to\kappa(Q)\otimes_A F\to\kappa(Q)\otimes_A M\to\kappa(Q)\otimes_A N\to 0$ for all prime ideals Q of A. By assumption we have that $\dim_{\kappa(Q)}(\kappa(Q)\otimes_A M)=d=\operatorname{rank}_A F$ when Q is minimal. Hence we have that $\kappa(Q)\otimes_A N=0$ for all minimal prime ideals Q of A.

Since $\sqrt{0}\subseteq Q$ we have that $\kappa(Q)\otimes_A N=\kappa(Q)\otimes_A(N/\sqrt{0}N)$, and thus that $0=\kappa(Q)\otimes_A(N/\sqrt{0}N)=(N/\sqrt{0}N)_{Q(A/\sqrt{0})}$. It follows from Corollary (2.3) applied to the flat $A/\sqrt{0}$-module $N/\sqrt{0}N$ that $N=\sqrt{0}N$. By assumption $\sqrt{0}$ is nilpotent. Hence we have that $N=0$ and we have proved the Theorem.

Remark. For a local ring A with maximal ideal P and an A-module M, Theorem (2.4) shows that if $\dim_{\kappa(P)}(\kappa(P)\otimes_A M)=\dim_{\kappa(Q)}(\kappa(Q)\otimes_A M)$ for all prime ideals Q of A, then M is a free A-module of finite rank when A is noetherian or possibly non-notherian but reduced. The following example shows that for non-noetherian and non-reduced rings it is not necessarily true that M is finitely generated. In the example the ring A has only one prime ideal P, with a non-zero flat module M such that $M=PM$, and thus $\kappa(P)\otimes_A M=0$. A more complicated, but similar, example where M is an ideal in A was given by D. Lazard ([La1], Example 4.3, p. 91).

2.5 Example. Let $B=k[y_1,y_2\dots,]$ be the polynomial ring in the independent variables $y_1,y_2,\dots$ over a field k. Let $I\subset B$ be the ideal generated by the elements y_i^2, for $i=1,2,\dots$. Let A be the residue ring B/I, and let x_i be the class of y_i in A. The ring A is a local ring, where the only prime ideal P of A is the ideal generated by $(x_1,x_2,\dots)$.

Denote by E the free A-module with basis $e_1,e_2,\dots$ and by F the submodule of E generated by the elements $e_1-x_1e_2,e_2-x_2e_3,\dots$. It is clear that the elements $e_1-x_1e_2,e_2-x_2e_3,\dots$ are linearly independent over A. Thus F is a free A module. It is clear that the element e_1 of E is not in F so that F is a proper sub-module. Moreover we have that the residue map $\kappa(P)\otimes_A F\to\kappa(P)\otimes_A E$ is an isomorphism. It follows from Proposition (2.1) that the cokernel of the inclusion $F\subseteq E$ is a flat A-module. From the exact sequence $0\to F\to E\to M\to 0$ we obtain an exact sequence $0\to\kappa(P)\otimes_A F\to\kappa(P)\otimes_A E\to\kappa(P)\otimes_A M\to 0$ and thus that $\kappa(P)\otimes_A M=0$.

2.6 Note. As can be seen from the proof of Theorem (2.4), the statement of the Theorem can be strenghtened in cases when tensor products commute with products. This is only true in general if the product is finite. That commutation may fail for infinite products can be seen from the simple example

$$\mathbf{Q} \otimes_{\mathbf{Z}} \prod_{p \text{ prime}} \mathbf{Z}/p\mathbf{Z} \to \prod_{p \text{ prime}} (\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}/p\mathbf{Z}).$$

The target is obviously zero and the source is non-zero because we have an injection $\mathbf{Z} \to \prod_{p \text{ prime}} \mathbf{Z}/p\mathbf{Z}$ sending n to $(n, n, \dots)$, and thus an injection $\mathbf{Q} = \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z} \to \mathbf{Q} \otimes_{\mathbf{Z}} \prod_{p \text{ prime}} \mathbf{Z}/p\mathbf{Z}$.

