INFINITE INTERSECTIONS OF OPEN SUBSCHEMES AND THE HILBERT SCHEME OF POINTS.

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Abstract. We study infinite intersections of open subschemes and the corresponding infinite intersection of Hilbert schemes. If $\{U_\alpha\}$ is the collection of open subschemes of a variety X containing the fixed point P, then we show that the Hilbert functor of flat and finite families of $\mathrm{Spec}(\mathcal{O}_{X,P}) = \bigcap_\alpha U_\alpha$ is given by the infinite intersection $\bigcap_\alpha \mathcal{H}ilb_{U_\alpha}$, where $\mathcal{H}ilb_{U_\alpha}$ is the Hilbert functor of flat and finite families on U_α. In particular we show that the Hilbert functor of flat and finite families on $\mathrm{Spec}(\mathcal{O}_{X,P})$ is representable by a scheme.

1. - Introduction

We will consider in this article infinite intersections of open subschemes $\{U_\alpha\}$ of a fixed ambient scheme X. We are interested in the corresponding Hilbert scheme and in particular in the Hilbert scheme of $\mathrm{Spec}(\mathcal{O}_{X,P})$ the intersection of the open subschemes containing a point P in X.

For a scheme X the Hilbert scheme of n-points Hilb^n_X (if it exists) represents the functor of finite flat families of length-n closed subschemes of X. Grothendieck constructed Hilb^n_X for X quasi-projective over a noetherian base scheme, but we wish to look at $\mathrm{Spec}(\mathcal{O}_{X,P})$ for P a point of such an X. We know that if U is an open subscheme of X then Hilb^n_U is an open subscheme of Hilb^n_X, so there is a natural candidate for the Hilbert scheme of points on an infinite intersection $\bigcap U_\alpha$ of open subschemes of X, namely the corresponding infinite intersection $\bigcap \mathrm{Hilb}^n_{U_\alpha}$. Note (see Proposition (2.3)) however that an infinite intersection of open subschemes is not necessarily a scheme!

We restrict ourselves to infinite intersection of locally principal open subschemes. The technical heart of the paper is the study of such infinite intersections, which we call localized schemes. The notion of localized schemes and generalized fraction rings is carried out in Section (3). These concepts are thereafter applied to show that the Hilbert functor of points on a localized scheme $\mathcal{S}^{-1}X$ is representable, if the Hilbert scheme of points on X exists. A special case of that statement gives the following.

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Result. *Let $X \to S$ be a projective morphism of noetherian schemes. Let P be a point on X, with stalk $\mathcal{O}_{X,P}$. Then the Hilbert functor of n-points on* $\operatorname{Spec}(\mathcal{O}_{X,P})$ *is representable by a noetherian scheme* $\operatorname{Hilb}^n_{\mathcal{O}_{X,P}}$. *Furthermore, if $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is the collection of open subschemes of X containing the point P, then the Hilbert scheme of n-points on* $\operatorname{Spec}(\mathcal{O}_{X,P})$ *is given as the infinite intersection* $\operatorname{Hilb}^n_{\mathcal{O}_{X,P}} = \bigcap_{\alpha\in\mathcal{A}} \operatorname{Hilb}^n_{U_\alpha}$.

The above localization property for Hilbert functors of points was known to hold for the affine line $X = \operatorname{Spec}(k[x])$ (see [LS] and [S]) where the Hilbert scheme of points on fraction rings of $k[x]$ were constructed explicitly. Here we show that the localization property of the Hilbert functors of points hold for localized schemes.

What happens is the following. If L is a line bundle on X, then we get by pulling back L to the universal family of n-points on X, a vector bundle of rank n over the Hilbert scheme Hilb^n_X. From each global section of L we get a determinant section of the norm bundle $N(L)$ on Hilb^n_X. If $U_s \subseteq X$ is the open subscheme defined by the non-vanishing of a section $s \in \Gamma(X, L)$, then we show that the Hilbert scheme of n-points on U_s is the open subscheme of Hilb^n_X given by the non-vanishing of the corresponding determinant section of the norm bundle $N(L)$ on Hilb^n_X.

2. - Infinite Intersections of open subschemes

Let X be a scheme, and let $\{U_\alpha \subseteq X \mid \alpha \in \mathcal{A}\}$ be a collection of open subschemes of X. The set-theoretic intersection $\bigcap_{\alpha\in\mathcal{A}} U_\alpha$ can be made into a locally ringed space by giving it the topology induced by the Zariski topology of X and by using as structural sheaf the inverse image sheaf $i^{-1}\mathcal{O}_X$, where $i : \bigcap_{\alpha\in\mathcal{A}} U_\alpha \to X$ is the inclusion.

In the category of locally ringed spaces we have that $\bigcap_{\alpha\in\mathcal{A}} U_\alpha = \varprojlim_{\alpha\in\mathcal{A}} U_\alpha$. When $\bigcap_{\alpha\in\mathcal{A}} U_\alpha$ is a scheme, we also have that $\bigcap_{\alpha\in\mathcal{A}} U_\alpha = \varprojlim_{\alpha\in A} U_\alpha$ in the category of schemes. However, $\bigcap_{\alpha\in\mathcal{A}} U_\alpha$ is not necessarily a scheme; indeed $\varprojlim_{\alpha\in A} U_\alpha$ does not always exist in the category of schemes (see Proposition 2.3 below).

2.1. Theorem (Grothendieck)**.** *Let $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection of open subschemes of a scheme X. If the inclusion maps $i_\alpha : U_\alpha \to X$ are affine morphisms, then the locally ringed space $\bigcap_\alpha U_\alpha$ is a scheme. Moreover, the inclusion $\bigcap_\alpha U_\alpha \to X$ is an affine monomorphism.*

Proof. All the assertions of the theorem are proven in [EGA] **IV** §8.2 when the system $\{U_\alpha\}$ of open subsets is *filtered*, i.e. for any α, β there exists a γ such that $U_\gamma \subseteq U_\alpha \cap U_\beta$. But we may reduce to the filtered case by replacing $\{U_\alpha\}$ with the system of all finite intersections $\{U_{\alpha_1} \cap \cdots \cap U_{\alpha_r}\}$ because the inclusion maps remain affine morphisms while the categorical limit is unchanged.

