HILBERT SCHEMES OF POINTS ON SMOOTH SURFACES

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Introduction

These notes were prepared for the mini-course on Hilbert scheme of points given at IMPA (Instituto Nacional de Matemática Pura e Aplicada, Brazil) during two hot weeks of January 2004.

The purpose of these notes was to introduce the Hilbert schemes of points to graduate students in Algebraic Geometry knowing the basics about schemes.

Hilbert schemes are certain parameter schemes, that is objects defined by universal properties and I use the first three sections to explain and explore this. The representability theorem due to A. Grothendieck we will not prove but take for granted. In the first section I set up the language of representable functors, and in Section 2 the definition of the Hilbert functor of points is given. In the third section I discuss the relationship between the parameter schemes given a pair of ambient schemes, one a subscheme of the other.

In the fourth section I recall the important norm map developed by P. Deligne. The norm map is a morphism from the Hilbert scheme of n-points on X to the symmetric quotient $\mathrm{Sym}^n(X)$ that point wise forgets the structure but remembers the multiplicity.

In the fifth section I discuss the Theorem of J. Fogarty that states that the Hilbert scheme of n-points on a smooth surface X is itself smooth and birational to $\mathrm{Sym}^n(X)$.

In the sixth section the notion of local Lie algebras is introduced and I write down the oscillator representation of the Heisenberg algebra.

In the last section I recall the operators and the Theorem of H. Nakajima which relates the cohomologies of the Hilbert scheme of points to the Heisenberg representation of the cohomologies of the surface.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

§1. - Representable functors

The idea of viewing schemes as functors goes back to A. Grothendieck. Even though the idea is well know and important in algebraic geometry the set up is not presented in the standard texts in Algebraic Geometry. Our references are ([19]), the appendix in EGA 1 [12] and [6].

1.1. Notation. We fix a base scheme S and we consider some category $\mathbf{C}/S$ of schemes over S (typically noetherian schemes or affine schemes). When we in the sequel refer to a scheme we always mean an object in the fixed category $\mathbf{C}/S$. To any scheme X over S we have the *contravariant* functor $h_X : \mathbf{C}/S \to \text{Sets}$ that maps an object T to the set $h_X(T) = \text{Hom}_S(T, X)$, and any morphism $f : T' \to T$ of schemes gives by composition a map of sets

$$h_X(f) : h_X(T) \to h_X(T').$$

Let $F : \mathbf{C}/S \to \text{Sets}$ be a given contravariant functor, and let X be an S-scheme. An element $\xi \in F(X)$ is an *X-valued point of* F and determines a natural transformation

$$\Phi_\xi : h_X \longrightarrow F$$

in the following way. For any scheme T, and any $f \in h_X(T)$ we let $\Phi_\xi(T)(f) := F(f)(\xi) \in F(T)$.

1.2. Yoneda Lemma. *The assignment $\xi \mapsto \Phi_\xi$ described above determines a bijection between the set $F(X)$ and the set of natural transformations $h_X \to F$.*

Proof. We have that each X-valued point ξ of F determines a natural transformation Φ_ξ. Let now $\Phi : h_X \to F$ be a natural transformation, and set $\xi := \Phi(\text{id}_X)$, where $\text{id}_X : X \to X$ is the identity morphism. It is readily checked that the two assignments are mutually inverses, proving the claim. $\square$

1.2.1. Clearly a morphism of schemes $\varphi : X \to Y$ induces a morphism of functors $\Phi : h_X \to h_Y$ and it follows that we can embed the category of schemes in the larger category of functors. By the Yoneda Lemma any morphism $h_X \to h_Y$ comes from a morphism of schemes $X \to Y$. We say that the category $\mathbf{C}/S$ is faithfully embedded in the category of contravariant functors from $\mathbf{C}/S$ to sets.

1.3. Representing pair. Let $F : \mathbf{C}/S \to \text{Sets}$ be a contravariant functor. Assume that ξ is an X-valued point of F such that the induced natural transformation $\Phi_\xi : h_X \to F$ is a bijection for all objects T. Then we say that the pair (X, ξ) *represents* F.

1.4. Note. If F is represented by (X, ξ), then the pair is unique up to canonical isomorphism.

1.5. Example. Let $S = \text{Spec}(A)$ be an affine scheme. A contravariant functor F from the category $\mathbf{C}/S$ of affine schemes is the same as a *covariant* functor from the category of A-algebras. Consider the co-variant functor F from the category of A-algebras to sets that maps an A-algebra B to the set

$$F(B) = \{A\text{-module morphisms } M \to B\}.$$

Recall that for a B-module N we have that the set of B-module maps from M to B is the same as B-algebra homomorphisms from $\text{S}_B(N)$ to B, where $\text{S}_B(N)$ denotes the quotient of the full-tensor algebra $\mathrm{T}_B(N)$ with the two sided ideal generated by elements of the form $x \otimes y - y \otimes x$, with $x, y \in N$. The B-algebra $\mathrm{S}_B(N)$ is a commutative ring with a canonical B-module map $\xi : N \to \mathrm{S}_B(N)$. Note that the map $\xi : M \to \mathrm{S}_A(M)$ described above is an $\mathrm{S}_A(M)$-valued point of F and consequently we have a natural transformation

$$h_{\mathrm{S}_A(M)} \longrightarrow F.$$

Furthermore, note that $\mathrm{S}(M) \otimes_A B$ is canonically isomorphic to $\mathrm{S}_B(M \otimes_A B)$. It follows that the set of A-algebra homomorphisms from $\mathrm{S}_A(M)$ to B equals the set of B-algebra homomorphisms from $\mathrm{S}_B(M \otimes_A B)$ to B. And finally as noted above the set of B-algebra homomorphisms from $\mathrm{S}_B(M \otimes_A B)$ to B equals the set B-module maps from $M \otimes_A B$ to B, or equivalently the set of A-module maps from M to B. Summa summarum we have that the pair $(\mathrm{S}_A(M), \xi)$ represents F.

1.5.1. We are of course familiar with several other affine schemes that are determined by its defining properties. To mention some we have fraction rings $S^{-1}A$ of a ring with respect to a multiplicatively closed set $S \subseteq A$, the tensor product $B \otimes_A C$ of two A-algebras or modules B and C, the full tensor algebra $T_A(M)$ of an A-module M, and the enveloping algebra $\mathcal{U}(M)$.

1.6. Fibered product. Let $\Phi_i : G_i \to F$ be two $(i = 1, 2)$ natural transformations of functors. The fibered product $G_1 \times_F G_2$ is the functor that to a scheme T assigns the set

$$G_1 \times_F G_2(T) = \{(x_1, x_2) \in G_1(T) \times G_2(T) \mid \Phi_1(x_1) = \Phi_2(x_2) \text{ in } \quad F(T)\}.$$

1.6.1. Subfunctors. If a natural transformation $\Phi : G \to F$ is such that $\Phi(T)$ is injective for all objects T, we say that G is a subfunctor of F. Note that if $h_X \to h_Y$ is a subfunctor, it does follow that the corresponding morphism of schemes $X \to Y$ is a monomorphism, but not that X is a subscheme of Y. It is of interest to formulate the notion of a subscheme in functorial language. Let $G \to F$ be a subfunctor, and let X be a scheme and ξ an element in $F(X)$. Assume that we have a morphism of schemes $i : X_\xi \to X$ such that $F(i)(\xi)$ is in $G(X_\xi) \subseteq F(X_\xi)$. Then we have an induced natural transformation

$$h_{X_\xi} \longrightarrow h_X \times_F G \tag{1.6.2}$$

as $(i, F(i)(\xi))$ is an X_ξ-valued point of $h_X \times_F G$. If the induced transformation (1.6.2) is bijective for all schemes T we say that X_ξ represents $h_X \times_F G$.

1.7. Definition. Let $G \to F$ be a subfunctor. If for any given scheme X and any $\xi \in F(X)$ there exists a locally closed immersion of schemes $X_\xi \subseteq X$ such that X_ξ represents $h_X \times_F G$, then we say that G is a locally closed subfunctor of F.

1.7.1. Remark. By replacing *locally closed* with *closed* or *open*, we obtain the notion of a subfunctor being closed or open, respectively.

1.7.2. Remark. The definition of a locally closed subfunctor G of F is equivalent with the statement that a morphism $f : T \to X$ factors via $X_\xi \subseteq X$ if and only if $F(f)(\xi) \in G(T)$.

1.7.3. Remark. Let F be represented by (X, ξ), and let $G \to F$ be a locally closed, closed or open subfunctor. Then there is a subscheme $X_\xi \subseteq X$ representing $h_X \times_{h_X} G = G$. Thus, if G is a locally closed, closed or open subfunctor of a functor F represented by a scheme X, then G is represented by a locally closed, closed or open subscheme X_ξ of X.

1.8. Open cover. Let $\{F_i\}_{i\in I}$ be a family of open subfunctors of a functor F. We say that $\{F_i\}_{i\in I}$ is an *open cover* if for any scheme X, and any $\xi \in F(X)$, the corresponding open subschemes $\{X_{\xi_i}\}_{i\in I}$ form an open covering of the scheme X.

1.8.1. A functor F is a *Zariski sheaf* if for each object T, and any open cover $\{T_i\}_{i\in I}$ of the scheme T the induced sequence of sets

$$F(T) \longrightarrow \prod_{i\in I} F(T_i) \longrightarrow \prod_{i,j\in I} F(T_i \cap T_j) \tag{1.8.1}$$

is exact. That is, the leftmost arrow is injective and its image is the set

$$\{(x_i)_{i\in I} \mid \rho(x_i) = \rho'(x_j) \mid i,j \in I\}$$

where ρ and ρ' are the maps induced by the two inclusions $T_i \cap T_j \to T_i$ and $T_i \cap T_j \to T_j$.

1.8.2. Note that representable functors h_X are Zariski sheaves.

1.9. Proposition. *Let F be a Zariski sheaf, and let $\{F_i\}_{i\in I}$ be an open covering of representable functors. Then F itself is representable.*

Proof. Let (X_i, ξ_i) represent the open subfunctors F_i, for $i \in I$. We then have that $\{h_{X_i}\}_{i\in I}$ is an open cover of F. For each pair of indices i, j there is an open subset $X_{i,j}$ of X_i representing the open subfunctor $h_{X_i} \times_F h_{X_j}$ of h_{X_i}. The canonical isomorphism $\rho_{i,j} : X_{i,j} \to X_{j,i}$ is such that $\rho_{j,k} \circ \rho_{i,j} = \rho_{i,k}$ for any $i, j, k \in I$. Consequently we can glue together X_i with X_j along the identification $X_{i,j} = X_{j,i}$ and obtain a scheme X. As F is a Zariski sheaf the X_i-valued points ξ_i form an X-valued point ξ. We claim that the induced natural transformation $\Phi_\xi : h_X \to F$ is an isomorphism.

Let $f : T \to X$ be a morphism of schemes, and let $f_i : T_i \to X_i$ denote the induced morphism of open sets with $T_i = f^{-1}(X_i)$. As $f_i \in h_{X_i}(T_i) = F_i(T_i) \subseteq F(T_i)$ injectivity of Φ_ξ follows by the left exactness of (1.8.1).