3. Finitely generated flat modules over algebras.

One of the main results of Section (2) is that when A is local and noetherian and M is a flat A-module with fixed finite fiber dimension $\dim_{\kappa(P)}(\kappa(P) \otimes_A M)$ for all prime ideals P of A, then M is free of finite rank.

Example (1.7) and Example (2.5) show that this is not necessarily true when A is not local but noetherian, or when A is not noetherian but local. The main result of this section is that when we are given a finitely generated A-algebra B and M is a finitely generated B-module, then M is locally free of finite rank over A if M is flat over A and the dimension of the fiber $\kappa(P) \otimes_A M$ is finite and fixed for all prime ideals P of A.

We note that some condition on the A-algebra B is necessary. Indeed, let $B = A \oplus M$ be the A-algebra with multiplication $(a, m)(a', m') = (aa', am' + a'm)$. Then $N = B$ is a finite B-module which is flat as an A-module when M is flat as an A-module. Moreover $\dim_{\kappa(P)}(\kappa(P) \otimes_A M) = 1 + \dim_{\kappa(P)}(\kappa(P) \otimes_A M)$ for all prime ideals P of A. However N is not locally free of finite rank as an A-module when M is not locally free of finite rank as an A-module.

3.1 Lemma. *Let M be a finitely generated and faithful A-module. Moreover let $I \subseteq A$ be an ideal. Then we have that* $\operatorname{ann}_A(M/IM) \subseteq \operatorname{Rad}(I)$.

Proof. Let b be an element of $\operatorname{ann}_A(M/IM)$. Then b corresponds to an endomorphism on M such that $bM \subseteq IM$. Since M is finitely generated the endomorphism b satisfies an equation of the form $f(b) = b^n + a_1 b^{n-1} + \cdots + a_n = 0$, where $a_1, \dots, a_n$ are in I ([A-M], Proposition 2.4, p. 21). Thus $f(b)M = 0$. The module M is assumed to be faithful such that $f(b) = 0$ in A, from which it follows that b^n is in I. We have proved our claim.

3.2 Lemma. *Let A be a ring with radical R and B an A-algebra. Moreover, let M be a finitely generated faithful B-module. If M/RM is a finitely generated A/R-module, then B is integral over the image of A in B.*

Proof. To prove that B is integral over A it suffices to prove that B/I is integral over the image of A, where I is an ideal contained in the nilradical of B. Indeed, if an element b in B is such that the image of b in B/I satisfies a monic polynomial $f(x)$ in the polynomial ring $A[x]$, we have that $f(b)$ is in I. Consequently we have that $f(b)^n = 0$ in B for big enough n, and hence b is integral over the image of A in B.

Clearly we have that RB is contained in the nilradical of B. It follows from Lemma (3.1) that $\operatorname{ann}_B(M/RM)$ is contained in the nilradical of B. To prove the Lemma we may consequently replace B by the ring $B/\operatorname{ann}_B(M/RM)$, and we may therefore assume that M/RM is a faithful B-module. However, we know that an element b in B is integral over the image of A if and only if there is a faithful finitely generated $A[b]$-module ([L], §1, p. 334), and if M/RM is a finitely generated A/R-module we have that M/RM is such a module.

3.3 Proposition. *Let A be a ring with radical R and B an A-algebra essentially of finite type. Moreover, let M be a finitely generated B-module. If M/RM is a finitely generated A/R module, then M is a finitely generated A-module.*

Proof. We can obviously replace B by its residue ring $B/\operatorname{ann}_B M$. Consequently we may assume that M is a faithful B-module. However, then it follows from Lemma (3.2) that B is integral over the image of A.