The construction of [EGA] is that if the inclusions $U_\alpha \subseteq X$ come locally from maps of commutative rings $A \to B_\alpha$, then $\bigcap_\alpha U_\alpha \to X$ comes from $A \to \varinjlim_\alpha B_\alpha$. We will use this in later arguments. $\square$

2.1.1. Locally principal subschemes. An open subscheme $U \subseteq X$ is *locally principal* if X can be covered by affine open subschemes $\operatorname{Spec}(A_i)$ such that each $U \cap \operatorname{Spec}(A_i)$ is a principal affine open subscheme of $\operatorname{Spec}(A_i)$ (i.e. of the form $\operatorname{Spec}(A_{i,f_i})$ for some $f_i \in A_i$). The inclusion $U \subseteq X$ of a locally principal open subscheme is an affine morphism, so Theorem (2.1) applies.

2.2. Corollary. *If the $U_\alpha \subseteq X$ are locally principal open subschemes for all $\alpha \in \mathcal{A}$, then the locally ringed space $\bigcap_{\alpha\in\mathcal{A}} U_\alpha$ is a scheme.*

2.3. Proposition. *Let $X = \operatorname{Spec} k[x,y]$ be the affine plane over a field k, and let $\{E_\alpha\}_{\alpha\in\mathcal{A}}$ be the collection of finite subsets of closed points of X. Then the locally ringed space $Y = \bigcap_\alpha (X \setminus E_\alpha)$ is not a scheme.*

Proof. As a set Y is the union $\{\xi\} \cup X_1$ where ξ is the generic point of the plane, and X_1 is the set of generic points of irreducible plane curves. The open subsets of Y are induced by the open subsets of X, and they are all of the form $Y \cap U_f$ where $U_f = \operatorname{Spec} k[x,y]_f$ is a principal open subset of X. If Y were a scheme, it would be covered by affine open subschemes, and there would exist an $f \in k[x,y] \setminus \{0\}$ such that $Y \cap U_f$ is an affine scheme. We claim this is impossible.

Let $S = k[x] \setminus \{0\}$ and $T = k[y] \setminus \{0\}$. These are multiplicative systems such that (i) any maximal ideal of $k[x,y]$ meets both S and T, and (ii) $S \cap T = k^*$. Because of Property (i) the schemes $\operatorname{Spec} k[x,y]_{f,S}$ and $\operatorname{Spec} k[x,y]_{f,T}$ contain none of the closed points of X, and so we have a commutative diagram of locally ringed spaces.

$$\begin{array}{ccccc}
\operatorname{Spec} k(x,y) & \longrightarrow & \operatorname{Spec} k[x,y]_{f,S} & & \\
\downarrow & & \downarrow & & \\
\operatorname{Spec} k[x,y]_{f,T} & \longrightarrow & Y \cap U_f & \longrightarrow & U_f
\end{array}$$

All the maps are inclusions between infinite intersections of open subschemes of X. Because of Property (ii) one has an equality $k[x,y]_{f,S} \cap k[x,y]_{f,T} = k[x,y]_f$ of subrings of $k(x,y)$. Since the intersection of two subrings gives a pullback in the category of commutative rings, dually U_f is the pushout in the category of affine schemes. Hence if $Y \cap U_f$ were an affine scheme, then the universal property of pushouts would give us maps $U_f \to Y \cap U_f \hookrightarrow U_f$ whose composition is the identity. But the second map is the natural inclusion, which is not surjective. Contradiction. So no nonempty $Y \cap U_f$ can be an affine scheme, and Y is not a scheme. $\square$.

2.4. Infinite intersection of Noetherian schemes.

If B is an A-algebra we denote with IB the extension of an ideal $I \subseteq A$, and with $J \cap A$ the contraction of an ideal $J \subseteq B$.

2.5 Lemma. *Let $\varphi : A \to B$ be a homomorphism of commutative rings. Assume that the corresponding morphism of schemes* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *is an open immersion. Then any ideal $J \subseteq B$ is the extension of its contraction to A.*

Proof. Let $J \subseteq B$ be an ideal. The extension of the contraction $(J \cap A)B$ is trivially contained in J and we need only to show that $J \subseteq (J \cap A)B$.

Since affine schemes are quasi-compact, $\operatorname{Spec}(B)$ can be covered by a finite number of principal affine open subschemes of $\operatorname{Spec}(A)$. Thus there exist $f_1, \dots, f_r$ in A such that $\operatorname{Spec}(B) = \bigcup_{i=1}^r \operatorname{Spec}(A_{f_i})$. The induced maps $A_{f_i} \to B_{\varphi(f_i)}$ are isomorphisms, and one deduces that for any element x in the ideal $J \subseteq B$ there exist elements $a_1, \dots, a_r$ in A such that $\varphi(a_i) = \varphi(f_i)^N x$, for some $N \gg 0$. In particular we have that $a_i \in \varphi^{-1}(x) \subseteq J \cap A$ for each $i = 1, \dots, r$. Since the $\operatorname{Spec}(B_{\varphi(f_i)})$ cover $\operatorname{Spec}(B)$ it follows that there exist $b_1, \dots, b_r$ such that $\sum_{i=1}^r b_i \varphi(f_i)^N = 1$. Then we have that $x = \sum_{i=1}^r b_i \varphi(a_i)$ is in the extension of $J \cap A$, hence $J \subseteq (J \cap A)B$. $\square$

2.6. Lemma. *Suppose we are given a direct (or filtered) system of commutative rings A_i and transition maps $\varphi_{ij} : A_j \to A_i$ such that any ideal in A_i is the extension of the contraction to A_j. Then any ideal in $\varinjlim_i A_i$ is the extension of the contraction to A_j via the natural homomorphism $A_j \to \varinjlim_i A_i$.*

Proof. Let J be an ideal of the direct colimit $A = \varinjlim_i A_i$. From the the assumption of the transition maps φ_{ij} we have $J \cap A_i = (\varphi_{ij}^{-1}(J \cap A_i))A_i = (J \cap A_j)A_i$. By the exactness of the direct colimit we get that $\varinjlim_i (J \cap A_i)$ is an ideal in $\varinjlim_i A_i$, easily seen to coincide with J. □

2.7. Proposition. *In the situation of Theorem (2.1), if X is a noetherian scheme then so is $\bigcap_{\alpha \in \mathcal{A}} U_\alpha$.*

Proof. Assume first that $X = \mathrm{Spec}(A)$ is affine. Then $\bigcap_{\alpha \in \mathcal{A}} U_\alpha \to X$ is given by $A \to \varinjlim_\alpha B_\alpha$. We must show that $\varinjlim_\alpha B_\alpha$ is noetherian. By Lemma (2.5) we have that the homomorphism of rings $\varphi_\alpha : A \to B_\alpha$ is such that the extension of the contraction of an ideal $J \subseteq B_\alpha$ equals J. It follows from Lemma (2.6) that any ideal $J \subseteq \varinjlim_\alpha B_\alpha$ is the extension of its contraction to A. Since A is noetherian and consequently any ideal of A is finitely generated, it follows that any ideal of $\varinjlim_\alpha B_\alpha$ is finitely generated. Hence $\varinjlim_\alpha B_\alpha$ is noetherian.