To see the surjectivity of Φ_ξ let η be an T-valued point of F, and let $U_i \subseteq T$ be the open subset such that h_{U_i} represents $h_T \times_F h_{X_i}$. Let $\eta_i \in F(U_i)$ denote the image of η by the map $F(T) \to F(U_i)$. As the inclusion $U_i \to T$ factors via itself we have, by Remark (1.7.2) that $\eta_i \in F_i(U_i) = h_{X_i}(U_i)$. By the right exactness of (1.8.1) the morphisms $\eta_i : U_i \to X_i$ agree on intersections and form a morphism $\eta' : T \to X$. As both η' and η restrict to η_i over U_i they are equal by the left exactness of (1.8.1), and surjectivity follows. □

1.9.1. Remark. Observe that Zariski sheaves are determined by their values on affine schemes. Thus when studying a Zariski sheaf (and separated schemes) we can always assume that our objects are affine.

1.10. Grassmannians. Let $\mathcal{F}$ be a quasi-coherent sheaf on S, and let $\mathcal{F}_T$ denote the pull-back of $\mathcal{F}$ along a given morphism $f : T \to S$. For each fixed positive integer n we define the Grassmann functor by assigning any scheme T to the set $Gr^n_{\mathcal{F}}(T)$ of coherent $\mathcal{O}_T$-module quotients $\mathcal{F}_T \to Q$ that are locally free of rank n. We identify two quotients Q and Q' if they have the same kernel.

1.11. Proposition. *Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme S. We have that the Grassmann functor $Gr^n_{\mathcal{F}}$ is representable by a scheme.*

Proof. It is clear that the Grassmann functor is a Zariski sheaf, and consequently, by Proposition (1.9), we may assume that $S = \mathrm{Spec}(A)$ is an affine scheme. Let $x = x_1, \dots, x_n$ be global sections of $M = \Gamma(S, \mathcal{F})$, and consider the induced morphism of $\mathcal{O}_S$-modules $V = \oplus \mathcal{O}_S \to \mathcal{F}$. We define the subfunctor E of $Gr^n_{\mathcal{F}}$ by defining the T-valued points of E as the set

$$E(T) = \{Q \in Gr^n_{\mathcal{F}} \mid V_T \to \mathcal{F}_T \to Q \text{ is surjective}\}.$$

We claim that E is an open subfunctor of $Gr^n_{\mathcal{F}}$. Indeed, let Q be a T-valued point of $Gr^n_{\mathcal{F}}$, and let $T_Q \subseteq T$ denote the open subset where the map $V_T \to Q$ is surjective. Now, by using the fact that determinants are compatible with pull-backs gives that a morphism $g : U \to T$ factors via $T_Q \subseteq T$ if and only if $V_U \to Q_U$ is surjective.

It is furthermore clear that the open functors E for different n-choices of global sections of $\mathcal{F}$ cover the Grassmann functor. Hence, by (1.9) it suffices to show that E is representable. It will turn out that E is representable by an affine scheme, and consequently we may assume that situation is affine. Let B be an A-algebra, and consider a $\mathrm{Spec}(B)$-valued point of E;

$$V_B = \oplus_{i=1}^n B \longrightarrow M_B = M \otimes_A B \longrightarrow Q.$$

As the composition above is surjective it is an isomorphism. By identifying V_B with Q and noting that the sequence above splits, we get that the $\mathrm{Spec}(B)$-valued points of E is precisely the set of B-module homomorphisms $M_B/V_B \to V_B$. Moreover, as V_B is free of finite rank we have that a B-module morphism $N \to V_B$ is equivalent with $N \otimes_B V_B^* \to B$, where $V_B = \mathrm{Hom}_B(V_B, B)$ is the dual module. Then finally (Example 1.5) one observes that the set $\mathrm{Hom}_B(M_B/V_B \otimes_B V_B^*, B)$ is the same as A-algebra homomorphisms from $\mathrm{S}(M/V \otimes_A V^*)$ to B. Hence $\mathbf{E} = \mathrm{Spec}(\mathrm{S}(M/V \otimes_A V^*))$ represents E. $\square$

1.13. Remark. When $n = 1$ we usually write $\mathbf{P}(\mathcal{F})$ for the scheme representing $Gr^1_{\mathcal{F}}$, and the universal element or the universal quotient we denote by $\mathcal{O}(1)$. When $\mathcal{F}$ is free of finite rank $r+1$ the scheme representing $Gr^1_{\mathcal{F}}$ is simply denoted by $\mathbf{P}^r_S$ - the projective r-space over S.

§2. - Hilbert functor of points

We will here state a particular case of A. Grothendieck's fundamental result about the existence of Hilbert schemes ([7] and [30]).

2.1. The Hilbert functor. We now fix a scheme $X \to S$ and a positive integer n. We define the Hilbert functor of n-points on X as the contravariant functor from $\mathbf{C}/S \to$ Sets, that to any S-scheme T assigns the set

$$\mathrm{Hilb}^n_X(T) = \{\text{closed subschemes } Z \subseteq X \times_S T \text{ such that the projection is flat, finite and of rank } n.$$

2.1.1. Remark. A flat and finitely generated module over a local ring is a free module. The T-valued points of Hilb^n_X are therefore closed subschemes $Z \subseteq X \times_S T$ that are finite over T such that $\mathcal{O}_Z$ is a locally free $\mathcal{O}_T$-module of rank n.

2.1.2. Remark. Let $X \to \mathrm{Spec}(k)$ be a scheme defined over a field k, and let $x_1, \dots, x_n$ be distinct points of X with residue field k. Then the union of $x_1, \dots, x_n$ is a well-defined subscheme of X - and a $\mathrm{Spec}(k)$-rational point of the Hilbert functor Hilb^n_X.

2.2. Example. The affine line. Let $X = \operatorname{Spec}(A[T]) \to S = \operatorname{Spec}(A)$ be the affine line over a commutative ring A. Let E be the free A-module of rank n. Assume that there is an ideal $I \subset A[T]$ such that the quotient $A[T]/I = E$. The A-module structure is given, hence the variable T is mapped to an A-linear endomorphism on E. By the Cayley-Hamilton Theorem we have that the endomorphism satisfies its characteristic polynomial $c(T)$. Consequently we obtain a surjective morphism of A-algebras $A[T]/(c(T)) \to E$. As $c(T)$ is monic of degree n it follows that the ideal I is generated by $c(T)$. Note that the residue classes of $1, T, \dots, T^{n-1}$ form an A-module basis of E. It follows that if $A[T]/I$ is *locally* free of rank n it is actually free. Furthermore as $B \otimes_A A[T] = B[T]$ for any A-algebra B we see that the Hilbert functor Hilb^n_X is represented by the affine scheme

$$\operatorname{Spec}(\mathrm{S}_A(E)) \backsimeq \operatorname{Spec}(A[t_1, \dots, t_n])$$

where the coordinate ring $\mathrm{S}_A(E)$ is isomorphic to the polynomial ring in n-variables $t_1, \dots, t_n$ over A. The universal family is given by the principal ideal generated by $T^n + t_1T^{n-1} + \dots + t_n$ in $A[t_1, \dots, t_n][T]$.

2.3. Theorem. *[A. Grothendieck] Let $X \to S$ be a projective morphism of noetherian schemes. Then the Hilbert functor* Hilb^n_X *is representable by a projective scheme* $\mathbf{H}^n_X$.

2.3.1. Remark. The statement above is a special case of a result of A. Grothendieck which says that if we have a projective scheme X with a very ample sheaf $\mathcal{O}(1)$ then the Hilbert functor Hilb^P_X is representable by a projective scheme, for any numerical polynomial P. When the polynomial $P = n$ is the constant polynomial any fiber Z is a zero dimensional scheme. And consequently as the higher cohomology vanishes we have that the vector space dimension of its global sections is $\Gamma(Z, \mathcal{O})$ is precisely n. Furthermore, as the space of global sections is independent of the actual embedding we do not need to choose a very ample sheaf $\mathcal{O}(1)$.

2.3.2. Remark. The proof of the above result does not give information how to construct the scheme $\mathbf{H}^n_X$, only that it exists.

2.3.3. Remark. To see the similarities with the Grassmannian and the Hilbert functor consider the affine case $\operatorname{Spec}(F) = X \to \operatorname{Spec}(A) = S$. We then have a obvious natural transformation $\operatorname{Hilb}^n_X \to \operatorname{Grass}^n_{\mathcal{O}_X}$.

§3.- Open and closed immersions and the Hilbert functor

In this section we will study the relation between Hilbert scheme of points on Y with the Hilbert scheme of X, given $Y \to X$ is a closed or an open subscheme.

3.1. Fitting ideals. Let A be a noetherian ring and M a finitely generated A-module. Choose a free presentation of M.

$$A^r \xrightarrow{N} A^s \longrightarrow M \longrightarrow 0.$$

For each fixed integer n the $(s-n)$-minors of N generate an ideal $F_n(M) \subseteq A$, the n'th *Fitting ideal* of M. We remark that the ideal $F_n(M)$ is independent of the chosen free presentation, a verification we leave for the reader. Clearly we have the ascending chain $0 = F_{-1}(M) \subseteq F_0(M) \subseteq \dots \subseteq F_s(M) = A$ of ideals in A where we define $F_{-1}(M) = 0$ and $F_s(M) = A$.

3.2. Proposition. *Let A be a noetherian local ring and M a finitely generated A-module. Then n is the minimal number of generators for M if and only if $F_{n-1}(M)$ is a proper ideal and $F_n(M) = A$. Furthermore we have that M is free of finite rank n if and only if $F_{n-1}(M) = 0$ and $F_n(M) = A$.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of A. If n is the minimal number of generators of M then it follows by Nakayama's Lemma that we have a free presentation of the form

$$A^r \xrightarrow{\ N\ } A^n \longrightarrow M \longrightarrow 0, \tag{3.2.1}$$

for some integer r. Furthermore, as the $A/\mathfrak{m}$-vector space $M/\mathfrak{m}M$ is of dimension n it follows that the coefficients describing the map N are all in the maximal ideal $\mathfrak{m}$ of A. The 1-minors of N generate by definition $F_{n-1}(M)$ which therefore is a proper ideal. If in addition the module M were free then we could take a presentation with $N = 0$.

In order to prove the converse we assume now that $F_{n-1}(M)$ is a proper ideal with $F_n(M) = A$, and we chose a free presentation $A^p \to A^q \to M$. Note that if the matrix N describing the map $A^p \to A^q$ contains an invertible coefficient, then we can construct a map $A^{p-1} \to A^{q-1}$ with the same cokernel M. As the Fitting ideals were independent of the resolution we may assume that the matrix N has only coefficients that are not invertible. If now $q > n$ there is an invertible $(q-n)$ block in N as $F_n(M) = A$, which contradicts our assumption. Hence $q \leq n$. However if q is strictly smaller than n then we would have $F_{n-1}(M) = A$, and consequently $q = n$. Thus we have reduce the situation to having a resolution of the form (3.2.1) where the coefficients of N are in the maximal ideal. As we have the vector space isomorphism $(A/\mathfrak{m})^n \backsimeq M/\mathfrak{m}M$ the minimal number of generators of M is n. Furthermore, if $F_{n-1}(M) = M$ the in particular the generators of $F_{n-1}(M)$ are zero. That is the 1-minors of N are zero and we have that M is free. □

3.2. If $\mathcal{F}$ is a coherent sheaf on a noetherian scheme X, the the n'th Fitting ideal sheaf is defined locally on affines by the n'th Fitting ideal of the module of sections of $\mathcal{F}$.

3.3. Proposition. *Let $Y \subseteq X$ be a closed immersion of schemes. Then* Hilb^n_Y *is a closed subfunctor of* Hilb^n_X.

