NON-EFFECTIVE DEFORMATIONS OF GROTHENDIECK'S HILBERT FUNCTOR

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Abstract. Let X be a scheme that does not satisfy the valuative criterion of separatedness. We show that the Hilbert functor parametrizing closed families of X that are flat, finite and of rank one is not represented by a scheme or an algebraic space.

Introduction

One of Artin's criteria for representability of functors is the condition of effectivity of formal deformations [Art69]. We will in this note show that the Hilbert functor of Grothendieck [FGA] does not always fulfill this criterion, and in particular that those functors will not be representable by an algebraic space, or a scheme.

For a fixed scheme X we let $\mathcal{H}^1_X$ denote the Hilbert functor of 1-points on X; the functor that parametrizes families of closed subschemes of X that are flat and of finite rank one over the base. The crucial fact in the definition is that the families are closed.

We let A be a complete local ring, and consider the natural map

$$\mathcal{H}^1_X(A) \longrightarrow \varprojlim \mathcal{H}^1_X(A/\mathfrak{m}^{n+1}).$$

A formal deformation $\{\xi_n\}$ is a collection of compatible families $\xi_n \in \mathcal{H}^1_X(A/\mathfrak{m}^{n+1})$, and the deformation is called effective if it is in the image of the above map. It is easy to see that the map above is injective, but that the map should be surjective in general is a misconception. When the fixed scheme X does not satisfy the valuative criterion of separatedness we show that there exists a complete valuation ring A such that surjectivity of the above map fails. In particular we have that the Hilbert functor $\mathcal{H}^1_X$ is not representable for such schemes.

An explanation for the above mentioned result is as follows. It is easy to see that $\mathcal{H}^1_X$ parametrizes *closed* sections of the structure map $f : X \to S$. And when the morphism f is separated we have that any section is closed.

On the other hand, schemes $X \to S$ that do not satisfy the valuative criterion of separatedness have non-closed sections. Replacing S with the spectrum of a complete valuation ring A, we have that different extensions of the generic point of the curve S yield sections $\xi : S \to$

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X that are not closed. However, the infinitesimal truncations ξ_n : $\mathrm{Spec}(A/\mathfrak{m}^{n+1}) \to X$ of a non-closed section ξ are closed. Consequently the infinitesimal truncations ξ_n form a formal deformation, which is not effective since $\xi \notin \mathcal{H}^1_X(A)$ and the section ξ is uniquely determined by the ξ_n.

Examples of schemes X such that the Hilbert functor of n-points on X is not representable by a scheme have appeared in the literature (see e.g. [Knu71, p. 14], and [FGA, Sec. 7, 221-27]). The examples we know of uses the fact that the symmetric product $\mathrm{Sym}^n_S(X)$ does not exist as a scheme, and are consequently complicated as it is not easy to construct such schemes X. It is therefore surprising, and interesting, that we can find simple examples of schemes X, such as the line with the double point, whose Hilbert functor is not representable.

1. The Hilbert functor of one point

We will in this first section define Grothendieck's Hilbert functor of points, and recall what is known for the Hilbert functor of one point.

1.1. **Sections.** We fix a morphism of schemes $f : X \to S$. A section ξ is a morphism of schemes $\xi : S \to X$ such that the composition $f \circ \xi$ is the identity on S. Sections are always immersions, and we say that a section is *closed* if it is a closed immersion.

We let $\mathcal{S}_X$ denote the contravariant functor that to any S-scheme T assigns the set $\mathcal{S}_X(T)$ of sections $T \to X \times_S T$ of the projection map $X \times T \to T$.

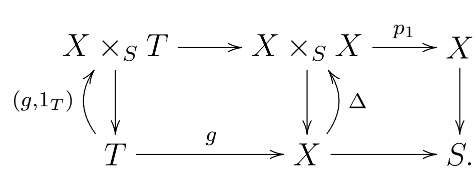
Lemma 1.2. *Composing a section $T \to X \times_S T$ with the projection map $X \times_S T \to X$ gives a map of functors*

$$n : \mathcal{S}_X \longrightarrow \mathrm{Hom}_S(-, X), \tag{1.2.1}$$

which is an isomorphism. The inverse of (1.2.1) is induced by the diagonal map $\Delta : X \to X \times_S X$.

Proof. Let $\delta : \mathrm{Hom}_S(-, X) \to \mathcal{S}_X$ denote the map of functors induced from the diagonal $\Delta \in \mathcal{S}_X(X)$, and denote by p_1 the first projection $X \times_S X \to X$.