If X is simply a noetherian scheme, then $Y = \bigcap_{\alpha \in \mathcal{A}} U_\alpha$ is given locally by the construction above, so Y is locally noetherian. Since X is quasi-compact and the morphism $Y \to X$ is affine and hence quasi-compact, Y is also quasi-compact. Hence Y is a noetherian scheme. □

3. - Localized schemes and generalized fraction rings

3.1. Localized schemes. Let X be a scheme. We will write sections of invertible sheaves on X as pairs (s, L), where $s : \mathcal{O}_X \to L$ is a global section of the invertible sheaf L. We let $U_s \subseteq X$ denote the open subscheme where the section s is non-vanishing, that is the complement of the support of s.

3.2. Theorem. *Let $\mathcal{S} = \{(s_\alpha, L_\alpha)\}_{\alpha \in \mathcal{A}}$ be a collection of sections of invertible sheaves on X. Then there exists a morphism of schemes $i_{\mathcal{S}} : \mathcal{S}^{-1}X \to X$ such that the following two assertions hold.*

(1) *The pull-back $i_{\mathcal{S}}^*(s_\alpha) : \mathcal{O}_{\mathcal{S}^{-1}X} \to i_{\mathcal{S}}^* L_\alpha$ is nowhere vanishing on $\mathcal{S}^{-1}X$, for all $\alpha \in \mathcal{A}$.*

(2) *Any homomorphism $f : T \to X$ of schemes such that $f^*(s_\alpha) : \mathcal{O}_T \to f^* L_\alpha$ is nowhere vanishing on T for all $\alpha \in \mathcal{A}$, has a unique factorization via $i_{\mathcal{S}}$.*

Moreover, $i_{\mathcal{S}} : \mathcal{S}^{-1}X \to X$ is unique up to unique isomorphism.

Proof. Each $U_{s_\alpha} \subseteq X$ is a locally principal open subscheme, thus by Corollary (2.2) we have that the inclusion $\bigcap_{\alpha \in \mathcal{A}} U_{s_\alpha} \to X$ is a morphism of schemes. Let $\mathcal{S}^{-1}X = \bigcap_{\alpha \in \mathcal{A}} U_{s_\alpha}$ and let $i_{\mathcal{S}}$ be the inclusion $\mathcal{S}^{-1}X \to X$.

Note that $\mathcal{S}^{-1}X = \varprojlim_{\alpha \in \mathcal{A}} U_{s_\alpha}$ such that a morphism of schemes $f : T \to X$ factors via $i_{\mathcal{S}} : \mathcal{S}^{-1}X \to X$ if and only if f factors via $i_{s_\alpha} : U_{s_\alpha} \to X$ for all $\alpha \in \mathcal{A}$. Assertion (1) then follows since it is clear that the pull-back of a section $s : \mathcal{O}_X \to L$ along the inclusion $i_s : U_s \to X$ is non-vanishing.

To show Assertion (2) it suffices to show that for a given section (s, L) on X a morphism $f : T \to X$ factors via $i_s : U_s \to X$ if and only if $f^*(s)$ is non-vanishing on T. We can cover X by open affine subschemes $\{\mathrm{Spec}(A_i)\}_{i \in \mathcal{I}}$, such that $U_s \cap \operatorname{Spec}(A_i)$ is given by some principal open subschemes $\operatorname{Spec}(A_{i,f_i})$ of $\operatorname{Spec}(A_i)$. Assertion (2) now follows from the universal properties of fraction rings.

It is clear that the condition on the morphism $f : T \to X$ given in Assertion (2) defines a functor which is represented by the scheme $\mathcal{S}^{-1}X$ with universal element $i_{\mathcal{S}} : \mathcal{S}^{-1}X \to X$, hence uniqueness follows. □

3.3. Lemma. *Let $\mathcal{S} = \{(s_\alpha, L_\alpha)\}_{\alpha\in\mathcal{A}}$ be a collection of sections of invertible sheaves on X, and $p^*\mathcal{S} = \{(p^*(s_\alpha), p^*L_\alpha)\}_{\alpha\in\mathcal{A}}$ the pull-back of $\mathcal{S}$ along a given morphism of schemes $p : Z \to X$. Then the localization map $i_{p^*\mathcal{S}} : (p^*\mathcal{S})^{-1}Z \to Z$ and the pull-back $\mathcal{S}^{-1}X \times_X Z \to Z$ of the localization map on X, coincide up to unique isomorphism.*

Proof. One immediately checks that the map $\mathcal{S}^{-1}X \times_X Z \to Z$ satisfies the two conditions (1) and (2) of Theorem (3.2), which proves the claim. □

3.3.1. Remark. Let X be a scheme over some base S, and let $T \to S$ be a morphism of schemes. Let $\mathcal{S} = \{(s_\alpha, L_\alpha)\}_{\alpha\in\mathcal{A}}$ be a collection on X, and let U_{s_α} be the locally principal open subscheme defined by the section (s_α, L_α). The natural morphism of schemes

$$\bigcap_{\alpha\in\mathcal{A}} (U_{s_\alpha} \times_S T) = \varprojlim_{\alpha\in\mathcal{A}} (U_{s_\alpha} \times_S T) \to (\varprojlim_{\alpha\in\mathcal{A}} U_{s_\alpha}) \times_S T = (\bigcap_{\alpha\in\mathcal{A}} U_\alpha) \times_S T$$

is an isomorphism by Lemma (3.3).