Proof. As the composition of closed immersions is closed it follows immediately that Hilb^n_Y is a subfunctor of $F = \mathrm{Hilb}^n_X$. Let T be a scheme and Z an element of $F(T)$. Denote by Z' the intersection of Z and $Y \times_S T$. Then Z' is closed in both Z and $Y \times_S T$. As the projection $p : Z \to T$ is finite we have a surjective morphism

$$p_*\mathcal{O}_Z \longrightarrow p_*\mathcal{O}_{Z'} \tag{3.3.1}$$

of coherent $\mathcal{O}_T$-modules. Let $T_Z \subseteq T$ be the closed subscheme defined by the $(n-1)$'th Fitting ideal of $p_*\mathcal{O}_{Z'}$. As $p_*\mathcal{O}_Z$ is of rank n the surjective morphism above combined with Proposition (3.2) imply that the restriction of (3.3.1) to T_Z is an isomorphism. In particular we have that $Z'' = Z \times_T T_Z$ is closed in $Y \times_T T_Z$, flat and of finite rank n over T_Z. Consequently we have an induced natural transformation

$$h_{T_Z} \longrightarrow h_T \times_F \mathrm{Hilb}^n_Y,$$

which we claim is an isomorphism. Let $f : U \to T$ be a morphism of schemes. Let Z_U denote the pull-back of the fixed T-valued point Z of Hilb^n_X. Assume that Z_U

is an U-valued point of Hilb^n_Y. We then have that $Z'_U = Z_U$. We need to show that f factors via the closed immersion $T_Z \to T$. As $p_U : Z_U \to U$ is flat and finite of rank n, we have by Proposition (3.2) that the $n-1$'th Fitting idealsheaf of $p_{U*}\mathcal{O}_{Z_U} = p_{U*}\mathcal{O}_{Z'_U}$ is zero. Furthermore as the Fitting ideal sheaf of $p_{U*}\mathcal{O}_{Z'_U}$ is the pull-back of the Fitting ideal sheaf of $p_*\mathcal{O}_{Z'}$ on T, we obtain our desired factorization $f : U \to T_Z$. $\square$

3.4. Example. Consider the following situation. We fix the closed subscheme $Y = \mathrm{Spec}(A[T]/(T^N))$ of the affine line $X = \mathrm{Spec}(A[T])$. The Hilbert scheme of points on X we have as the spectrum of the polynomial ring $H = A[t_1, \dots, t_n]$ in n-variables over A. Instead of computing the Fitting ideal, which would require a free resolution, we do the following. The universal family of n-points on X is given by the polynomial $\Delta = T^n + t_1T^{n-1} + \cdots + t_n$. Introduce new $N-n$ variables $u_1, \dots, u_{N-n}$ and the polynomial

$$U_{N-n}(T) = T^{N-n} + u_1T^{N-n-1} + \ldots + u_{N-n}.$$

Then in the polynomial ring in the variables $t = t_1, \dots, t_n$, $u = u_1, \dots, u_{N-n}$ and T we consider the product $U(T)\Delta(T)$ and require it to equal T^N. That is

$$U_{N-n}(T)\Delta(T) = T^N + (u_1 + t_1)T^{N-1} + \cdots + t_nu_{N-n} = T^N.$$

The coefficients of the product polynomial above are functions in t and u. These functions can be recursively solved eliminating the variables $u_1, \dots, u_{N-n}$. After doing so we end up with n functions in $t_1, \dots, t_n$ that generate an ideal I in $H = A[t_1, \dots, t_n]$. The Spectrum of the quotient ring H/I is then our Hilbert scheme of n-points on Y.

3.4.1. Remark. The above example is also interesting when $A = K$ is a field as the ambient scheme $X = \mathrm{Spec}(K[T]/(T^N))$ is simply a fat point on the line. Furthermore, as X is projective it follows that the Hilbert scheme of n-points on X is projective. Thus the projective Hilbert scheme is only one fat point in affine n-space.

3.5. Incidence schemes. We will later need certain incidence schemes of the Hilbert schemes. Let $m \geq n > 0$ be two integers and consider the subfunctor $\mathcal{H}^{m,n}_X$ of the product $\mathrm{Hilb}^m_X \times \mathrm{Hilb}^n_X$ whose T-valued points are

$$\mathcal{H}^{m,n}_X(T) = \{(Z_1, Z_2) \in \mathrm{Hilb}^m_X(T) \times \mathrm{Hilb}^n_X(T) \mid Z_2 \subseteq Z_1\}.$$

We are thus considering pairs (Z_1, Z_2) where both are closed subschemes of $X \times_S T$, Z_1 of length m and Z_2 of length n - such that Z_2 is a closed subscheme of Z_1.

3.6. Lemma. *The functor $\mathcal{H}^{m,n}_X$ is a closed subfunctor of* $\mathrm{Hilb}^m_X \times \mathrm{Hilb}^n_X$.

Proof. Let $F = \mathrm{Hilb}^m_X \times \mathrm{Hilb}^n_X$ and let $\xi = (Z_1, Z_2)$ be a T-valued point of F. Note that Z_2 being closed in Z_1 is then equivalent with Z_2 being a T-valued point of $\mathrm{Hilb}^n_{Z_1}$ (as a functor from T-schemes to sets). As $Z_1 \subseteq X \times_S T$ is closed it follows by Proposition (3.9) that $\mathrm{Hilb}^n_{Z_1}$ is closed in $\mathrm{Hilb}^n_{X \times_S T}$, and closedness of $\mathcal{H}^{m,n}_X$ follows being the product of two closed subfunctors. $\square$

3.7. Open immersions. We will in this section in detail explore the open immersion $\mathrm{Hilb}^n_U \subseteq \mathrm{Hilb}^n_X$ coming from certain open immersions $U \subseteq X$.

3.8. Proposition. *Let E be an A-algebra which is free of finite rank n as an A-module. For any $x \in E$ we let $d(x) \in A$ denote the determinant of the A-linear multiplication map $e \mapsto ex$ on E. For any homomorphism of rings $\varphi : A \to B$ the following two statements are equivalent.*

1. *The induced map $E \otimes_A B \to E_x \otimes_A B$ is an isomorphism.*
2. *The homomorphism $\varphi : A \to B$ factors via $A \to A_{d(x)}$.*

In particular we have that $E \otimes_A A_{d(x)} \to E_x \otimes_A A_{d(x)}$ is an isomorphism.

Proof. The morphism $E \otimes_A B \to E_x \otimes_A B$ we obtain by localizing the element $x \otimes 1$. A localization is an isomorphism if and only if the element we are inverting already is invertible. Thus the first map (1) is an isomorphism if and only if the determinant $\det(x \otimes 1) = d(x) \otimes 1$ is invertible. But the statement that the determinant $d(x)$ is invertible is equivalent with (2).

3.9. Principal open subschemes. Let $s : \mathcal{O}_X \to \mathcal{L}$ be a global section of an invertible sheaf $\mathcal{L}$ on a scheme X. We then have the open subset $X_s \subseteq X$ where the section is non-vanishing. We will explore how the Hilbert scheme of points on X_s is related to the Hilbert scheme of points on X.

Let Z be an H-valued point of Hilb^n_X. As Z is closed in $X \times_S H$ we have the two projections from Z to X and H. The pull-back of $s : \mathcal{O}_X \to \mathcal{L}$ along the projection $p : Z \to X$ gives a global section $p^* s : \mathcal{O}_Z \to p^*\mathcal{L}$. The other projection $q : Z \to H$ is flat, finite and of rank n and consequently we have that $q_*p^*s : q_*\mathcal{O}_Z \to q_*p^*\mathcal{L}$ is an $\mathcal{O}_H$-module map of locally free sheaves of rank n. Let

$$d(s) : \wedge^n q_*\mathcal{O}_Z \longrightarrow \wedge^n q_* p^* \mathcal{L} \tag{3.9.1}$$

denote the induced map. We have that $d(s)$ is a global section of the invertible $\mathcal{O}_H$-module $\mathcal{N} = \mathcal{H}om(\wedge^n q_*\mathcal{O}_Z, \wedge^n q_*p^*\mathcal{L})$.

3.10. Proposition. *Let $s : \mathcal{O}_X \to \mathcal{L}$ be a global section of an invertible sheaf $\mathcal{L}$ on X. Assume that H represents Hilb^n_X. Then the Hilbert functor of n-points on X_s, the open subscheme of X where s does not vanish is represented by $H_{d(s)}$, the open subscheme of H where $d(s)$ (3.9.1) does not vanish. The universal family is the restriction of the universal family of points on X.*

Proof. First we show that the universal family $Z \to H$ restricted to the open subscheme $H_d = H_{d(s)} \subseteq H$ is an H_d-valued point of the Hilbert functor of n-points on X_s. We clearly have that the restriction $Z_d \to H_d$ is flat, finite and of rank n. We need to check that Z_d is closed in $X_s \times H_d$ and not only in $X \times H_d$. However that we can do locally, and is then a consequence of Proposition (3.8). We thus have an induced map

$$\mathrm{Hom}_S(-, H_d) \longrightarrow \mathrm{Hilb}^n_{X_s}, \tag{2.3.1}$$

and we will construct an inverse to (2.3.1). Let T be an S-scheme and let Z' be an T-valued point of $\mathrm{Hilb}^n_{X_s}$. As the projection is finite it follows that Z' also is closed in $X \times_S T$ - and consequently Z' is an T-valued point of Hilb^n_X. Hence there exist a unique morphism $f : T \to H$ such that the pull-back of the universal family $Z \to H$ along f is Z'. To prove the proposition it suffices to show that the morphism $f : T \to H$ factors vi H_d. That statement is clearly local and using Proposition (3.8) again, we see that we have a factorization $f : T \to H_d$. □

3.11. Example. Let $U = \operatorname{Spec}(A[T]_f)$ be a principal basic open of the affine line $\operatorname{Spec}(A[T])$. The Hilbert scheme of points on the affine line we know from Example (2.2). We will however present the parameter scheme slightly differently here. Let $H = \operatorname{Spec} A[\sigma_1, \dots, \sigma_n]$, where

$$A[\sigma_1, \dots, \sigma_n] = A[t_1, \dots, t_n]^{\mathfrak{S}_n}$$

is the invariant ring of the polynomial ring in n-variables under the action of the symmetric group $\mathfrak{S}_n$-permuting the variables. The $\sigma_1, \dots, \sigma_n$ are the elementary symmetric functions, well-known to be algebraically independent over A. The universal family is the quotient of the ideal $(\Delta) \subset A[T, \sigma_1, \dots, \sigma_n]$, where

$$\Delta = T^n - \sigma_1 T^{n-1} + \cdots + (-1)^n \sigma_n.$$

To describe the Hilbert scheme of n-points on $\operatorname{Spec}(A[T]_f)$ we need, using the Proposition above, to compute the determinant of the multiplication map $e \mapsto ef$ on

$$E = A[T, \sigma_1, \dots, \sigma_n]/(\Delta).$$

We claim that the determinant $d(f) = f(t_1) \cdots f(t_n)$. Note that the universal polynomial Δ splits into linear factors $\Delta = \prod (T - t_i)$ when we do the ring extension

$$A[\sigma_1, \dots, \sigma_n] \subseteq A[t_1, \dots, t_n].$$

It easy to verify that the characteristic polynomial of the multiplication map with T on $A[t_1, \dots, t_n]/(\Delta)$ is $\Delta = \prod (T - t_i)$. It then follows by the Spectral Theorem (see i.e. [22]) that multiplication with f has characteristic polynomial $\prod (T - f(t_i))$. In particular we have that the determinant $d(f) = f(t_1) \cdots f(t_n)$. Thus we have that $\mathbf{H}^n_U$, with $U = \operatorname{Spec}(A[T]_f)$ is the Spectrum of the ring

$$A[\sigma_1, \dots, \sigma_n]_{f(t_1) \dots f(t_n)}.$$

3.11.1. The above Proposition can be extended from one global section of a fixed invertible sheaf, to an arbitrary set of global sections of invertible sheaves. We will not pursue that direction, but simply state a consequence.