Then for any S-morphism $g : T \to X$ we have $\delta(g) = (p_1 \circ \Delta \circ g, 1_T) = (g, 1_T)$, as seen by the diagram below:



Also, for any section $\xi = (g, 1_T) : T \to X \times_S T$ we have by definition $n(\xi) = g$. It is thus clear that n and δ are inverse to each other. □

Remark 1.3. It follows from Lemma (1.2) that any section of a separated morphism $f : X \to S$ is closed.

Lemma 1.4. *Let $f : X \to \mathrm{Spec}(A)$ be a morphism of schemes, where $(A, \mathfrak{m})$ is a complete local ring. For each $n \geq 1$ we let $A_n := A/\mathfrak{m}^{n+1}$ and $X_n := X \times_A \mathrm{Spec}(A_n)$. Furthermore we let $f_n : X_n \to \mathrm{Spec}(A_n)$ denote the induced morphism.*

(a) *Sections ξ_n of $f_n : X_n \to \mathrm{Spec}(A_n)$ are closed.*
(b) *Let ξ' and ξ be sections of $f : X \to \mathrm{Spec}(A)$, such that when restricted to $\mathrm{Spec}(A_n)$ we have $\xi'_n = \xi_n$ for all n. Then $\xi = \xi'$.*

Proof. As the underlying topological space $\mid \mathrm{Spec}(A_n) \mid$ is one point it follows that the image of the section ξ_n is closed in any open affine $U \subseteq X_n$. This proves the first statement.

To prove the second assertion we let $\mathrm{Spec}(B) \subseteq X$ be an open affine subscheme containing the image of the closed point $\mathrm{Spec}(A/\mathfrak{m})$ under ξ. The sections $\xi_n : \mathrm{Spec}(A_n) \to X_n$, composed with the closed immersions $X_n \to X$ factor through $\mathrm{Spec}(B)$. The corresponding ring homomorphism $B \to A_n$ determines a unique morphism to the inverse limit $\varprojlim A_n = A$. As ξ' coincides with ξ when restricted to $\mathrm{Spec}(A_n)$ it follows that $\xi' = \xi$. □

1.5. **The Hilbert functor of points.** For a fixed scheme X over some base S we let $\mathcal{H}^m_X$ be the Hilbert functor of m-points on X, as defined by Grothendieck [FGA, p. 221-26]. Thus, for any S-scheme T we have that the T-valued points of $\mathcal{H}^m_X$ is the set of closed subschemes $Z \subseteq X \times_S T$ such that the induced projection $p : Z \to T$ is flat and finite of rank m. In other words, p is flat and finite, and $p_*\mathcal{O}_Z$ is locally free of rank m as an $\mathcal{O}_T$-module.

Lemma 1.6. *Let $U \subseteq X$ be an open subscheme, where X is separated over the base scheme S. Then there is a natural transformation $\mathcal{H}^m_U \to \mathcal{H}^m_X$.*

Proof. As a T-valued point Z of $\mathcal{H}^m_U$ is finite over T it is in particular proper over T by [EGA$_{\mathrm{II}}$, Cor. 6.1.11]. Thus the composition $Z \to X \times_S T \to T$ is proper and the projection map $X \times_S T \to T$ is separated. It follows from [EGA$_{\mathrm{II}}$, Cor. 5.4.3] that the immersion $Z \to X \times_S T$ is proper and hence closed, and so we get an element of $\mathcal{H}^m_X(T)$. □

Remark 1.7. If $U \subset X$ is an open immersion then it is *not* true in general that we have a map of functors $\mathcal{H}^1_U \to \mathcal{H}^1_X$, as the following example shows.

Example 1.8. Let X denote the line with a double point. We obtain X by glueing two copies of the line along the open complement of a closed point. In particular we can let $U \subset X$ be the open subscheme given by one of the lines. Finally we let the base S be the line, with the natural projection morphism $X \to S$. Now it is clear that the whole line U itself is flat and finite of rank one over the base. However, the line U is not closed in X, but clearly closed in U. Thus there is not a natural map from $\mathcal{H}^1_U$ to $\mathcal{H}^1_X$. In fact $\mathcal{H}^1_U(U)$ is a singleton set, whereas $\mathcal{H}^1_X(U)$ is the empty set.