3.4. Generalized fraction rings. Let R be a ring (commutative with unit), and let $U = \{(s_\alpha, L_\alpha)\}_{\alpha\in\mathcal{A}}$ be a collection of pairs $s_\alpha \in L_\alpha$ with L_α an invertible R-module. Let $\mathbf{N}\cdot\mathcal{A}$ denote the subset of $\mathbf{N}^{\mathcal{A}}$ consisting of systems of non-negative integers $a = \{a_\alpha\}_{\alpha\in\mathcal{A}}$ having only a finite number of non-zero components. The set $\mathbf{N}\cdot\mathcal{A}$ is naturally partially ordered where we say that $a \leq b$ if for each component we have $a_\alpha \leq b_\alpha$. We define for any $a \in \mathbf{N}\cdot\mathcal{A}$ the invertible R-modules

$$\underline{L}^a = \bigotimes_{a_\alpha\neq 0} L_\alpha^{\otimes a_\alpha} \quad \text{and} \quad \underline{L}^0 = R. \tag{3.4.1}$$

For any $b \in \mathbf{N}\cdot\mathcal{A}$ we have a natural identification $\underline{L}^a \otimes_R \underline{L}^b = \underline{L}^{a+b}$. We have furthermore the element

$$s^b = \bigotimes_{b_\alpha\neq 0} s_\alpha^{\otimes b_\alpha} \in \underline{L}^b \tag{3.4.2}$$

The element s^b defines R-module homomorphisms

$$\underline{L}^a \to \underline{L}^{a+b} \tag{3.4.3}$$

sending $x \in \underline{L}^a$ to $x \otimes s^b \in \underline{L}^a \otimes_R \underline{L}^b = \underline{L}^{a+b}$. We denote the direct colimit of the R-modules (3.4.1) and the described transition maps (3.4.3) as

$$R_U = \operatorname*{\underrightarrow{colim}}_{a\in\mathbf{N}\cdot\mathcal{A}} \{\underline{L}^a\}. \tag{3.4.4}$$

Note that we have a natural product structure on R_U with $\underline{L}^a \cdot \underline{L}^b \subseteq \underline{L}^{a+b}$, given by $x^a \cdot y^b := x^a \otimes y^b$. As the R-modules $\underline{L}^a$ are invertible for all $a \in |\mathcal{A}|$, we have that $x^a \cdot y^b = y^b \cdot x^a$. Hence R_U is a commutative ring. As $R = \underline{L}^0$, we have that R_U is a commutative R-algebra. We call R_U a generalized fraction ring (with respect to $U = \{(s_\alpha, L_\alpha)\}_{\alpha\in\mathcal{A}}$). If we have $L_\alpha = R$ for all α, the direct colimit R_U is the fraction ring $V^{-1}R$, where $V \subseteq R$ is the multiplicative system generated by the s_α.

3.5. Properties of the generalized fraction rings. Let $U = \{(s_\alpha, L_\alpha)\}_{\alpha\in\mathcal{A}}$ be a collection of invertible modules. We will in this section list some properties of the generalized fraction rings R_U, properties that we will use later on in Section (4).

3.5.1. Remark. We have that R_U is an R-algebra, thus also an R-module. By definition R_U is the direct colimit of locally free, in particular flat, R-modules $\underline{L}^a$, hence R_U is a flat R-module.

3.5.2. Remark. If N is a R-module we denote by $N_U := R_U \otimes_R N$. We have that tensor product commute with direct colimit hence

$$N_U = \varinjlim_{a\in\mathbf{N}\cdot\mathcal{A}} \{\underline{L}^a\} \otimes_R N = \varinjlim_{a\in\mathbf{N}\cdot\mathcal{A}} \{\underline{L}^a \otimes_R N\}.$$

In particular we have the following. Let R be an A-algebra, and $A \to B$ a homomorphism of rings. Write $R \otimes_A B = R_B$ and let U_B be the collection on R_B coming from the collection U on R, that is $U_B = \{(s_\alpha \otimes 1, L_\alpha \otimes_A 1)\}_{\alpha\in\mathcal{A}}$. Then we have that

$$R_U \otimes_A B = \varinjlim_{a\in\mathbf{N}\cdot\mathcal{A}} (\underline{L}^a) \otimes_A B = \varinjlim_{a\in\mathbf{N}\cdot\mathcal{A}} (\underline{L}^a \otimes_A B) = (R_B)_{U_B}. \tag{3.5.2.1}$$

3.5.3. Remark. Let N be an R-module. For any element $x \in \underline{L}^a \otimes_R N$ we denote the image of x in the colimit N_U by x/s^a, where s^a is the element defined in (3.4.2). If $y \in \underline{L}^b \otimes_R N$ is another element then $x/s^a = y/s^b$ in N_U if and only if there exists $c \in \mathbf{N} \cdot \mathcal{A}$ such that

$$s^c(s^b x - s^a y) = 0 \quad \text{in } \underline{L}^{a+b+c} \otimes_R N.$$

In particular we have that $s_\alpha \in L_\alpha$ becomes a unit in R_U, namely $s_\alpha/s_\alpha = 1$.

3.5.4. Remark. An invertible R-module $\underline{L}^b$ is faithfully flat, hence a map

$$s^a : N \to \underline{L}^a \otimes_R N \tag{3.5.4.1}$$

is injective or surjective if and only if the R-module map

$$s^a : \underline{L}^b \otimes_R N \to \underline{L}^{a+b} \otimes_R N \tag{3.5.4.2}$$

is injective or surjective, respectively.

3.5.5. Remark. For any subset $J \subseteq \mathcal{A}$ we can consider the colimit

$$R_{U_J} = \varinjlim_{a\in\mathbf{N}\cdot J} \{\underline{L}^a\}.$$

The union of two subsets J_1 and J_2 of $\mathcal{A}$ again is a subset of $\mathcal{A}$, and we have that $\mathcal{A}$ is partially ordered by the union of its subsets. It is clear that R_U is the direct colimit

$$R_U = \varinjlim_{\text{finite } J\subseteq\mathcal{A}} \{R_{U_J}\}. \tag{3.5.5.1}$$

3.6. Proposition. *Let $U = \{(s_\alpha, L_\alpha)\}_{\alpha\in\mathcal{A}}$ be a collection of invertible modules on R. We have the following.*

1. *For any $\alpha \in \mathcal{A}$ we have that $1 \otimes s_\alpha \in R_U \otimes_R L_\alpha$ is nowhere vanishing.*
2. *If $R \to A$ is an R-algebra homomorphism such that $1 \otimes s_\alpha$ in $A \otimes_R L_\alpha$ is nowhere vanishing, for all $\alpha \in \mathcal{A}$, then the homomorphism $R \to A$ factors via the homomorphism $R \to R_U$.*

Proof. To show Assertion (1) we need to show that the map $R_U \to R_U \otimes_R L_\alpha$ determined by sending 1 to $1 \otimes s_\alpha$ is an isomorphism, for all $\alpha \in \mathcal{A}$. We have (3.5.2) that $R_U \otimes_R L_\alpha = R_U$, where we identify $1 \otimes s_\alpha$ with s_α. We have already remarked (3.5.3) that s_α is a unit in R_U for all $\alpha \in \mathcal{A}$, and Assertion (1) follows.