3.12. Theorem. *Let $X \to S$ be a projective morphism of noetherian schemes. Let $Y = \operatorname{Spec}(\mathcal{O}_{X,x})$ denote the Spectrum of the local ring at a point $x \in X$ - not necessarily a closed point. Then Hilb^n_Y is representable by a scheme given as the intersection of opens*

$$\mathbf{H}^n_Y = \cap_{x \in U} \mathbf{H}^n_U,$$

where the intersection runs over all open $U \subseteq X$ containing the point x.

Proof. See [29], for details.

3.13. Example. Let $Y = \operatorname{Spec}(k(T))$, where k is a field and $k(T)$ is the function field of the line $k[T]$. Note that Y is a point and in particular contains only the trivial subschemes; the empty set and Y itself. However, the Hilbert scheme $\mathbf{H}^n_Y$ is, using Example (2.4), the Spectrum of the fraction ring

$$k[\sigma_1, \dots, \sigma_n]_U \quad U = \{f(t_1) \dots f(t_n) \mid f(t) \in k[t] \setminus 0\}.$$

The above fraction ring is of dimension $(n-1)$, but contains no k-rational points.

3.14. Example. Let Y be the Spectrum of the local ring at the origin of the line $\mathrm{Spec}(k[T])$, where k is a field. We consider in detail the Hilbert scheme $\mathbf{H}^2_Y$. Note that the closed subschemes of Y have defining ideals (T^n), and in particular there is only one closed subscheme of length 2. As $\mathbf{H}^2_Y$ is obtained as a the Spectrum of a fraction ring of the polynomial ring in two variables we can draw a picture that describes $\mathbf{H}^2_Y$, and for that purpose we assume k to be algebraically closed. We observe the following. If $\mathfrak{m} = (\sigma_1 - \alpha, \sigma_2 - \beta)$ is a maximal ideal different from (σ_1, σ_2) we can always find $1 + aT \in k[T]$ such that

$$1 + a\alpha + a^2\beta = 0.$$

One solution to the quadratic equation above gives an $f = 1+aT$ not in the maximal ideal (T) of $k[T]$. Hence we have that any maximal ideal in $k[\sigma_1, \sigma_2]$ different from (σ_1, σ_2) becomes invertible in the fraction ring describing $\mathbf{H}^2_Y$. Furthermore, any $g \in k[\sigma_1, \sigma_2]$ that is irreducible in $k[t_1, t_2]$ is not of the form $f(t_1)f(t_2)$. Hence any prime ideal generated by such a polynomial g, not contained in (σ_1, σ_2), will become a maximal ideal in the coordinate ring of $\mathbf{H}^2_Y$.

§4.- The Norm map

We will in this section recall the Norm map which is a morphism from the Hilbert scheme of points on X to the symmetric quotient. The material is mostly taken from P. Deligne's article [5] and the the book [17] by B. Iversen, but also [3] and [10].

4.1. Divided powers. Let A be a ring and M an A-module. The ring of divided powers $\Gamma_A M$ is the polynomial ring $A[\gamma^n(x) \mid n \in \mathbf{N}, x \in M]$ modulo the relations

$$\gamma^0(x) - 1 \tag{3.1.1}$$
$$\gamma^n(ax) - a^n\gamma^n(x) \tag{3.1.2}$$
$$\gamma^n(x+y) - \textstyle\sum_{j=0}^n \gamma^j(x)\gamma^{n-j}(y) \tag{3.1.3}$$
$$\gamma^m(x)\gamma^n(x) - \tbinom{m+n}{m}\gamma^{m+n}(x) \tag{3.1.4}$$

for all non-negative integers m, n, all x, y in M, and all scalars $a \in A$. We have a natural grading on $\Gamma_A M$ where the residue class of the variable $\gamma^n(x)$ has degree n, and we write $\Gamma_A M = \oplus_{n \geq 0} \Gamma^n_A M$. The product on the algebra $\Gamma_A M$ we denote by $*$ and we refer to it as the external product.

4.2. Polynomial laws. A *polynomial law* g between two A-modules M and N is a collection of maps $g_B : M \otimes_A B \to N \otimes_A B$ indexed by A-algebras B such that for any A-algebra homomorphism $B \to B'$ we have the commutative diagram

$$\begin{tikzcd} M \otimes_A B \arrow[r, "g_B"'] \arrow[d] & N \otimes_A B \arrow[d] \\ M \otimes_A B' \arrow[r, "g_{B'}"'] & N \otimes_A B', \end{tikzcd}$$

where the vertical maps are the canonical ones. If, in addition, $g_B(bx) = b^n g_B(x)$ for all $b \in B$, for any A-algebra B, we say that the polynomial law g is *homogeneous* of degree n.

4.3. Example. We have $\Gamma^n_A M \otimes_A B = \Gamma^n_B(M \otimes_A B)$, and consequently the natural map $\gamma^n_B : M \otimes_A B \to \Gamma^n_B(M \otimes_A B)$ sending x to the residue class of $\gamma^n(x)$ is a polynomial law, homogeneous of degree n. The pair $(\Gamma^n_A M, \{\gamma\})$ is *universal* for homogeneous degree n polynomial laws from M.

The map $d_B : M \otimes_A B \to (T^n_B(M \otimes_A B))$ sending x to $x \otimes \cdots \otimes x$ in the n-fold tensor product of $M \otimes_A B$ gives a polynomial law of degree n. Consequently, by the universal property of $(\Gamma^n_A M, \{\gamma\})$ there exists a unique A-module homomorphism $\alpha_n : \Gamma^n_A M \to T^n_A M$ such that $d_B = (\alpha_n \otimes \mathrm{id}_B) \circ \gamma_B$.

By construction of d_B it is clear that the map factors via the invariant submodule $(T^n_B(M \otimes_A B))^{\mathfrak{S}_n}$, where the symmetric group $\mathfrak{S}_n$ of n-letters permutes the factors. When M is a flat A-module we have that

$$\alpha_n : \Gamma^n_A M \to (T^n_A M)^{\mathfrak{S}_n}$$

is an A-module isomorphism [26]. Note that $\alpha_n(\gamma(x)) = x \otimes \cdots \otimes x$.

4.4. Norms. A polynomial law $\{g\}$ between two A-algebras F and E is *multiplicative* if $g_B(xy) = g_B(x)g_B(y)$ for all x, y in $F \otimes_A B$, and all A-algebras B. A *norm* (of degree n) is a polynomial law $\{g\}$ between two algebras F and E, homogeneous of degree n and multiplicative. When $M = F$ the A-module $\Gamma^n_A F$ is an A-algebra [26]. The product on $\Gamma^n_A F$, which we refer to as the internal one, is different from the external product $*$ on $\Gamma_A F$. The polynomial law $\{\gamma\}$ from F to $\Gamma^n_A F$ is universal for norms from F.

4.5. Example. When F is an A-algebra it is clear that the homogeneous degree n polynomial law $\{d\}$ from F to $(T^n_A F)^{\mathfrak{S}_n}$ of Example (4.3), is a norm. Hence we obtain an A-algebra homomorphism

$$\alpha_n : \Gamma^n_A F \to (T^n_A F)^{\mathfrak{S}_n}$$

which is an isomorphism when F is flat as a module over A.

4.6. Canonical homomorphism. Let E be an A-algebra which is free of finite rank n as an A-module. For any A-algebra B we have the determinant map

$$d_B : E \otimes_A B \to B$$

sending x to $d_B(x) = \det(e \mapsto ex)$. The determinant is compatible with base change and consequently we have a polynomial law $\{d\}$ from E to A. It is furthermore clear that $\{d\}$ is homogeneous of degree n, and multiplicative. As $\{d\}$ is a norm we have an A-algebra homomorphism

$$\sigma_E : \Gamma^n_A E \to A$$

such that $\sigma(\gamma^n(x)) = \det(e \mapsto ex)$. We call σ_E the canonical homomorphism.

4.7. Proposition. *Let E be an A-algebra such that as an A-module E is free of finite rank n. For any $x \in E$ the characteristic polynomial of the endomorphism $e \mapsto ex$ on E is*

$$c(\Lambda) = \Lambda^n + \sum_{j=1}^{n} (-1)^j \Lambda^{n-j} \sigma_E(\gamma^j(x) * \gamma^{n-j}).$$

Proof. Let $E[\Lambda] = E \otimes_A A[\Lambda]$ where Λ is an independent variable over A. We have that the characteristic polynomial $c(\Lambda) \in A[\Lambda]$ of a given element $x \in E$ is the determinant $\det(\Lambda - x)$. We have furthermore that $\det(\Lambda - x) = (\sigma_E \otimes 1) \circ \gamma^n_{A[\Lambda]}(\Lambda - x)$. Using the defining relations (4.12) and (4.13) we have

$$\gamma^n_{A[\Lambda]}(\Lambda - x) = \sum_{j=0}^{n} (-1)^j \Lambda^{n-j} \gamma^j_A(x) * \gamma^{n-j}_A(1).$$

Our claim now follows by applying the A-algebra homomorphism $\sigma_E \otimes \mathrm{id}$ to the expression above. $\square$

4.8. Proposition. *[Iversen] Let $A = k$ be an algebraically closed field and let E be an k-algebra of rank n. Write $E = \prod E_i$ as a product of local rings, and let $\rho_i : E \to k$ denote the k-algebra homomorphism that factors via E_i. We have the commutative diagram*

$$\begin{tikzcd}
(T^n_k E)^{\mathfrak{S}_n} \arrow[r] & (T^n_k E) \arrow[d, "{(\rho_1,\dots,\rho_1,\dots,\rho_p,\dots,\rho_p)}"] \\
\Gamma^n_k E \arrow[u, "\alpha_n"] \arrow[r, "\sigma_E"'] & k
\end{tikzcd}$$

where ρ_i occurs $\operatorname{rank} E_i$ times.

Proof. Let $\sigma : (T^n_k E)^{\mathfrak{S}_n} \to k$ denote the composite map $\sigma_E \circ \alpha_n^{-1}$. The kernel of σ is a maximal ideal $\mathfrak{m}$, and in particular a prime. As $(T^n_k E)^{\mathfrak{S}_n}$ is the invariant ring of a finite group we have that $T^n_k E$ is integral over the invariant ring, hence by the Going-up Theorem there exists a prime ideal $\mathfrak{n}$ in $T^n_k E$ that restricts down to $\mathfrak{m}$. As E is finite dimensional over k, it follows that $T^n_k E$ is of finite dimension and consequently that $\mathfrak{n}$ is a maximal ideal. As k is algebraically closed we have that $T^n_k E/\mathfrak{n} = k$, and thereby we have that the residue class map $\rho : T^n_k E \to k$ extends the homomorphism σ.