We have a factorization $A \to C \to B$, where C is a finitely generated A-algebra and B is a localization of C in a set S. Since B is integral over the image of A we have that C is integral over the image of A, and thus is a finite A-module. On the other hand, we have that $C = B$. Indeed, for each $c \in S$ we have that $\frac{1}{c}$ in B is integral over the image of A, and thus there is a relation $(\frac{1}{c})^n + a_{n-1}(\frac{1}{c})^{n-1} + \cdots + a_0 = 0$ in B with $a_0, \ldots, a_{n-1}$ in A. Consequently we have that $\frac{1}{c} = -a_{n-1} - ca_{n-2} - \cdots - a_0c^{n-1}$ which is in C.

We have proved that B is a finitely generated A-module. Hence we have that M is a finitely generated A-module.

3.4 Proposition. *Let B be an A-algebra essentially of finite type and let M be a finitely generated B-module. Assume that M_P is a finitely generated A_P-module for every prime ideal P in A. Then we have that M is a finitely generated A-module.*

Proof. We can, if necessary, replace B by the residue ring $B/\operatorname{ann}_B(M)$ of B modulo the annihilator of M in B. Thus we may assume that M is a faithful B-module. Then we have for each prime ideal P of A that M_P is a faithful B_P-module which is finitely generated as an A_P-module. From Lemma (3.2) we have that B_P is integral over the image of A_P.

We have a factorization $A \to C \to B$ such that C is a finitely generated A-algebra and B is a localization of C. As we just proved B_P is integral over the image of A_P and thus B_P is integral over the image of C_P. Since B_P is a localization of C_P it follows that $C_P \to B_P$ is surjective. Hence the localization of the map $C \to B$ at each prime ideal P of A is surjective. It follows that the map $C \to B$ is surjective and thus that B is an A-algebra of finite type.

Let b be an element in B. The class of b in B_P is integral over A_P. Thus there exists f not in P such that the class of b in B_f is integral over A_f. Since B is finitely generated A-algebra we can for each prime P in A find f_P not in P such that B_{f_P} is integral over A_{f_P}. The elements f_P for all primes P in A, generate the trivial ideal in A. We have that $\sum_{i=1}^m a_i f_{P_i} = 1$ for some prime ideals $P_1, \ldots, P_m$. Let $b_1, \ldots, b_n$ in B be such that the classes of $b_1, \ldots, b_n$ generate $B_{f_{P_i}}$ as an $A_{f_{P_i}}$-module, for $i = 1, \ldots, m$. Then $b_1, \ldots, b_n$ generate B as an A-module. We have shown that B is a finitely generated A-module and it follows that M is finitely generated over A. We have proved the Proposition.

3.5 Theorem. *Let A be a ring and B an A-algebra which is essentially of finite type. Moreover, let M be a finitely generated B-module. Assume that M is a flat A module such that*

$$d = \dim_{\kappa(P)}(\kappa(P) \otimes_A M)$$

for all minimal and maximal prime ideals P of A and some integer d. Then we have that M is a locally free A-module.

Proof. It follows from Corollary (1.5) that it suffices to prove that M is a finitely generated A-module. Consequently it follows from Proposition (3.3) that we can replace A, B and M by A/R, B/RB and M/RM, where R is the radical of A. We may thus assume that A is reduced.

Let P be a prime ideal in A. It follows from Theorem (2.4) applied to the localizations A_P and M_P that M_P is a finitely generated A_P-module.Thus it follows from Proposition (3.4) that M is a finitely generated A-module.

Remark. Our approach in the first part of this section was to use an integral dependence argument allowing a globalization of the local results in Theorem (2.4) when the module was finite over an A-algebra essentially of finite type. For the rest of this section we will take a different approach, using the Zariski Main Theorem. Although we can only prove Theorem (3.5) in the particular case when B is an A-module of finite type, we think it is worth wile to present the methods involved in this approach.