We then show Assertion (2). From the assumption we have that $A \to A \otimes_R L_\alpha$ sending $x \to x \otimes s_\alpha$ is an isomorphism of A-modules for all $\alpha \in \mathcal{A}$. It follows that $A \to A \otimes \underline{L}^a$ is an isomorphism for all $a \in \mathbf{N} \cdot \mathcal{A}$, hence the colimit $R_U \otimes_R A$ is isomorphic to A. We have an R-algebra homomorphism $R_U \to R_U \otimes_R A$ that composed with the inverse of the isomorphism $A \to R_U \otimes_R A$ gives our desired map. □

3.7. Corollary. *Let $\mathcal{S} = \{(s_\alpha, \tilde{L}_\alpha)\}_{\alpha\in\mathcal{A}}$ be a collection of invertible sheaves on a affine scheme $X = \mathrm{Spec}(R)$. Let $L_\alpha = \Gamma(X, \tilde{L}_\alpha)$. Then $i : \mathcal{S}^{-1}X \to X$ is canonically identified with $\mathrm{Spec}(R_U) \to X$.*

Proof. By the proposition we have that $\mathrm{Spec}(R_U) \to \mathrm{Spec}(R)$ satisfies the universal defining properties of $\mathcal{S}^{-1}X \to X$. □

3.7.1. Remark. If the collection $U = (s, L)$ consists of one pair only then we have $\mathrm{Spec}(R_{(s,L)}) = U_s$, where $U_s \subseteq \mathrm{Spec}(R)$ is the locally principal affine open subscheme defined by the non-vanishing of the section $s \in L$.

3.7.2. Remark. If the collection $U = \{(s_i, L_i)\}_{i=1,\dots,r}$ is finite then we can reduce the situation to the single pair (s, L), where

$$s = s_1 \otimes \cdots \otimes s_r \in L_1 \otimes_R \cdots \otimes_R L_r = L.$$

Then we have that $R_U = R_{(s,L)}$. On the level of Spec we have that the finite intersection of locally principal open subschemes $U_{s_i} \subseteq \mathrm{Spec}(R)$ is the locally principal open subscheme $U_s \subseteq \mathrm{Spec}(R)$.

3.8. Proposition. *Let N be an R-module, and U a collection of invertible R-modules. If the map $N \to N_U$ is an isomorphism of R-modules, then the maps $N \to \underline{L}^a \otimes_A N$ are isomorphisms for all $a \in \mathbf{N} \cdot \mathcal{A}$.*

Proof. As N_U is the direct colimit (3.5.2) it is clear that the assumed injectivity of $N \to N_U$ implies that the maps $N \to \underline{L}^a \otimes_R N$ are injective for all $a \in \mathbf{N} \cdot \mathcal{A}$. In particular the maps (3.5.4.2) are injective. We need only to show surjectivity of the maps $N \to \underline{L}^a \otimes_R N$. Let $x \in \underline{L}^a \otimes_R N$. The map $N \to N_U$ to the direct colimit is assumed to be surjective. Hence there exists $y \in N$ having the same image as x in N_U. Thus $y = x/s^a$. By (3.5.3) we have that there exists $c \in \mathbf{N} \cdot \mathcal{A}$ such that

$$s^c(ys^a - x) = 0 \quad \text{in } \underline{L}^{a+c} \otimes_R N.$$

As the maps (3.5.4.2) are injective we have that $ys^a = x$ in $\underline{L}^a \otimes_R N$, hence we have proven the surjectivity of $N \to \underline{L}^a \otimes_R N$. □

3.9. Lemma. *Let $f : M \to N$ be a homomorphism of R-modules. Assume that N is finitely generated and that the induced map $N \to N_U$ is an isomorphism of R-modules. If the R_U-linear map $f_U : M_U \to N_U$ is surjective, then the homomorphism $f : M \to N$ is surjective.*

3.9.1. In particular, if M is an R-module, and M_U is a finitely generated R-module, then the localization map $M \to M_U$ is surjective.

Proof. Let $x_1, \dots, x_r$ generate the R-module N. For each $a \in \mathbf{N} \cdot \mathcal{A}$ we let $f_a : \underline{L}^a \otimes_A M \to \underline{L}^a \otimes_A N$ denote the induced R-linear maps. The map f_U between the direct colimits is assumed to be surjective, hence there exists $a \in \mathbf{N} \cdot \mathcal{A}$ and elements $y_1, \dots, y_r$ in $\underline{L}^a \otimes_R M$ such that $f_a(y_i) \in \underline{L}^a \otimes_R N$ has the same image as x_i in N_U for each $i = 1, \dots, r$. By Proposition (3.8) we have that $N \to \underline{L}^a \otimes_R N$ is surjective, thus the images of $x_1, \dots, x_r$ generate $\underline{L}^a \otimes_R N$. It follows that the R-module homomorphism $f_a : \underline{L}^a \otimes_R M \to \underline{L}^a \otimes_R N$ is surjective. As $\underline{L}^a$ is a faithfully flat R-module we obtain that $M \to N$ is surjective. □

3.10. Lemma. *Let $U = \{(s_\alpha, L_\alpha)\}_{\alpha \in \mathcal{A}}$ be a collection of invertible R-modules. Let $I_U \subseteq R_U$ be an ideal of the generalized fraction ring R_U and let $I = I_U \cap R$ denote its contraction. Then the localization map $R/I \to R_U/I_U$ is an isomorphism if and only if R_U/I_U is finitely generated as an R-module.*

Proof. One direction of the lemma is trivial. In addition the map $R/I \to R_U/I_U$ is always injective (we have taken $I = I_U \cap R$), and we claim that $(R/I)_U = R_U/IR_U$ and R_U/I_U are isomorphic. So if R_U/I_U is finitely generated, then $R/I \to R_U/I_U$ is surjective by (3.9.1), and the lemma follows.