Write $\rho = (\tau_1, \dots, \tau_n)$, where each $\tau_i : E \to k$ is a k-algebra homomorphism $(i = 1, \dots, n)$. Each τ_i is of course of one the morphisms ρ_j, and we need to calculate the number they occur. We have, for any $x \in E$ that the characteristic polynomial of x is

$$c(\Lambda) = \rho(\alpha_n(\gamma^n_{k[\Lambda]}(\Lambda - x))) = \prod_{i=1}^{n} (\Lambda - \tau_i(x)).$$

We can however compute the characteristic polynomial by a different approach. We have that $E = \prod_{i=1}^r E_i$ is a product of local rings E_i. Let $x_i \in E$ be such that $p_j(x_i) = \delta_{i,j}$, where $p_j : E \to E_j$ is the projection. The characteristic polynomial of x_i we have as

$$c(\Lambda) = (\Lambda - 1)^{n_i} \Lambda^{n-n_i}$$

where $n_i = \operatorname{rank} E_i$. Comparing the two expression for the characteristic polynomial $c(\Lambda)$ for $x = x_i$, obtained above, proves the claim. $\square$

4.9. The Norm map. What we have now is a morphism from the Hilbert scheme of points on X to the symmetric product of X. We will describe the morphism locally. Let $X = \operatorname{Spec}(F) \to \operatorname{Spec}(A)$ be the fixed ambient scheme. For any A-algebra H we have that a $\operatorname{Spec}(H)$-valued point is an algebra quotient $F \otimes_A H \to E$ which is locally free of finite rank n as a H-module. We thus have the following sequence of algebras $F \to F \otimes_A H \to E$, which yields

$$\Gamma^n_A(F) \to \Gamma^n_A(F) \otimes_A H = \Gamma^n_H(F \otimes_A H) \to \Gamma^n_H E \to H.$$

The sequence above is a sequence of A-algebras and where the last morphism is the canonical $\sigma_H : \Gamma^n_H E \to H$. In other words we have a natural transformation of functors $n_X : \mathrm{Hilb}^n_X \to \operatorname{Hom}(-, \operatorname{Spec}(\Gamma^n_A(F)))$. And, if Hilb^n_X is represented by a scheme $\mathbf{H}^n_X$ we then have a morphism of schemes

$$n_X : \mathbf{H}^n_X \longrightarrow \operatorname{Spec}(\Gamma^n_A(F)).$$

The above morphism is the *Grothendieck-Deligne* norm map, or the norm map for short.

4.9.1. Remark. When $X = \operatorname{Spec}(F) \to S = \operatorname{Spec}(A)$ is flat we write $\operatorname{Sym}^n_S(X)$ instead of $\operatorname{Spec}(\Gamma^n_A(F))$. Indeed, by (4.5) we have that the coordinate ring of $\operatorname{Sym}^n_S(X)$ is identified with $\Gamma^n_A(F)$.

4.10. Theorem. *When $X \to S$ is a smooth family of irreducible curves, then the norm map $n_X : H^n_X \to \operatorname{Sym}^n(X)$ is an isomorphism.*

Proof. See [5].

4.11. Example. When $X = \operatorname{Spec}(A[T]_f) \to S = \operatorname{Spec}(A)$ we have already seen that the Hilbert scheme $\mathbf{H}^n_X$ is the Spectrum of the ring $A[\sigma_1, \dots, \sigma_n]_{f(t_1)\cdots f(t_n)}$, which however is isomorphic to the invariant ring

$$(A[T]_f \otimes \cdots \otimes A[T]_f)^{\mathfrak{S}_n}$$

as stated in the theorem.

4.12. Distinct points. We will give a description of the open subset of the Hilbert scheme $\mathbf{H}^n_X$ that parameterizes the distinct n-points on X. Let $X = \operatorname{Spec}(F) \to S = \operatorname{Spec}(A)$ be fixed. We define a closed subscheme $\Delta \subseteq \operatorname{Spec}(\Gamma^n_A F)$ by giving generators for its defining ideal. For any $2n$-elements $x = x_1, \dots, x_n$ and $y = y_1, \dots, y_n$ in F we define the element

$$\delta(x, y) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{|\sigma|} \gamma^1(x_1 y_{\sigma(1)}) * \cdots * \gamma^1(x_n y_{\sigma(n)}) \quad \in \Gamma^n_A F.$$

Note that $*$ is the external product on $\Gamma_A F$ such that each summand in the expression above is in $\Gamma^n_A(F)$. We define the ideal $I \subseteq \Gamma^n_A F$ to be the ideal generated by the elements $\delta(x, y)$ for any $2n$ elements x and y in F.

4.13. Proposition. *Let $X = \operatorname{Spec}(F) \to \operatorname{Spec}(A)$ be a flat morphism of finite type of noetherian affine schemes, and assume that the Hilbert scheme $\mathbf{H}^n_X$ exists. FUrthermore we let $\operatorname{Sym}^n_A(X) = \operatorname{Spec}(\Gamma^n_A(F))$. Then we have that following*

(1) *The inverse image $n_X^{-1}(\Delta)$ of the closed subscheme $\Delta \subseteq \operatorname{Sym}^n_A(X)$ by the norm map is the discriminant of the universal family $Z \to \mathbf{H}^n_X$.*
(2) *The restriction of the norm map n_X gives an isomorphism of schemes*

$$\mathbf{H}^n_X \setminus n_X^{-1}(\Delta) \to \operatorname{Sym}^n_A(X) \setminus \Delta.$$

Proof. We refer [7] for details, but we remark that the second statement is part of the folklore as the scheme Δ is supported on the diagonals in $\mathrm{Sym}^n_A(X)$.

4.14. Punctual Hilbert schemes. There has been a lot of interest in the fibers of the norm map n_X, in particular when X is a smooth surface over an algebraically closed field. By Proposition (4.9) we have that the norm map sends a closed point $Z \subseteq X$ to the associated cycle $\sum l(\mathcal{O})_{Z,P}[P]$. The worst fiber is the the fiber over the cycle nP.

4.15. Proposition. *Let $X \to \mathrm{Spec}(K)$ be a smooth surface over an algebraically closed field of characteristic zero. For a point $P \in X$ the fiber $n_X^{-1}(nP)$ of the Grothendieck-Deligne norm map is reduced, irreducible and of dimension $(n-1)$.*

Proof. That the fiber is irreducible and of dimension $n-1$ was proven by J. Briancon [4], whereas the reducedness was established by M. Haiman [15].

4.16. Examples. We give examples when $n = 2$ and $n = 3$. We may assume that X is the plane. The idea is to fix a closed point P in the plane and let another point Q approach the fixed point along any curve. When the two points collide only the support P with the tangent line of the curve remains. The set of tangent lines we identify with the projective line which equals $n_X^{-1}(2P)$.

For $n = 3$ we fix a fat 2 point supported at P - which is a point P with a tangent line. The variable, third point Q we move along a curve. If the tangent line of that curve is different from the fixed one we obtain a point P with two different tangent lines, otherwise the collision remembers the second order tangent line. Anyhow over the fixed two point there is a projective line of second order tangent lines where the one at infinity corresponds to a tangent line different from the fixed one. What we get is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$. However the section corresponding fiber vise to the two different tangent lines give the same 3 point. When the $\mathbf{P}^1$ is contracted along that section one can show that the set we get is the cone over a quadric.

4.17. Relative norm map. Recall the incidence functor $\mathcal{H}_X^{m,n}$ defined in (3.5). When $X = \mathrm{Spec}(F) \to S = \mathrm{Spec}(A)$ is a morphism of affine schemes we have a natural transformation

$$d : \mathcal{H}_X^{m,n} \to \mathrm{Hom}(-, \mathrm{Spec}(\Gamma_A^{m-n}(F))).$$

Indeed, let B be an A-algebra and fix an $\mathrm{Spec}(B)$-valued point of $\mathcal{H}_X^{m,n}$. That is B-algebras E_1 and E_2, and surjective homomorphisms

$$F \otimes_A B \to E_1 \to E_2$$

where E_1 and E_2 are flat and finite over B, of ranks m and n respectively. It follows that the kernel I of $E_1 \to E_2$ is flat and (as we are in the noetherian situation) finitely generated. Thus the B-module I is locally free of rank $m - n$. As the B-module I also is an ideal in E_1 it follows that we can take the determinant of multiplication $x \to ex$ with any element $x \in E_1$. We then get a norm of degree $(m - n)$ from E_1 to B. Thus, analogously to the situation (4.7) we have that the composition of A-algebras and A-algebra homomorphism

$$\Gamma_A^{m-n} F \to \Gamma_B^{m-n} E_1 \to B$$

determines the relative norm map d.

4.18. Note. We end this section with a note about the generators of $\Gamma^n_K F$, when K is a field and $F = K[x_1, \dots, x_m]$ is the polynomial ring in a finite set of variables over K. We have that $\Gamma^n_K F = (T^n_K F)^{\mathfrak{S}_n}$, but still it is hard figure out generators for that particular invariant ring. When K is of characteristic zero one can use an old result of H.Weyl ([31]) and obtain that $\gamma^1(x^\alpha) * \gamma^{n-1}(1)$ generate $\Gamma^n_A F$ with x^α being monomials in $x_1, \dots, x_n$ of degree less or equal to n. The similar statement to arbitrary characteristic is false. On the other hand, when $m = 1$, we have that $\gamma^{n-i}(x) * \gamma^i(1)$, with $i = 1, \dots, n$ generate the algebra $\Gamma^n_K(K[x])$ for any ring K.

§5.- A Theorem of Fogarty

The material in this section is all around a result of J. Fogarty, whose proof presented below follows the original proof line by line.

5.1. The tangent space. Let k be a field and denote by $k[\epsilon] = k[t]/(t^2)$ the ring of dual numbers. Recall that an A-module M is flat if and only if the natural map $I \otimes_A M \to IM \subseteq M$ is injective for all ideals $I \subset A$. It follows that a given $k[\epsilon]$-module $M$is flat if and only if

$$(\epsilon) \otimes_{k[\epsilon]} M \longrightarrow M \qquad (5.1.1)$$

is injective.

We consider the following problem. Let $I \subseteq A$ be a fixed ideal in a k-algebra A, where k is a field. Let J be an ideal in $A[\epsilon] = A \otimes_k k[\epsilon]$ such that the quotient ring $A[\epsilon]/J$ is flat as a $k[\epsilon]$-module and such that $A[\epsilon]/J \otimes_{k[\epsilon]} k = A/I$. Such an ideal J must be of the form

$$J = (f_1 + \epsilon g_1, \dots, f_r + \epsilon g_r),$$

where $(f_1, \dots, f_r)$ is a generator set for the ideal I, and where $g_1, \dots, g_r$ are some elements in A.

5.2. Lemma. *Let A be an algebra over a field k, and $I \subseteq A$ an ideal $I = (f_1, \dots, f_r)$. For any $g_1, \dots, g_r$ in A we consider the ideal*

$$I_g = (f_1 + \epsilon g_1, \dots, f_r + \epsilon f_r) \subseteq A[\epsilon] = A \otimes_k k[\epsilon].$$

Then the quotient ring $M = A[\epsilon]/I_g$ is flat over $k[\epsilon]$ if and only if the map

$$\varphi_g : I \to A/I$$

determined by $\varphi_g(f_i) = \bar{g}_i$ for $I = 1, \dots, r$, is well-defined.

Proof. Assume that $M = A[\epsilon]/I_g$ is flat over $k[\epsilon]$. To show that the map φ_g is well-defined it suffices to show that if $\sum a_i f_i = 0$ in A, then we have that $\sum a_i \bar{g}_i = 0$ in A/I. If $\sum a_i f_i = 0$ we have

$$\epsilon \cdot \sum_{i=1}^r a_i g_i = \sum_{i=1}^r a_i(f_i + \epsilon g_i) \in I_g.$$

It follows that the element $\sum a_i \bar{g}_i$ in $M = A[\epsilon]/I_g$ is mapped to zero by the multiplication map (5.1.1). Hence, by flatness of M we obtain that $\sum a_i \bar{g}_i = 0$ in M, but then also in A/I.