Remark 1.9. Let $\mathcal{F}$ be a coherent sheaf on X, and let $\mathrm{Quot}(\mathcal{F}/X/S)$ denote the Quot-functor (see e.g. [FGA] or [Art69]). When $f : X \to S$ is locally of finite presentation, Artin applies the algebraization theorem to show that the Quot-functor is representable by an algebraic space in [Art69, §6] - but the additional hypothesis that $f : X \to S$ is separated is required (see corrections in the appendix of [Art74]). The mistake in the proof of [Art69, Thm 6.1, p.61] appears to be the reduction to the Quot-functor $\mathrm{Quot}(\mathcal{F}'/X'/S)$, where X' is open in X. Because, as pointed out in Remark (1.7), there does not always exist a map $\mathrm{Quot}(\mathcal{F}'/X'/S) \to \mathrm{Quot}(\mathcal{F}/X/S)$.

1.10. **The Hilbert functor of one point.** We will now focus on a particular case of the Hilbert functor of points, namely when $m = 1$. In that case we have that the projection $p : Z \to T$ is finite and flat of rank 1, and then p must be an isomorphism. The inverse to p gives a closed section $T \to X \times_S T$ and thus we may identify the set $\mathcal{H}^1_X(T)$ with the set of closed sections $T \to X \times_S T$. In particular we see that $\mathcal{H}^1_X$ is a subfunctor of $\mathcal{S}_X$.

Proposition 1.11. *The map of functors (1.2.1) induces a natural map*

$$n_X : \mathcal{H}^1_X \longrightarrow \mathrm{Hom}_S(-, X), \tag{1.11.1}$$

which is an isomorphism if and only if $f : X \to S$ is separated.

Proof. When $f : X \to S$ is separated we have that any section is closed. Consequently we have that $\mathcal{H}^1_X = \mathcal{S}_X$, and the proposition is then a special case of Lemma (1.2). When $f : X \to S$ is not separated there exist non-closed sections, e.g. the diagonal map $X \to X \times_S X$. Therefore $\mathcal{H}^1_X(X) \subset \mathcal{S}_X(X)$ is a proper subset and the map n_X is not an isomorphism. □

Remark 1.12. In [FGA, p. 221-26] Grothendieck introduced a norm map from the Hilbert scheme $\mathrm{Hilb}^m_{X/S}$ to the m-fold symmetric product $\mathrm{Sym}^m_S(X)$. The map (1.11.1) is this norm map for $m = 1$.

Remark 1.13. In the definition of the Hilbert functor $\mathcal{H}^m_X$, one could replace closed subschemes with locally closed subschemes. In that case we would have equality $\mathcal{H}^1_X = \mathcal{S}_X$, and in particular the norm map (1.11.1) would be an isomorphism. See [Kle90, Prop 2.2, Cor. 2.3]. However, it is not clear that the refined definition of $\mathcal{H}^m_X$ would prove to be representable (for $m > 1$). For another definition see [Art74, p. 186].

2. Formal deformations of the Hilbert functor

Before we prove our main result, we will, for the sake of completeness, prove that deformations of algebraic spaces are effective. A similar proof for a restricted class of algebraic spaces[1] can be found in [Art69, p. 37].

2.1. **Effective deformations.** Let S be a scheme, and let F be a contravariant functor from the category of S-schemes to sets. If $X = \operatorname{Spec}(A)$ is an affine scheme, we write $F(A)$ instead of $F(X)$.

Given a field k and an element $\xi_0 \in F(k)$. A *formal deformation* of ξ_0 is a pair $(A, \{\xi_n\}_{n\geq 0})$ where A is a complete local ring with residue field k and $\{\xi_n\}_{n\geq 0}$ is a collection of elements with $\xi_n \in F(A/\mathfrak{m}^{n+1})$ such that ξ_{n-1} is induced from ξ_n and ξ_0 is the original element. The deformation is called *effective* if there is an element $\xi \in F(A)$ inducing the elements $\{\xi_n\}$.

Remark 2.2. Any formal deformation of a scheme $F = \operatorname{Hom}_S(-, Y)$ is effective. Indeed, if $\xi_n : \operatorname{Spec}(A/\mathfrak{m}^{n+1}) \to Y$ is a compatible collection of morphisms, then all ξ_n factor through any open affine $U \subseteq Y$ containing the image of the point $\operatorname{Spec}(A/\mathfrak{m})$. Consequently the collection of maps ξ_n can be reduced to the affine case where it is clear.