3.6 Lemma. *Let A be a local ring and B a finitely generated A-algebra. Moreover, let M be a finitely generated faithful B-module. If there is an integer d such that $d = \dim_{\kappa(P)}(\kappa(P)\otimes_A M)$ for all prime ideals P of A, then we have that $\dim_{\kappa(P)}(\kappa(P)\otimes_A B)$ is finite for all prime ideals P of A.*

Proof. Since M is a finitely generated and faithful B-module we have that M_P is a faithful B_P-module. Write $C = B_P/\operatorname{ann}_{B_P}(M_P/PM_P)$. We have $PB_P \subseteq \operatorname{ann}_{B_P}(M_P/PM_P)$. It follows from Lemma (3.1) that $\operatorname{ann}_{B_P}(M_P/PM_P)$ is included in the radical of PB_P. Hence we have surjections

$$B_P/PB_P \to C \to (B_P/PB_P)_{\text{red}}, \tag{3.6.1}$$

where we for each ring R write R_{red} for the residue ring modulo its nilradical. The module M_P/PM_P is finitely generated and faithful C-module, and by assumption we have that $M_P/PM_P = \kappa(P)\otimes_A M$ is a finitely generated $\kappa(P)$-module. It follows from Lemma (3.2) that C is integral over $\kappa(P)$. The A-algebra B is of finite type, hence C is a $\kappa(P)$-algebra of finite type. Hence C is a finite $\kappa(P)$-module. It follows from the surjections (3.6.1) that $(B_P/PB_P)_{\text{red}}$ is a finite dimensional as $\kappa(P)$-vector space. Finite dimensional vector spaces satisfy the descending chain condition and are noetherian rings of Krull dimension zero. Consequently B_P/PB_P is noetherian with Krull dimension zero. Since B_P/PB_P in addition is finitely generated as a $\kappa(P)$-algebra we get that B_P/PB_P is finite dimensional as a $\kappa(P)$-vector space. We have proved our claim.

3.7 Theorem. *Let A be a ring and B a finitely generated A-algebra. Moreover, let M be a finitely generated B-module which is flat as an A-module. Assume that there is an integer d such that*

$$d = \dim_{\kappa(P)}(\kappa(P) \otimes_A M)$$

for all prime ideals P of A. Then M is a finitely generated A-module. In particular M is locally a free A-module.

Proof. By Proposition (3.4) it is sufficient to show that M_P is finitely generated over A_P for all prime ideals P of A. Hence we may assume that A is a local ring. Furthermore we may replace B with $B/\operatorname{ann}_B(M)$ and assume that M is a faithful B-module. It follows from Lemma (3.6) that $A \to B$ is quasi-finite, that is for all prime ideals P in A the $\kappa(P)$-vector space $\kappa(P) \otimes_A B$ is finite dimensional. By the Zariski Main Theorem ([R], Chapter 4, Theorem 1 and Corollary 2, p. 41 et seq.) we have a factorization

$$A \to C \to B \tag{3.7.1}$$

where the A-algebra C is a finite A-module and $C \to B$ has the property that for each prime ideal R in B there is an element f in C, not in the contraction of R to C, such that $C_f = B_f$.

We will next reduce to the case when A is a henselian ring. Recall ([N], Chapter VII or [R], Chapter VIII) that the Henselization A^h of a local ring A is a henselian local ring and that $A \to A^h$ is faithfully flat. It is clear that $A^h \otimes_A B$ is an A^h-algebra of finite type and that $A^h \otimes_A M$ is a finitely generated $A^h \otimes_A B$-module which is flat over A^h. If we have that $A^h \otimes_A M$ is finitely generated as an A^h-module it follows by faithfully flatness of $A \to A^h$ that M is a finitely generated A-module. Hence we can replace A by A^h.

We note that if a ring A has a decomposition $A = \prod_{i=1}^m A_i$, then an A-module M has a decomposition $M = \prod_{i=1}^m M_i$. Indeed, let $e_1, \dots, e_m$ be the idempotents in A corresponding to the decomposition $A = \prod_{i=1}^m A_i$. Write $M_i = e_i M$. It is readily checked that the injective map $\prod_{i=1}^m M_i \to M$ is surjective, hence we have a decomposition $M = \prod_{i=1}^m M_i$. When B is an A-algebra we obtain a decomposition $\prod_{i=1}^m B_i$ by A_i-algebras B_i.