Assume now that the map $\varphi_g : I \to A/I$ is well-defined. We need to show that the quotient module M is flat over $k[\epsilon]$. Let $f = F_1 + \epsilon F_2$ be any element in $A[\epsilon]$, with $F_1, F_2 \in A$, and denote by $\bar{f}$ the residue class of f modulo I_g. By the flatness criterium (5.1.1) we need to show that if $\epsilon \bar{f} = 0$ in M then $\bar{f} = 0$. Note that we may assume that $f = F_1$, that is there is only a constant term.

Hence, we let $f \in A$ be any element such that $\epsilon \bar{f} = 0$ in $M = A[\epsilon]/I_g$. We shall show that $\bar{f} = 0$ in M, but in fact we will show that $\bar{f} = 0$ in A/I. By assumption we have $\epsilon f \in I_g$, hence

$$\begin{aligned} \epsilon f &= \sum_{i=1}^{r} (a_i + \epsilon b_i)(f_i + \epsilon g_i) \\ &= \sum_{i=1}^{r} a_i f_i + \epsilon \sum_{i=1}^{r} (a_i g_i + b_i f_i) \quad \in A[\epsilon]. \end{aligned}$$

As $A[\epsilon]$ is a free A-module with basis $1, \epsilon$ the equations above give us that $\sum a_i f_i = 0$ and that $f = \sum (a_i g_i + b_i f_i)$. Now modulo the ideal I we have that f is congruent to $\sum a_i g_i$. That is $\bar{f} = \sum a_i \bar{g}_i$ in A/I. As the morphism φ_g was well- defined we have that $\sum a_i g_i = \varphi_g(\sum a_i f_i)$. Above we noted that $\sum a_i f_i = 0$, hence $\varphi_g(\sum a_i f_i = 0)$ and it follows that $\bar{f} = 0$. $\square$

5.3. Proposition. *Let $X = \mathrm{Spec}(F) \to S = \mathrm{Spec}(A)$ be a morphism of schemes, and let K be a field. Let $I \subset F \otimes_A K$ denote the defining ideal of a fixed $\mathrm{Spec}(K)$-valued point of the Hilbert scheme H^n_X. Then the tangent space of H^n_X at the fixed point is naturally identified with $\mathrm{Hom}_{F \otimes_A K}(I, F \otimes_A K/I)$.*

Proof. The Zariski tangent space of a point x at a scheme H is the dual of the $\kappa(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$ at x. Now, let H be a scheme defined over a field K, and let x be a fixed K-rational point. Then we have that the Zariski tangent space at x equals the set of K-linear maps from Spectrum of the dual numbers into X, extending the fixed point $\mathrm{Spec}(K) \to X$. By Lemma (5.2) we therefore obtain our claim. $\square$

5.4. Regular sequences and depth. Let A be a local (noetherian) ring with maximal ideal $\mathfrak{m}$, and let M be a non-zero A-module. An element $x \in \mathfrak{m}$ is M-regular if $xm \neq 0$ for all non-zero elements m in M. An ordered sequence $(x_1, \dots, x_n)$ is M-regular if x_1 is M-regular, and x_{i+1} is $M/(x_1, \dots, x_i)M$-regular, for $i = 0, n-1$. (Note that we require $(x_1, \dots, x_n)M$ to be a proper submodule of M).

The depth of a finitely generated (non-zero) A-module M is the maximal length of M-regular sequences $(x_1, \dots, x_r)$ with $x_i \in \mathfrak{m}$ $(i = 1, \dots, r)$. The depth is denoted $\mathrm{depth}(M)$. We have that the depth of the ring A equals the dimension of A when A is regular, and the ring has depth 0 if it is artinian.

5.5. Note. There is an equivalent description of depth, which we will need. We have that $\mathrm{depth}(M) = r$ if and only if $\mathrm{Ext}^q_A(N, M) = 0$ for all A-modules N of finite length, all $q < r$ (See i.e. [1]).

5.6. Projective dimension. Let A be a local ring and M a finitely generated A-module. The projective dimension of M is the smallest integer s such that there is an exact sequence

$$0 \to P_s \to \cdots \to P_1 \to P_0 \to M \to 0 \tag{5.6.1}$$

with free A-modules P_i, $(i = 0, \dots, s)$. We denote with $\operatorname{proj.dim}(M)$ the projective dimension of M.

5.7. Theorem. *Let A be a regular local ring, M a non-zero finitely generated A-module. Then we have* $\operatorname{depth}(M) + \operatorname{proj.dim}(M) = \dim A$.

Proof. See [1] for a proof. A more general statement for not regular rings A, replacing $\dim A$ with the depth of A is due to M. Auslander and D. Buchsbaum [2].

5.8. Proposition. *Let A be a regular local ring of dimension 2, and let I be an ideal such that the quotient ring A/I has length n as an A-module. Then we have that the length*

$$l_A(\operatorname{Hom}_A(I, A/I)) \leq 2n.$$

Proof. By Theorem (5.7) we have the equality $\operatorname{proj.dim}(I) = 2 - \operatorname{depth}(I)$. We clearly have that the ideal I contains at least one I-regular element. Hence the depth of I is at least 1, and consequently the projective dimension of I is either 0 or 1. As the ideal I is not generated by one element it is not free as an A-module. It follows then that the projective dimension can not be 0, but has to be 1. We therefore have a free resolution (5.6.1) of I consisting of 2 terms P_0 and P_1. As the ideal I is non-zero it has rank 1, and consequently we have a short exact sequence

$$0 \to A^r \to A^{r+1} \to I \to 0, \tag{5.8.1}$$

For some integer r. Applying $\operatorname{Hom}_A(-, A/I)$ yields the exact sequence

$$0 \to \operatorname{Hom}_A(I, A/I) \to \operatorname{Hom}_A(A^{r+1}, A/I) \to \operatorname{Hom}_A(A^r, A/I) \to \operatorname{Ext}^1_A(I, A/I) \to 0.$$

The length function is additive, and since we have that the length of $\operatorname{Hom}(A^r, N)$ is $r \cdot l_A(N)$ we obtain the equation

$$l_A \operatorname{Hom}_A(I, A/I) = n + l_A(\operatorname{Ext}^1_A(I, A/I)).$$

Therefore we need only show that the length of $\operatorname{Ext}^1_A(I, A/I)$ is at most n. From the long exact sequence of Ext-groups associated to $0 \to I \to A \to A/I \to 0$ we obtain that $\operatorname{Ext}^1_A(I, A/I) = \operatorname{Ext}^2_A(A/I, A/I)$. As A has global homological dimension 2 (being regular) we have that all Ext groups $\operatorname{Ext}^q_A(M, N)$, with $q > 2$ vanish. It follows that we have a surjection

$$\operatorname{Ext}^2_A(A/I, A) \to \operatorname{Ext}^2_A(A/I, A/I) \to 0,$$

and we are reduced to showing that the length of $\operatorname{Ext}^2_A(A/I, A)$ is at most n. As the ring A is Gorenstein we have that $\operatorname{Ext}^2_A(k, A) = k$, where k is the residue field of A. It now follows by Remark (5.5) that $\operatorname{Ext}^2_A(-, A)$ is an exact functor on the category of finite length A-modules, and consequently the result follows by induction on the length of A/I. □

5.9. Theorem. *[Fogarty] Let $X = \mathrm{Spec}(F)$ be a non-singular quasi-projective surface over a field k. The the Hilbert scheme H_X^n is regular of dimension $2n$.*

Proof. By a result of R. Hartshorne ([16]) we have that the Hilbert scheme $\mathbf{H}_X^n$ is connected. By (4.13) we have that $U = \mathrm{Sym}^n(X) \setminus \Delta$ is an open subset. As U clearly has dimension $2n$ it suffices to show that the tangent space of any closed point x of $\mathbf{H}_X^n$ has dimension at most $2n$, because a singularity would increase the tangent space.

We may assume that k is algebraically closed, and as $\mathbf{H}_F^n$ is quasi-projective, a closed point of $\mathbf{H}_X^n$ is the same as a k-valued point. Thus, let Z be a $\mathrm{Spec}(k)$-valued point of $\mathbf{H}_X^n$, and let $I \subseteq F = F \otimes_k k$ denote its defining ideal where $E = F/I$ is of dimension n as a k-vector space. We can write the artinian ring $E = \prod_{i=1}^p E_i$ as a product of local rings, where each E_i is the quotient of a local 2-dimensional ring F_i by an ideal I_i. As we have

$$\mathrm{Hom}_F(I, F/I) = \prod_{i=1}^{p} \mathrm{Hom}_{F_i}(I_i, E_i = F_i/I_i)$$

the result follows from Proposition (5.8). $\square$

5.10. A construction. In the last part of this section we will state without proof a way of constructing the Hilbert scheme of points on smooth surfaces $X = \mathrm{Spec}(F)$. Recall (4.12) that for fixed n we define the ideal $I \subseteq \Gamma_A^n F$ as the ideal generated by $\delta(x, y)$ for any $2n$-elements $x = x_1, \dots, x_n$ and $y = y_1, \dots, y_n$ in F.

5.11. Theorem. *Let $X = \mathrm{Spec}(F)$ be a smooth quasi-projective surface over a field K. Then the blow-up of $\mathrm{Sym}_K^n(X)$ along the closed subscheme Δ defined by the ideal $I \subseteq \Gamma_A^n F$ is isomorphic to the Hilbert scheme $\mathbf{H}_X^n$.*

Proof. See [7].

§6. - Distributions and the Heisenberg representation

We will in this section recall some basic material about distributions. The material is mainly from ([18]) and ([27]).

6.1. Distributions. Let $\mathcal{U}$ be a vector space over a field K of characteristic zero. We let z and w be variables over $\mathcal{U}$. A distribution, with values in U, is an element of the form

$$a(z, w) = \sum_{m,n \in \mathbf{Z}} a_{m,n} z^{-m-1} w^{-n-1},$$

with $a_{m,n} \in \mathcal{U}$. The space of all distributions we denote by $\mathcal{U}[[z^{\pm}, w^{\pm}]]$. The space of distributions is a $K[z, w]$-module. If $\mathcal{U}$ is an algebra the space of distributions is an $\mathcal{U}[z, w]$-module, but not an algebra. We can not multiply two distributions, but we can however multiply a distribution $a(z)$, a series only in z, with a distribution $b(w)$, a series only in w. The *formal delta distribution* is defined as

$$\delta(z - w) = \sum_{m \in \mathbf{Z}} z^{-m-1} w^m.$$

6.2. Power series expansion. We will do to different power series expansions of a simple function which will give us an important identity. Let $\mathcal{U}_z = K[z, z^{-1}]$. We consider the rational function $1/(z-w)$ as an element in the power series ring $\mathcal{U}_z[[w]]$. We then get the power series expansion

$$\frac{1}{(z-w)} = \frac{z^{-1}}{1-\frac{w}{z}} = \sum_{n\geq 0} z^{-n-1}w^n \in \mathcal{U}_z[[w]].$$

Similarly we can consider $1/(z-w)$ as an element in $\mathcal{U}_w[[z]]$ giving the power series expansion

$$\frac{1}{(z-w)} = -\sum_{m>0} w^{-m}z^{m-1} = -\sum_{n\leq -1} z^{-n-1}w^n \in \mathcal{U}_w[[z]].$$

Let $i_z : \mathcal{U}_z[[w]] \to K[[z^{\pm}, w^{\pm}]]$ denote the natural map of $K[z,w]$-modules, and similarily we let $i_w : \mathcal{U}_w[[z]] \to K[[z^{\pm}, w^{\pm}]]$.