Lemma 2.3. *Let A be a complete local ring with maximal ideal $\mathfrak{m}$, and let $A_n = A/\mathfrak{m}^{n+1}$. For each n, we assume that we have a free A_n-module E_n of rank k, such that $E_n \otimes_{A_n} A_{n-1} = E_{n-1}$.*

(a) *The inverse limit $E = \varprojlim E$ is a free A-module of rank k.*
(b) *Let E'_n be another system of free A_n-modules of rank k', with inverse limit E'. Then the natural map $E \otimes_A E' \to \varprojlim(E_n \otimes_{A_n} E'_n)$ is an isomorphism of A-modules.*

Proof. By assumption we have that $E = \varprojlim E_n = \varprojlim(\oplus^k A_n)$. It is readily checked that products commute with finite direct sums, and consequently that $E = \oplus^k(\varprojlim A_n) = \oplus^k A$. To prove the second statement, we have from the first statement that $\varprojlim(E_n \otimes_{A_n} E'_n)$ is a free A-module of rank $k \cdot k'$. It is then clear that the canonical map $E \otimes_A E' \to \varprojlim(E_n \otimes_{A_n} E'_n)$ is an isomorphism of free A-modules. □

Proposition 2.4. *A formal deformation of an algebraic space F is effective.*

Proof. Let A be a complete local ring, and let $A_n = A/\mathfrak{m}^{n+1}$. To a given collection $\{\xi_n\}$ of compatible elements $\xi_n \in F(A_n)$, we shall show that there exist $\xi \in F(A)$ inducing ξ_n. As the algebraic space F has an open covering of quasi-compact spaces F_α, and as any morphism

[1]Algebraic spaces that are locally of finite type over the base S, and where the base is of finite type over a field, or over an excellent Dedekind domain.

$\mathrm{Spec}(A_n) \to F$ would have to factor through some F_α, we can assume that F is quasi-compact [Knu71, Ch. II, Prop. 3.13].

Let $\pi : U \to F$ be a representable étale covering, and let $U_n = U \times_F \mathrm{Spec}(A_n)$. Then the morphism of schemes $U_n \to \mathrm{Spec}(A_n)$ is an étale covering, and consequently $\mathrm{Spec}(A_n)$ is the categorical quotient of $U_n \times_{A_n} U_n \rightrightarrows U_n$. Furthermore, the projection map $z_n : U_n \to U$ is a lifting of $\xi_n : \mathrm{Spec}(A_n) \to F$.

As U is a quasi-compact scheme, we have that $U_0 \to \mathrm{Spec}(A/\mathfrak{m})$ is the finite disjoint union of field spectra, and in particular U_0 is affine. We also have that $U_n \times_{A_n} \mathrm{Spec}(A_0) = U_0$, and consequently the underlying topological space $\mid U_n \mid$ is as U_0. Hence U_n is the finite disjoint union of spectra of local rings, and in particular U_n is affine. Let U_n be the spectrum of the A_n-algebra E_n. We have that E_n is étale, and hence flat over A_n. Furthermore, the maximal ideal of A_n is nilpotent and so E_n is free of finite rank $k = \dim_{A_0} E_0$ over A_n [Bou61, Ch. II, §3, Prop 5].

The rings $E_n = \prod_j F_n^j$ are finite products of local rings F_n^j and the morphisms $E_n \to E_{n-1}$ induce maps $F_n^j \to F_{n-1}^j$ for each j. For a fixed index j the morphisms $\mathrm{Spec}(F_n^j) \to U$ factor through an open affine of U, inducing a canonical morphism $\mathrm{Spec}(F^j) \to U$ where $F^j = \varprojlim F_n^j$. Furthermore, since products commute with inverse limits we have that $E = \varprojlim E_n = \prod_j F^j$. Thus we obtain a morphism z from $\hat{U} = \mathrm{Spec}(E) = \coprod_j \mathrm{Spec}(F^j)$ to U.

We have by Lemma (2.3) that E is a free A-module, and we claim that E is in fact an étale extension of A. As E is free of finite rank we can determine étaleness by the determinant d_E of the bilinear trace form of E/A [$\mathrm{EGA_{IV}}$, 18.2.1, Prop. 18.2.3]: The A-algebra E is étale if and only if the determinant d_E is a unit in A. Clearly we have that the image of d_E in each A_n is d_{E_n}. As each E_n is étale over A_n it follows that d_{E_n} is a unit for each n, and hence E is étale. Thus $\mathrm{Spec}(A)$ is the quotient of the equivalence relation $\hat{U} \times_A \hat{U} \rightrightarrows \hat{U}$.