Now by (3.5.5.1) R_U is the direct colimit of generalized fraction rings R_{U_J} where J is a finite subset of $\mathcal{A}$, and by (3.7.2) each map $\operatorname{Spec}(R_{U_J}) \to \operatorname{Spec}(R)$ is an open immersion. So by Lemma (2.5) any ideal in R_{U_J} is the extension of its contraction to R, and then by Lemma (2.6) the ideal $I_U \subset R_U$ is the extension of its contraction to R. This gives $IR_U = I_U$, and the claim follows. □

3.11. Proposition. *Let $X \to S$ be a scheme over a base scheme S, and let $\mathcal{S}$ be a collection of sections of invertible sheaves on X. Let $f : T \to S$ be a morphism of schemes, and let $j : Z \subseteq \mathcal{S}^{-1}X \times_S T$ be a closed subscheme such that the projection map $Z \to T$ is finite. Then Z is a closed subscheme of $X \times_S T$ via the composite map $(i_{\mathcal{S}} \times \mathrm{id}_T) \circ j$.*

Proof. We may assume that $T = \operatorname{Spec}(A)$ is affine since closedness is a local property. We may also, by Lemma (3.3) assume that $T = S$. Finally, it is clear that we may assume that $X = \operatorname{Spec}(R)$ is affine. The Proposition now follows from Lemma (3.10). □

4. - Determinants and Localized Schemes

There exists a notion of noncommutative localization and σ-inverting rings, for any ring R and any set σ of morphisms $s : P \to Q$ of finitely generated projective modules P and Q ([C], [NR]). We will our commutative situation obtain those σ-inverting rings as generalized fraction rings of a collection of determinants and norm bundles.

4.1. Notation. Let $s : E \to L$ be an A-module homomorphism between two locally free A-modules E and L of finite rank n. We take the highest exterior power of the A-module map $s : E \to L$ and obtain en element

$$\det(s) = \wedge s \in \mathrm{Hom}_A(\wedge^n E, \wedge^n L) = N(E, L).$$

The element $\det(s)$ is an element of the invertible A-module $N(E, L)$. We clearly have that $s : E \to L$ is an isomorphism if and only if $\det(s) : A \to N(E, L)$ is nowhere vanishing.

Let $\varphi : A \to B$ be an A-algebra homomorphism, and let $E_B = E \otimes_A B$ and $s_B = s \otimes 1 \in L_B = L \otimes_A B$. Then we have that

$$\det(s_B) = \det(s) \otimes_A 1 = \varphi(\det(s)). \tag{4.1.1}$$

Let now E be an A-algebra such that E is locally free of finite rank as an A-module. Let $U = \{(s_\alpha, L_\alpha)\}_{\alpha \in \mathcal{A}}$ be a collection of elements s_α in invertible E-modules L_α. We denote by

$$N_E(U) = \{(\det(s_\alpha), N(E, L_\alpha))\}_{\alpha \in \mathcal{A}}$$

the corresponding collection on A. If U is a collection of invertible modules on E we refer to $N_E(U)$ as the corresponding collection of norms on A.

4.2. Proposition. *Let E be an A-algebra such that E is locally free of finite rank n as an A-module. Let U be a collection on E and let $N_E(U)$ be the corresponding collection of norms on A. For any homomorphism of rings $\varphi : A \to B$ the following two statements are equivalent.*

(1) *The induced homomorphism $E \otimes_A B \to E_U \otimes_A B$ is an isomorphism*
(2) *The homomorphism $\varphi : A \to B$ factors via $A \to A_{N_E(U)}$.*

In particular we have that $E \otimes_A A_{N_E(U)} \to E_U \otimes_A A_{N_E(U)}$ is an isomorphism.

Proof. By (3.5.2.1) we have $E_U \otimes_A B = (E \otimes_A B)_{U_B}$, where U_B is the collection $\{(s_\alpha \otimes 1, L_\alpha \otimes_A B)\}_{\alpha \in \mathcal{A}}$. The Assertion (1) then reads by Propositon (3.6) that the sections $s_\alpha \otimes_A 1$ are nowhere vanishing, for all $\alpha \in \mathcal{A}$. Hence their determinants $\det(s_\alpha) \in N(E, L_\alpha)$ are nowhere vanishing. It then follows by the universal property of the generalized fraction rings, Proposition (3.6), that the homomorphism $f : A \to B$ factors via $A \to A_{N_E(U)}$. We have proven that Assertion (1) implies Assertion (2). Assume now that Assertion (2) holds. By Proposition (3.6) we have that the sections $1 \otimes \det(s_\alpha) \in A_{N(U)} \otimes_A N(E, L_\alpha)$ are nowhere vanishing for all $\alpha \in \mathcal{A}$. Then we have that $f(1 \otimes \det(s_\alpha)) \in B$ are invertible, for all $\alpha \in \mathcal{A}$. It follows from (4.1.1) that the sections $s_\alpha \otimes 1$ in $L_\alpha \otimes_A B$ are nowhere vanishing, for all $\alpha \in \mathcal{A}$. Consequently $E_B = E \otimes_A B$ is isomorphic to the direct colimit $(E_B)_{U_B}$, which by (3.5.2.1) equals $E_U \otimes_A B$. □

4.3. Definition. A flat and finite morphism of schemes $q : Z \to H$ is *of relative rank n*, if the quasi-coherent $\mathcal{O}_H$-module $q_*\mathcal{O}_Z$ is locally free of finite rank n.