6.3. Lemma. *We have the following identity of distributions*

$$i_z\frac{1}{(z-w)^{j+1}} - i_w\frac{1}{(z-w)^{j+1}} = \sum_{m\in\mathbf{Z}} \binom{m}{j} z^{-m-1}w^{m-j} = \partial_w^{(j)}\delta(z-w)$$

where $j!\partial_w^{(j)} = \partial_w^j$.

Proof. We have that $i_z(z-w)^{-1} - i_w(z-w)^{-1} = \delta(z-w)$ the formal delta distribution. Formal differentiation gives the stated identity. □

6.4. Lemma. *In the space of distributions* $K[[z^{\pm}, w^{\pm}]]$*we have the following identities*

(1) $\delta(z-w) = \delta(w-z)$.
(2) $\partial_z^j\delta(z-w) = (-1)^j\partial_w^j(z-w)$.
(3) $(z-w)\partial_w^{(j+1)}\delta(z-w) = \partial_w^{(j)}\delta(z-w)$.
(4) $(z-w)^{j+1}\partial_w^{(j)}\delta(z-w) = 0$.

Proof. The proof of Assertion (1) is immediate. The Assertion (2) follows as we by formal differentation obtain

$$\begin{aligned}\partial_z^j\delta(z-w) &= \sum_{m\in\mathbf{Z}}(-1)^j(m+1)\cdots(m+j)w^m z^{-m-j-1}\\ &= (-1)^j\sum_{n\in\mathbf{Z}}(n-j+1)\cdots n\, w^{n-j}z^{-n-1} = (-1)^j\partial_w^j\delta(z-w).\end{aligned}$$

The third and fourth statement follows from the identity in Lemma (6.3) and the fact that the maps i_z and i_w are $K[z,w]$-module homomorphisms. □

6.5. Locality. We are interested in distributions $a(z,w) \in \mathcal{U}[[z^{\pm}, w^{\pm}]]$ that can be written on the form

$$a(z,w) = \sum_{j=0}^{N} c^j(w)\partial_w^{(j)}\delta(z-w), \tag{6.5.1}$$

for some positive integer N, and distributions $c^j(w) \in \mathcal{U}[[w^{\pm}]]$.

When we write a distribution $a(z,w) = \sum_{m,n} a_{m,n} z^{-m-1} w^{-n-1}$ we define the distribution

$$\operatorname{Res}_z(a(z,w)) = \sum_{n \in \mathbf{Z}} a_{0,n} w^{-n-1} \in \mathcal{U}[[w^{\pm}]].$$

When we have a distribution in only one variable $a(z) = \sum a_n z^{-n-1}$ we have that $\operatorname{Res}_z a(z) = a_0$. Clearly we have that $\operatorname{Res}_z(\partial_z(a(z,w))) = 0$. Furthermore, for any $f(z) \in K[z, z^{-1}]$ we get that $\operatorname{Res}_z f(z)\delta(z-w) = f(w)$.

6.6. Lemma. *If a distribution $a(z,w)$ is of the form (6.5.1) then we have that*

$$c^j(w) = \operatorname{Res}_z((z-w)^j a(z,w)).$$

Proof. We multiply the distribution $a(z,w)$ with $(z-w)^j$. By Assertion (4) of Lemma (6.4) we see that multiplication with $(z-w)^j$ annihilates the first j-terms $i = 0, \dots, j-1$ in the sum, such that

$$(z-w)^j a(z,w) = \sum_{i=j}^{N} c^i (z-w)^j \partial_w^{(i)} \delta(z-w) = \sum_{i=0}^{N-j} c^{i+j}(w)(z-w)^j \partial_w^{(i+j)} \delta(z-w).$$

Now by using Assertion (3) of Lemma (6.4) we can rewrite the product as

$$(z-w)^j a(z,w) = \sum_{i=0}^{N-j} c^{i+j}(w) \partial_w^{(i)} \delta(z-w).$$

Then finally when taking the residues we get $\operatorname{Res}_z(a(z,w)(z-w)^j) = c^j(w)$. □

6.7. Definition. Let $\mathcal{U}$ be a K-algebra, which simply means a K-vector space $\mathcal{U}$ with a product - not necessarily associative. A distribution $a(z)$ is mutually local of order $N+1$ to a distribution $b(z)$ in $\mathcal{U}[[z^{\pm}]]$ if

$$[a(z), b(w)] = \sum_{j=0}^{N} c^j(w) \partial_w^{(j)} \delta(z-w).$$

6.8. Proposition. *A distribution $a(z)$ is mutually local to a distribution $b(z)$ if and only if*

$$[a_m, b_n] = \sum_{j=0}^{N} \binom{m}{j} c^j_{m+n-j}, \quad m, n \in \mathbf{Z}.$$

Proof. Assume that we have a distribution $\alpha(z,w) = \sum \alpha_{m,n} z^{-m-1} w^{-n-1}$ that can be written on the form $\sum_{j=0}^{N} c^j(w) \partial_w^{(j)} \delta(z-w)$. We then get using Lemma (6.3) and by comparing coefficients that

$$\alpha_{m,n} = \sum_{j=0}^{N} \binom{m}{j} c^j_{m+n-j}. \tag{6.8.1}$$

And also, if the coefficients of a distribution $\alpha(z,w)$ can be expressed as in (6.8.1) then it follows that the distribution $\alpha(z,w)$ is of the form (6.5.1). Applying that observation to the distribution

$$\alpha(z,w) = [a(z), b(w)] = \sum_{m,n\in\mathbf{Z}} [a_m, b_n] z^{-m-1} w^{-n-1}$$

yields the result. □

6.8.1. Note that a distribution $a(z,w)$ of the form (6.5.1) is by Assertion (4) of Lemma (6.4) annihilated by $(z-w)^{N+1}$. The converse is also true; if a distribution $a(z,w)$ is annihilated by $(z-w)^{N+1}$ then it can be written on the form (6.5.1) ([18], p. 18. Corollary 2.2). Consequently, a distribution $a(z)$ is mutually local to $b(z)$ if and only if

$$(z-w)^{N+1}[a(z), b(w)] = 0.$$

In particular we see that the notion of two distributions being local is symmetric, i.e. if $a(z)$ is mutually local with $b(z)$ then also $b(z)$ is mutually local with $a(z)$. We therefore simply say that two distributions $a(z)$ and $b(z)$ are *mutually local* if Definition (6.7) is satisfied.

6.9. A local Lie algebra. Let $\mathfrak{g}$ be a Lie algebra, and let $F \subseteq \mathfrak{g}[[z^{\pm}]]$ be a subset of the vector space of distributions satisfying the following conditions.

(1) The coefficients of distributions in F span the vector space $\mathfrak{g}$.
(2) All distributions in F are mutually local.
(3) There exists a K-linear derivation T of $\mathfrak{g}$ such that $T(a(z)) = \partial_z(a(z))$ for all distributions $a(z)$ in F.

Then we say that $\mathfrak{g}$ is a local Lie algebra.

6.10. Current algebra. Let $\mathfrak{g}$ be a Lie-algebra with a symmetric pairing $(|)$ which also is invariant with respect to the bracket, that is $([a,b] \mid c) = (a \mid [b,c])$. Consider the vector space

$$\hat{\mathfrak{g}} = K[t, t^{-1}] \otimes_K \mathfrak{g} \oplus K\hat{c}.$$

We will abbreviate $a_m = t^m \otimes a$, where $a \in \mathfrak{g}$ and any integer m. We make $\hat{\mathfrak{g}}$ into a Lie algebra by defining the bracket

$$[a_m, b_n] = [a,b]_{m+n} + m(a \mid b)\delta_{m,-n}\hat{c} \tag{6.10.1}$$

and $[\hat{c}, \hat{\mathfrak{g}}] = 0$. To check that the bilinear and skew-symmetric bracket (6.10.1) also sastifies the Jacobi identiy it sufficies to check for elements of the form a_p, b_q, c_r in $K[t,t^{-1}] \otimes_K \mathfrak{g}$. As the elements in $\mathfrak{g}$ satisfies the Jacobi identity we obtain that the expression $[a_p, [b_q, c_r]] + [b_q, [c_r, a_p]] + [c_r, [a_p, b_q]]$ reduces to the expression

$$\big(p(a \mid [b,c])\delta_{p,-(q+r)} + q(b \mid [c,a])\delta_{q,-(r+p)} + r(c \mid [a,b])\delta_{q,-(p+r)}\big)\hat{c}.$$

If $p+q+r \neq 0$ then the expression above is zero for trivial reasons, hence we may assume that $r = -p-q$. As the pairing $(|)$ is both symmetric and invariant we obtain that

$$(a \mid [b,c]) = ([b,c] \mid a) = -([c,b] \mid a) = -(c \mid [b,a]) = (c \mid [a,b]),$$

and it follows that the expression above equals zero. That is the Jacobi identity holds and $\hat{\mathfrak{g}}$ is a Lie algebra. The Lie algebra $\hat{\mathfrak{g}}$ is a central extension of the loop algebra $K[t,t^{-1}]\otimes_K \mathfrak{g}$ called the *current algebra*. If $\mathfrak{g}=K$ and the form is non-degenerate then $\hat{\mathfrak{g}}$ is the oscillator algebra.

For each $a\in\mathfrak{g}$ we have the currents

$$a(z)=\sum_{m\in\mathbf{Z}} a_m z^{-m-1} \qquad \in \hat{\mathfrak{g}}[[z^{\pm}]].$$

The current algebra $\hat{\mathfrak{g}}$ is an example of a local Lie-algebra. The first condition of Definition (6.9) is clear. By the Proposition (6.8) we see that

$$[a(z),b(w)]=\delta(z-w)c_0(w)+\delta_w\delta(z-w)(a\mid b)\hat{c},$$

where $c_0(w)=\sum[a,b]_m w^{-m-1}$. Hence any two currents are mutually local, and the second condition of (6.9) is satisfied. The map $T(a_m):=-ma_{m-1}$ is a well-defined K-linear derivation on the Lie algebra $\hat{\mathfrak{g}}$ having the effect that $T(a(z))=\partial_z(a(z))$.