Since we have assumed that A is henselian, we have that $C = \prod_{i=1}^m C_i$ where each C_i is a local ring which is a finite A-module. It follows by the above remark that $B = \prod_{i=1}^m B_i$, and that $M = \prod_{i=1}^m M_i$, where M_i is a finitely generated B_i-module.

Let P be the maximal ideal of A. We shall prove that $\dim_{\kappa(P)}(\kappa(P) \otimes_A M_i) = \dim_{\kappa(Q)}(\kappa(Q) \otimes_A M_i)$ for all prime ideals Q of A. First we note that each M_i is a flat A-module, as a direct summand of M, and since $\dim_{\kappa(P)}(\kappa(P) \otimes_A M)$ is finite we have that $\dim_{\kappa(P)}(\kappa(P) \otimes_A M_i)$ is finite. Let $F \to M_i$ be a homomorphism from a free A-module such that the residue map $\kappa(P) \otimes_A F \to \kappa(P) \otimes_A M_i$ is an isomorphism. It follows from Proposition (2.1) that $F \to M_i$ is injective with a cokernel which is flat over A. Hence we have that $\kappa(Q) \otimes_A F \to \kappa(Q) \otimes_A M_i$ is injective for all primes Q of A. It follows that $\dim_{\kappa(Q)}(\kappa(Q) \otimes_A M_i) \geq \dim_{\kappa(P)}(\kappa(P) \otimes_A M_i)$ for all prime ideals Q of A. Furthermore we have that

$$\sum_{i=1}^m \dim_{\kappa(Q)}(\kappa(Q) \otimes_A M_i) = \dim_{\kappa(Q)}(\kappa(Q) \otimes_A M)$$
$$= \dim_{\kappa(P)}(\kappa(P) \otimes_A M) = \sum_{i=1}^m \dim_{\kappa(P)}(\kappa(P) \otimes_A M),$$

from which it follows that $\dim_{\kappa(Q)}(\kappa(Q) \otimes_A M_i) = \dim_{\kappa(P)}(\kappa(P) \otimes_A M_i)$ for all prime ideals Q of A. If we have that $\dim_{\kappa(P)}(\kappa(P) \otimes_A M_i) = 0$ it follows ([M], Theorem 4.9, p. 27) that $M_i = 0$.

We shall finally prove that $C_i = B_i$ when $\dim_{\kappa(P)}(\kappa(P) \otimes_A M_i) > 0$. First we note that we can not have that $\kappa(P) \otimes_A B_i = 0$ when $\dim_{\kappa(P)}(\kappa(P) \otimes_A M_i) > 0$, because we have a surjection $B_i^n \to M_i$, and thus a surjection $(\kappa(P) \otimes_A B_i)^n \to \kappa(P) \otimes_A M_i$. Since $\kappa(P) \otimes_A B_i$ is not zero there is a prime ideal R in B_i that contracts to the maximal ideal P of A. Denote by Q the contraction of R to C_i. Then Q contracts to P in A. Since C_i is a finite A-module and thus integral over A the only ideal that contracts to the maximal ideal P in A is the maximal ideal in C_i ([A-M], Corollary 5.9, p. 61). Hence R contracts to the maximal ideal in C_i. It follows from the properties of the factorization (3.7.1), that there is an element $f \in C_i$ not in the maximal ideal such that $(C_i)_f = (B_i)_f$. That is $C_i = B_i$ as we wanted to prove.

We have proved that we have a product decomposition $M = \prod_{i=1}^m M_i$ such that each M_i is a finite B_i-module, and where $B_i = C_i$ whenever $M_i \neq 0$. Since each C_i is a finitely generated A-module it follows that M is a finitely generated A-module, and we have proved first part of the Theorem. The last part follows from Corollary (1.5).

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