We have that $U_n \times_{A_n} U_n$ is the spectrum of $E_n \otimes_{A_n} E_n$, and by the same reasoning as above we get an induced morphism of schemes $\mathrm{Spec}(\varprojlim(E_n \otimes_{A_n} E_n)) \to U \times_F U$. By Lemma (2.3) we have that $\varprojlim(E_n \otimes_{A_n} E_n) = E \otimes_A E$, and it follows that we obtain a morphism $\hat{U} \times_A \hat{U} \to U \times_F U$. Summarizing, we have shown that there is a commutative diagram

$$\begin{array}{ccccc} \hat{U} \times_A \hat{U} & \rightrightarrows & \hat{U} & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow{\scriptstyle z} & & \\ U \times_F U & \rightrightarrows & U & \xrightarrow{\pi} & F. \end{array}$$

In other words we have a morphism of algebraic spaces $\xi : \mathrm{Spec}(A) \to F$, whose restriction to $\mathrm{Spec}(A/\mathfrak{m}^{n+1})$ is ξ_n. $\square$

2.5. **Formal deformations of the Hilbert functor.** For any complete local ring $(A, \mathfrak{m})$ we consider the natural map

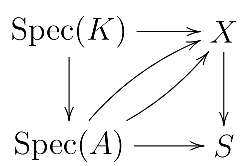
$$\mathcal{H}^1_X(A) \longrightarrow \varprojlim \mathcal{H}^1_X(A/\mathfrak{m}^{n+1}), \tag{2.5.1}$$

where $\mathcal{H}^1_X$ is the Hilbert functor defined in (1.10). A consequence of Lemma (1.4) is that the map (2.5.1) is injective. Our main result is that that the map (2.5.1) is not always surjective.

Theorem 2.6. *Let $f : X \to S$ be a morphism of schemes that does not satisfy the valuative criterion of separatedness. Then the Hilbert functor $\mathcal{H}^1_X$ has non-effective formal deformations. In particular, for such X, the functor $\mathcal{H}^1_X$ is not representable by a scheme or an algebraic space.*

Proof. We first construct a formal deformation of $\mathcal{H}^1_X$. By definition, as $X \to S$ does not satisfy the valuative criterion of separatedness, there exists a valuation ring A with fraction field K, and a commutative diagram of schemes

(2.6.1)



allowing at least two extensions $\mathrm{Spec}(A) \to X$ of $\mathrm{Spec}(K) \to X$. Furthermore, by [EGA$_{\mathrm{II}}$, Rem. 7.2.4] we can assume that the ring A is complete. Consider now one of the sections $\xi : \mathrm{Spec}(A) \to X_A := X \times_S \mathrm{Spec}(A)$ that we obtain from the diagram (2.6.1). As we have at least one other section extending the map from the generic point $\mathrm{Spec}(K) \to X_A$ it follows that ξ is not closed. Hence ξ is not a $\mathrm{Spec}(A)$-valued point of $\mathcal{H}^1_X$.

By Lemma (1.4), the induced sections $\xi_n : \mathrm{Spec}(A/\mathfrak{m}^{n+1}) \to X \times_S \mathrm{Spec}(A/\mathfrak{m}^{n+1})$ are closed. Therefore the collection of sections $\{\xi_n\}$ form a formal deformation of $\xi_0 : \mathrm{Spec}(A/\mathfrak{m}) \to X \times_S \mathrm{Spec}(A/\mathfrak{m})$. If there was an element $\xi' \in \mathcal{H}^1_X(A)$ whose truncations would form the constructed formal deformation, it would have to be ξ by Lemma (1.4). However, as $\xi \notin \mathcal{H}^1_X(A)$ we have that the constructed deformation is not effective. $\square$

Remark 2.7. A scheme $f : X \to S$ such that the diagonal map $X \to X \times_S X$ is quasi-compact is called *quasi-separated*. For such schemes we have by [EGA$_{\mathrm{II}}$, Prop. 7.2.3] that $X \to S$ is separated if and only if the valuative criterion holds. A consequence of Theorem (2.6) is therefore that if $X \to S$ is quasi-separated, then $\mathcal{H}^1_X$ is representable if and only if $X \to S$ is separated. The Hilbert functor of non-quasi-separated schemes such that the valuative criterion holds we were not able to say anything conclusive about.

For a discussion of quasi-separatedness in relation to algebraic spaces, see [Knu71, p. 8].

Remark 2.8. It is notable that the only thing we need in order to prove Theorem (2.6), is the existence of a non-closed section from a complete local ring A. We do not use the fact that the ring is a valuation ring.

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