6.11. The Heisenberg super algebra. Another example of a local Lie algebra is the Heisenberg algebra which we discuss below. Let $\mathfrak{h}=\mathfrak{h}_0\oplus\mathfrak{h}_1$ be a finite dimensional vector space, and assume that we have a non-degenerate supersymmetric pairing on $\mathfrak{h}$. That is a non-degenerate pairing such that for any a and b of parity $p(a)$ and $p(b)$, respectively, we have

$$(a\mid b)=(-1)^{p(a)p(b)}(b\mid a). \tag{6.11.1}$$

As in the example with the current algebra we now form the vector space $\hat{\mathfrak{h}}$. We extend the $\mathbf{Z}_2$-grading from $\mathfrak{h}$ to $\hat{\mathfrak{h}}$ by declaring that the parity $p(t)=p(\hat{c})=0$ (even). Then we define the bracket on $\hat{\mathfrak{h}}$ as

$$[a_m,b_n]:=m(a\mid b)\delta_{m,-n}\hat{c},$$

for elements a_m and b_n in $K[t,t^{-1}]\otimes_K\mathfrak{h}$, and finally that $[\hat{\mathfrak{h}},\hat{c}]=0$. For any homogeneous elements a and b in $\hat{\mathfrak{h}}$ we have that $[a,b]=(-1)^{p(a)p(b)}[b,a]$. The bracket also satisfies the super Jacobi identity, making $\hat{\mathfrak{h}}$ a super Lie algebra. The Heisenberg algebra $\mathfrak{h}'$ is the subalgebra of $\hat{\mathfrak{h}}$ consisting of the vector space

$$\mathfrak{h}'=\bigoplus_{n\neq 0} t^n\otimes_K\mathfrak{h}\oplus K\hat{c}.$$

6.12. Enveloping algebra. To a Lie algebra $\mathfrak{g}$ we let $\mathcal{U}(\mathfrak{g})$ denote the enveloping algebra of $\mathfrak{g}$. If $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ is $\mathbf{Z}_2$-graded we let $\mathcal{U}_2(\mathfrak{g})$ be the graded enveloping algebra of $\mathfrak{g}$. That is the quotient of the full tensor algebra $\mathrm{T}(\mathfrak{g})$ module the relations of the form

$$a\otimes b-(-1)^{p(a)p(b)}b\otimes a-[a,b].$$

We note that when $\mathfrak{h}=\mathfrak{h}_0\oplus\mathfrak{h}_1$ is Abelian we have that $\mathcal{U}_2(\mathfrak{h})=\oplus\mathcal{U}(\mathfrak{h})_m$ is a graded vector space, and furthermore that

$$\mathcal{U}_2(\mathfrak{h})=\mathrm{S}(\mathfrak{h}_0)\otimes_K\wedge(\mathfrak{h}_1), \tag{6.12.1}$$

where $\mathrm{S}(\mathfrak{h}_0)$ is the symmetric quotient of the full tensor algebra of $\mathfrak{h}_0$, and where $\wedge(\mathfrak{h}_1)$ is the anti-symmetric quotient - or the exterior algebra. The symmetric quotient $\mathrm{S}(\mathfrak{h}_0)$ can be identified with the polynomial ring in $\dim(\mathfrak{h}_0)$-variables, and it follows that the Poincaré series of $\mathrm{S}(\mathfrak{h}_0) = (1-q)^{-\dim(\mathfrak{h}_0)}$. The exterior algebra $\wedge(\mathfrak{h}_1)$ is a finite dimensional algebra with Poincaré series $(1+q)^{\dim(\mathfrak{h}_1)}$. From the splitting (6.11.1) it follows readily that the Poincaré series of $\mathcal{U}_2(\mathfrak{h})$ is

$$P(q) = \sum_{m\geq 0} \dim_K(\mathcal{U}(\mathfrak{h})_m)q^m = \frac{(1+q)^{\dim(\mathfrak{h}_1)}}{(1-q)^{\dim(\mathfrak{h}_0)}}. \tag{6.12.2}$$

6.13. The Heisenberg representation. We continue Example 6.11. Let $\mathfrak{h}_{\geq}$ be the subalgebra of $\mathfrak{h}'$ where $\mathfrak{h}_{\geq} = \oplus_{n>0} t^n \otimes \mathfrak{h} \oplus K\hat{c}$. We define the representation $\pi : \mathfrak{h}_{\geq} \to K = \mathrm{End}(K)$ by sending $\hat{c} \mapsto 1$ and $a_m \mapsto 0$ with $m > 0$. Consequently we obtain a $\mathcal{U}_2(\mathfrak{h}_{\geq})$-module structure on K. From that we obtain by extension of scalars an $\mathcal{U}_2(\mathfrak{h}')$-module

$$M^{\pi} = \mathcal{U}_2(\mathfrak{h}') \otimes_{\mathcal{U}_2(\mathfrak{h}_{\geq})} K.$$

The module M^{π} is the induced representation of the Heisenberg algebra $\mathfrak{h}'$ on K.

6.14. Lemma. *The graded vector space M^{π} has the following Poincaré series*

$$P(q) = \prod_{m>0} \frac{(1+q^m)^{\dim_K(\mathfrak{h}_1)}}{(1-q^m)^{\dim_K(\mathfrak{h}_0)}}.$$

Proof. The vector space M^{π} is a tensor product of infinite copies of $\mathcal{U}_2(\mathfrak{h})$, indexed by the negative integers $n < 0$. The Poincaré series of each copy $\mathcal{U}_2(\mathfrak{h})$ we obtain from (6.12.2) with q^m keeping track of the grading. The product of all theses Poincaré series gives the total series. $\square$

§7. - Nakajima operators

In this last section we will state the result of Nakajima and Grojnowski who independently managed to realize the direct summand of the cohomologies of the Hilbert scheme as a certain Heisenberg representation. The presentation we use is based on the article [23] authored by M. Lehn.

7.1. Göttsche's formula. We fix a smooth complex projective surface X. We consider the Hilbert scheme $\mathbf{H}_X^n$ as a real manifold and let $H^*(\mathbf{H}_X^n)$ denote its singular cohomology. As $\mathbf{H}_X^n$ is smooth the cohomology groups have a ring structure. There has been and is a notably interest in the cohomology ring of the Hilbert scheme and one of the striking results was L. Göttsche's formula for the generating function of its Betti numbers. As $\mathbf{H}_X^n$ has real dimension $4n$ we have the decomposition

$$H^*(\mathbf{H}_X^n) = \oplus_{j=0}^{4n} H^j(\mathbf{H}_X^n).$$

The Betti number $b_j(\mathbf{H}_X^n)$ is simply the vector space dimension of the j'th cohomology group of $\mathbf{H}_X^n$. The generating series for the Betti numbers for the Hilbert schemes of points is the formal sum

$$P(t,q) = \sum_{n\geq 0}\sum_{j=0}^{4n} \dim_{\mathbf{R}}(\mathbf{H}_X^n)t^j q^n \in \mathbf{Z}[[t,q]].$$

Setting $t=1$ we obtain that $P(1,q)$ is the Poincaré series of $D(\mathbf{H}) = \oplus_{n\geq 0} H^*(\mathbf{H}_X^n)$.

7.2. Theorem. *[Göttsche] Let X be a smooth complex surface, and for each non-negative integer n we let $\mathbf{H}_X^n$ denote the Hilbert scheme of n-points on X. Then we have that the generating series $P(t,q)$ for the Betti numbers for $D(\mathbf{H})$ equals*

$$\prod_{m>0} \frac{(1+t^{2m-1}q^m)^{b_1(X)}(1+t^{2m+1}q^m)^{b_3(X)}}{(1-t^{2m-2}q^m)^{b_0(X)}(1-t^{2m}q^m)^{b_2(X)}(1-t^{2m+2}q^m)^{b_4(X)}}.$$

Proof. There exist several proofs of the statement, but the original using the Weyl conjectures can be found in [14].

7.2.1. Remark. We consider $\mathfrak{h} = H^*(X)$ as an Abelian Lie algebra, with $\mathbf{Z}_2$-grading given by the even and odd degrees. The cup product in cohomology is super symmetric in the sense of (6.11.1). Note that the Poincaré series of the Heisenberg representation of $\mathfrak{h}$ by Lemma (6.14) equals the Poincaré series obtained by Göttsches Theorem. That observation is due to Wafa and Witten who therefore triggered the question whether the Heisenberg algebra module $D(\mathbf{H})$ could be given geometric meaning.

7.3. Let $m \geq n \geq 0$ be two non-negative integers, and let $\mathbf{H}_X^{m,n}$ denote the scheme representing the incidence functor $\mathcal{H}^{m,n}$ with X a smooth complex surface. Thus $\mathbf{H}_X^{m,n}$ is the closed subscheme of $\mathbf{H}_X^m \times \mathbf{H}_X^n$ consisting of pairs (ξ, ξ') with $\xi' \subseteq \xi$.

We have the relative norm map (4.17) which is a morphism of schemes

$$d : \mathbf{H}_X^{m,n} \to \mathrm{Sym}^{m-n}(X),$$

which sends a point (ξ, ξ') to the cycle $\xi - \xi'$. There is a natural embedding of X as the diagonal in $\mathrm{Sym}^{m-n}(X)$. The fiber $d^{-1}(X)$ contains one component $\mathrm{Q}^{m,n}$ of maximal dimension. Its dimension we can compute as follows. Fix a point $x \in X$, then the fiber $d^{-1}(x)$ consits of pairs (ξ, ξ') such that the cycle $\xi - \xi'$ is supported at the point x. We fix $\xi \in \mathbf{H}_X^n$, and then the other element ξ' corresponds (generically) to points in the punctual Hilbert scheme $n_X^{-1}((m-n)x)$. As $\mathbf{H}_X^n$ is of dimension $2n$ and the punctual Hilbert scheme $n_X^{-1}((m-n)x)$ is of dimension $(m-n-1)$ we get that the dimension

$$\dim(\mathrm{Q}^{m,n}) = 2n + (m-n-1) + 2 = m+n+1.$$

7.4. We will use the varieties $\mathrm{Q}^{m,n}$ to define operators

$$\mathfrak{q}_l : H^*(X) \to \mathrm{End}(\oplus H^*(\mathbf{H}_X^n))$$

We have that $\mathrm{Q}^{m,n}$ is closed in $\mathbf{H}_X^{m,n}$, hence closed in the product of $\mathbf{H}_X^m$ and $\mathbf{H}_X^n$. Therefore we have the following diagram of schemes and proper morphisms

$$\begin{array}{ccccc}
\mathbf{H}_X^m & \xleftarrow{p} & Q^{m,n} & \xrightarrow{q} & \mathbf{H}_X^n \\
 & & \downarrow{\scriptstyle d} & & \\
 & & X & &
\end{array}$$

where p and q are the induced projections, and d is the restriction of the relative norm map (4.17).

Assume first that $l \geq 0$, and let $\alpha \in H^*(X)$, and $y \in H^*(\mathbf{H}_X^n)$. We use the diagram above with $m = n + l$ to define the operator $\mathfrak{q}_l(\alpha)$ by

$$\mathfrak{q}_l(\alpha)(y) = p_*([\mathrm{Q}^{n+l,n}] \cap (d^*\alpha \cdot q^*y)) \in H_*(\mathbf{H}^{n+l}) = H^*(\mathbf{H}^{n+l}).$$

In the last step we use Poincaré-Duality to identify the homology with the cohomology. For negative $l < 0$ we switch the two components defining $\mathrm{Q}^{n+l,n}$ - and put in a sign factor.

7.5. Theorem. *[Nakajima] For any integers m and n, any class α and β in $H^*(X)$ we have that the following commutator relation*

$$[\mathfrak{q}_n(\alpha), \mathfrak{q}_m(\beta)] = n\delta_{n,-m} \int_X \alpha\beta.$$

Proof. The proof of Nakajima ([25]) was up to an universal constant which was later determined by Ellingsrud and Strømme ([9]).

7.6. Corollary. *Let $\mathfrak{h} = H^*(X)$. Then we have that $\oplus_{n\geq 0} H^*(\mathbf{H}_X^n)$ is a representation of the Heisenberg super algebra $\mathfrak{h}'$, and is as a module isomorphic to M^π.*

Proof. By the Theorem of Nakajima we obtain a map $M^\pi \to \mathbf{H}$. The Heisenberg module M^π is irreducible, hence the map is injective. As the Poincaré series of the two vector spaces M^π and $\mathbf{H}$ are equal it follows that the stated map is also surjective. □

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