4.4. Determinant sections. Let $s : \mathcal{O}_Z \to L$ be a section of an invertible sheaf L on Z. Let $q : Z \to H$ be a morphism of schemes that is flat, finite and of relative rank n. We then have that q_*L is a quasi-coherent $\mathcal{O}_H$-module, locally free of rank n. The highest exterior power of the $\mathcal{O}_H$-module homomorphism $q_*(s) : q_*\mathcal{O}_Z \to q_*L$ gives a global section $\det(s)$ of the invertible $\mathcal{O}_H$-module

$$\mathcal{N}_Z(L) = \mathcal{H}om_{\mathcal{O}_H-\mathrm{mod}}(\wedge^n q_*\mathcal{O}_Z, \wedge^n q_*L).$$

Let $\mathcal{S} = \{(s_\alpha, L_\alpha)\}_{\alpha \in \mathcal{A}}$ be a collection of sections of invertible sheaves on a scheme Z, and let $q : Z \to H$ be a morphism of schemes flat, finite and of relative rank n. We call $\mathcal{N}_Z(\mathcal{S})$ the corresponding collection of norms on H where

$$\mathcal{N}_Z(\mathcal{S}) = \{(\det(s_\alpha), \mathcal{N}_Z(L_\alpha))\}_{\alpha \in \mathcal{A}}.$$

4.5. Proposition. *Let $q : Z \to H$ be a morphism of schemes, flat, finite and of relative rank n. Let $\mathcal{S}$ be a collection of sections of invertible sheaves on Z, and let $\mathcal{N}_Z(\mathcal{S})$ be the corresponding collection of norms on H. A morphism of schemes $f : T \to H$ factors via $\mathcal{N}_Z(\mathcal{S})^{-1}H \to H$ if and only if the induced morphism of schemes $T \times_H \mathcal{S}^{-1}Z \to T \times_H Z$ is an isomorphism. In particular we have that $\mathcal{S}^{-1}Z \times_H \mathcal{N}_Z(\mathcal{S})^{-1}H \to Z \times_H \mathcal{N}_Z(\mathcal{S})^{-1}H$ is an isomorphism.*

Proof. This is a global version of Proposition (4.2). □

5. - An application to Hilbert schemes of points

We will in this last section apply results from the previous two sections about the generalized fraction rings to show the existence of Hilbert scheme of points on localized schemes $\mathcal{S}^{-1}X$, with X quasi-projective. We will use the fact that the Hilbert scheme of quasi-projective schemes X exists.

5.1. Set up. We fix a morphism of schemes $X \to S$, where we refer to S as the base scheme. Let H be an S-scheme, and let $Z \subseteq X \times_S H$ be closed subscheme such that the projection $q : Z \to H$ is flat, finite and of relative rank n. Let $p : Z \to X$ denote the other projection.

If $\mathcal{S}$ is a collection of sections and invertible sheaves on X we get by the construction (4.4) a collection $\mathcal{N} = \mathcal{N}_Z(p^*\mathcal{S})$ on H. We thus have the following diagram

$$\begin{array}{ccccc} & & (p^*\mathcal{S})^{-1}Z & \longrightarrow & \mathcal{S}^{-1}X \\ & & {\scriptstyle i_{p^*\mathcal{S}}}\downarrow & & {\scriptstyle i_{\mathcal{S}}}\downarrow \\ Z_{\mathcal{N}} & \longrightarrow & Z & \xrightarrow{p} & X \\ \downarrow & & {\scriptstyle q}\downarrow & & \\ \mathcal{N}^{-1}H & \xrightarrow{i_{\mathcal{N}}} & H & & \end{array} \tag{5.1.1}$$

where the upper right square in (5.1.1) is a fiber product by Lemma (3.3), and where the scheme $Z_{\mathcal{N}}$ is defined as the fiber product of the diagram to the down left.

5.2. Lemma. *The scheme $Z_{\mathcal{N}}$ in the diagram (5.1.1) is a closed subscheme of $\mathcal{S}^{-1}X \times_S \mathcal{N}^{-1}H$.*

Proof. We have that Z is closed in $X \times_S H$. It follows that $\mathcal{S}^{-1}X \times_X Z$ is closed in $\mathcal{S}^{-1}X \times_S H$. We have that $\mathcal{S}^{-1}X \times_X Z = (p^*\mathcal{S})^{-1}Z$, and thence that

$$(p^*\mathcal{S})^{-1}Z \times_H \mathcal{N}^{-1}H \tag{5.2.1}$$

is a closed subscheme of $\mathcal{S}^{-1}X \times_S \mathcal{N}^{-1}H$. By Proposition (4.5) we have that (5.2.1) is canonically isomorphic to $Z_{\mathcal{N}}$ since the morphism $Z_{\mathcal{N}} \to H$ factors via $\mathcal{N}^{-1}H$. We then have that $Z_{\mathcal{N}}$ is a closed subscheme of $\mathcal{S}^{-1}X \times_S \mathcal{N}^{-1}H$ as claimed. □

5.3. Lemma. *Let $f : T \to H$ be a morphism of schemes, and let $Z_T = Z \times_H T$, where Z is a closed subscheme of $X \times_S H$. If $Z_T \subseteq \mathcal{S}^{-1}X \times_S T$ then we have that the natural morphism $(p^*\mathcal{S})^{-1}Z \times_H T \to Z_T$ is an isomorphism.*

Proof. If Z_T is a subscheme of $\mathcal{S}^{-1}X \times_S T$ then we have that the projection morphism $Z_T \to X$ factors via the morphism $\mathcal{S}^{-1}X \to X$. Hence $Z_T \to Z$ factors via the fiber product $(p^*\mathcal{S})^{-1}Z$ and we obtain our isomorphism. □

5.4. Definition. The Hilbert functor $\mathcal{H}^n_X$ of n-points on X is defined ([G] p. 274) as the contravariant functor from the category of schemes over S to sets, sending a S-scheme T to the set

$$\mathcal{H}^n_X(T) = \{\text{closed subschemes } Z \subseteq X \times_Z T \text{ such that the projection map } q : Z \to T \text{ is flat, finite and of relative rank } n.\}$$

5.5. Theorem. *Let $X \to S$ be a fixed scheme, and assume that the Hilbert functor of n-points on X is represented by a scheme H^n_X with universal family $Z \to H^n_X$. Let $p : Z \to X$ denote the projection to X. For any collection $\mathcal{S}$ of sections of invertible sheaves on X, we let $\mathcal{N} = \mathcal{N}_Z(p^*(\mathcal{S}))$ be the corresponding collection of norms on H^n_X. We have that the scheme $\mathcal{N}^{-1}H^n_X$ is the Hilbert scheme of n-points on $\mathcal{S}^{-1}X$. The universal family $Z_{\mathcal{N}} \to \mathcal{N}^{-1}H^n_X$ is the pull-back of the family $Z \to H^n_X$ along the localization map $\mathcal{N}^{-1}H^n_X \to H^n_X$.*

Proof. By Lemma (5.2) we have that $Z_{\mathcal{N}}$ is an $\mathcal{N}^{-1}\mathrm{H}^n_X$-valued point of the Hilbert functor of n-points on $\mathcal{S}^{-1}X$. We then have a morphism of functors from the point functor of $\mathcal{N}^{-1}\mathrm{H}^n_X$ to the Hilbert functor $\mathcal{H}^n_{\mathcal{S}^{-1}X}$ of n-points on $\mathcal{S}^{-1}X$. A morphism we claim is an isomorphism.

Let $T \to S$ be a morphism of schemes, and let W be a T-valued point of $\mathcal{H}^n_{\mathcal{S}^{-1}X}$. It follows by Proposition (3.11) that W is a T-valued point of $\mathcal{H}^n_X$. Hence there exists a morphism $f : T \to \mathrm{H}^n_X$ such that the pull-back of the universal family $Z \to \mathrm{H}^n_X$ along f is the scheme W. We will show that f factors via $\mathcal{N}^{-1}\mathrm{H}^n_X$.

As W is a T-valued point of $\mathcal{H}^n_{\mathcal{S}^{-1}X}$ it is in particular a closed subscheme of $\mathcal{S}^{-1}X \times_S T$. Hence by Lemma (5.3) we have that $W = (p^*\mathcal{S})^{-1}Z \times_{\mathrm{H}^n_X} T$. That is the natural map

$$(p^*\mathcal{S})^{-1}Z \times_{\mathrm{H}^n_X} T \to Z \times_{\mathrm{H}^n_X} T \tag{5.5.1}$$

is an isomorphism. By Proposition (4.5) the isomorphism (5.5.1) is equivalent with $f : T \to \mathrm{H}^n_X$ factoring via $\mathcal{N}^{-1}\mathrm{H}^n_X \to \mathrm{H}^n_X$. We thus obtain a morphism of functors from $\mathcal{H}^n_{\mathcal{S}^{-1}X}$ to the point functor of $\mathcal{N}^{-1}\mathrm{H}^n_X$, a morphism that clearly is an inverse to the morphism of functors obtained by the $\mathcal{N}^{-1}\mathrm{H}^n_X$-valued point $Z_{\mathcal{N}}$. □

5.6. Corollary. *Let $U_\alpha \subseteq X$ be the open subscheme defined by the non-vanishing of the section $s_\alpha : \mathcal{O}_X \to L_\alpha$, for each $\alpha \in \mathcal{A}$. Then the Hilbert scheme of n-points on $\bigcap_{\alpha\in\mathcal{A}} U_\alpha$ is the corresponding intersection $\bigcap_{\alpha\in\mathcal{A}} \mathrm{Hilb}^n_{U_\alpha}$, where $Hilb^n_{U_\alpha} \subseteq Hilb^n_X$ is the open subscheme parameterizing n-points on $U_\alpha \subseteq X$.*

Proof. We have that $\mathcal{S}^{-1}X = \bigcap_{\alpha\in\mathcal{A}} U_{s_\alpha}$. It then follows from the theorem that the Hilbert scheme of points on $\mathcal{S}^{-1}X$ is the infinite intersection $\bigcap_{\alpha\in\mathcal{A}} H_{\det(s_\alpha)}$, where $H_{\det(s_\alpha)}$ is the open subscheme of Hilb^n_X defined by the non-vanishing of the section $\det(s_\alpha) : \mathcal{O}_{\mathrm{Hilb}^n_X} \to \mathcal{N}_Z(p^*\mathcal{L}_\alpha)$. Applying the theorem to the single pair (s_α, L_α) we get that $H_{\det(s_\alpha)}$ is the Hilbert scheme $\mathrm{Hilb}^n_{U_{s_\alpha}}$ of n-points on U_{s_α}. □

5.6.1. Noetherian schemes. The Hilbert functor defined in (5.4) restricts to a functor of noetherian schemes over a noetherian base scheme S.

5.7. Corollary. *Let $X \to S$ be a projective morphism of noetherian schemes. Let $P \in X$ be a point, and let $\mathcal{O}_{X,P}$ denote the stalk of the point. Then the Hilbert functor of n-points on* $\mathrm{Spec}(\mathcal{O}_{X,P})$ *is represented by a noetherian scheme. Furthermore, if we let $\{U_\alpha\}$ denote the set of open subschemes U_α in X containing the point P, then we have that* $\mathrm{Hilb}^n_{\mathcal{O}_{X,P}} = \bigcap_\alpha \mathrm{Hilb}^n_{U_\alpha}$.

Proof. We have [G] that the Hilbert functor of n-points on X is represented by a projective and in particular noetherian, scheme $\mathrm{Hilb}^n_{X/S}$. As X is projective we can always find a locally principal open affine subscheme $U \subseteq X$ containing the point P. Hence we have that $\mathrm{Hilb}^n_{U/S}$, the Hilbert scheme of n-points on U, is an open subscheme of $\mathrm{Hilb}^n_{X/S}$. We have that the basic open affines $D(f)$ form a basis for the topology on $U = \mathrm{Spec}(A)$, hence we can replace

$$U \cap \bigcap_{\alpha \in \mathcal{A}} U_\alpha = \bigcap_{f \in A, P \in D(f)} D(f).$$

Thus $\mathrm{Spec}(\mathcal{O}_{X,P})$ is the localized scheme $\mathcal{S}^{-1}U \subseteq U$, where $\mathcal{S}$ the collection $\{f, \mathcal{O}_U\}$, with $P \in D(f)$. We then have by the theorem that the Hilbert scheme of n-points on $\mathcal{S}^{-1}U$ is the infinite intersection of locally principal subschemes $\bigcap \mathrm{Hilb}^n_{D(f)}$. The only thing we need to verify is that the scheme $\bigcap \mathrm{Hilb}^n_{D(f)}$ is noetherian. This follows from Proposition (2.7). □

5.7.1. Remark. The Hilbert schemes of points on localized schemes are not generally varieties, even if the base scheme $S = \mathrm{Spec}(k)$ is the spectrum of a field. The Hilbert schemes are not always of finite type over the base, and consequently the underlying geometry is complicated if not bizarre (see [LS]).

5.7.2. Remark. Note that the point $P \in X$ in the Corollary, is not assumed to be a closed point. Thus for an integral scheme X the result also describes the Hilbert scheme of points on $\mathrm{Spec}(K_X)$, where K_X is the function field of X.

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