AN ELEMENTARY, EXPLICIT, PROOF OF THE EXISTENCE OF QUOT SCHEMES OF POINTS

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Abstract. We give an easy and elementary construction of quotient schemes of modules of relatively finite rank under very general conditions. The construction provides a natural, explicit description of an affine convering of the quotient schemes, that is useful in many situation.

Introduction

Quotient schemes were introduced by A. Grothendieck [G], and belong to the fundamental tools in algebraic geometry. They generalize simultaneously the Hilbert schemes and the Grassmann schemes. In most cases where these schemes appear it suffices to know that they exist. However, there are cases when it is crucial to have an explicit description of the schemes. We give here, for any morphism of schemes $\mathrm{Proj}(\mathcal{R}) \to S$, where S is arbitrary and $\mathcal{R}$ is a quasi-coherent graded $\mathcal{O}_S$-algebra, an elementary construction of quotient schemes parametrizing equivalence classes of surjections from a quasi-coherent $\mathcal{O}_{\mathrm{Proj}(\mathcal{R})}$-module to coherent modules that are of relatively finite rank over S. The construction provides a natural and explicit description of an affine covering of the quotient schemes. In a previous article [GLS] we indicated the usefulness of such a description in the case of Hilbert schemes of points, and further evidence of this is given by M. Huibregtse ([H1], [H2]). Our proof of the existence of the quotient schemes is a simplification and clarification of the constructions of these works.

The main new idea is the description of a local version of the quot functor. More precisely, let A be a ring, B an A-algebra and E and F modules over A with E free of finite rank, and fix an A-module homomorphism $s : E \to B \otimes_A F$. We parametrize B-module structures on E, together with B-module homomorphisms $u : B \otimes_A F \to E$ such that $us = \mathrm{id}_E$.

When S is locally noetherian and $\mathcal{R}$ is locally finitely generated by elements of degree one, our existence result for the quotient schemes of modules of relatively finite rank follows from the more general results of Grothendieck [G]. Apparently the first detailed existence proof of Grothendieck's result was provided by A. Altman

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and S. Kleiman [AK], valid under quite general conditions, where S is not assumed to be locally noetherian. A much used method for obtaining these results was given by D. Mumford [M] (see also E. Sernesi [S]). In contrast to these approaches our method rely on simple algebraic constructions and avoids embeddings into high dimensional grassmannians via Castelnuovo-Mumford regularity. As a consequence our local description of the quotient schemes is, in most cases appearing in applications, in terms of explicit natural equations in the affine space of commuting matrices of smallest possible size, that is, the size is equal to the finite rank of the modules.

1. The local quot functor

1.1 The local quot functor. Let A be a commutative ring with unit, and let $A \to B$ be an A-algebra. Moreover, let E and F be A modules where E is free of finite rank, and let

$$s : E \to B \otimes_A F$$

be an A-module homomorphism. We want to describe all B-module structures on E, together with B-module homomorphisms $u : B \otimes_A F \to E$ such that $us = \mathrm{id}_E$. Recall that two surjections are considered *equivalent* if their kernels coincide, and that a B-module structure on an A-module E corresponds to an A-algebra homomorphism $B \to \mathrm{End}_A(E)$.

More precicely, we want to describe, for every A-algebra $A \to A'$, the set consisting of an $A' \otimes_A B$-module structure on $A' \otimes_A E$ together with an $A' \otimes_A B$-module homomorphism $A' \otimes_A B \otimes_A F \xrightarrow{u} A' \otimes_A E$ such that the composite A'-module homomorphism

$$A' \otimes_A E \xrightarrow{\mathrm{id}_{A'} \otimes_A s} A' \otimes_A B \otimes_A F \xrightarrow{u} A' \otimes_A E$$

is the identity. This clearly defines a functor from A-algebras to sets.

The main objective of the three first sections of this article is to show that this functor is representable by an A-algebra Q^s, and to give a simple explicit description of this algebra:

We first find a slightly different description of this functor.

1.2 Notation. In the following we shall need *evaluation* and *trace* maps. We recall that when G and K are A-modules with K free of finite rank, we obtain for every submodule D of $\mathrm{Hom}_A(G, K)$ the *evaluation* homomorphisms

$$\mathrm{ev} : D \otimes_A G \to K \quad \text{or} \quad \mathrm{ev} : G \otimes_A D \to K$$

that maps $x \otimes_A \varphi$, or $\varphi \otimes_A x$, to $\varphi(x)$. Moreover, we have the *trace* homomorphism

$$\mathrm{tr} : A \to K \otimes_A K\check{}$$

obtained from the dual homomorphism of $\mathrm{ev} : K\check{} \otimes_A K \to A$.

1.3 Proposition. *There is a natural bijection between the following two sets*

(1) *The set of B-module structures on E together with a surjective map $u : B \otimes_A F \to E$ of B-modules.*
(2) *The set of A-algebra homomorphisms $\varphi : B \to \mathrm{End}_A(E)$ together with A-module homomorphisms $v : F \to E$ such that the composite map*

$$B \otimes_A F \xrightarrow{\varphi \otimes_A v} \mathrm{End}_A(E) \otimes_A E \xrightarrow{\mathrm{ev}} E$$

is surjective.

More precisely, we obtain a bijection that maps a pair (v, φ) in (2) to $\mathrm{ev}(\varphi \otimes_A v)$ in (1), where E has the B-module structure given by φ.

Proof. We first note that B-module structures on E correspond to A-algebra homomorphisms $\varphi : B \to \mathrm{End}(E)$.

Given a B-module structure on E, a surjection $u : B \otimes_A F \to E$ of B-modules determines this structure uniquely. Moreover an A-module homomorphism $u : B \otimes_A F \to E$ determines an A-module homomorphism $v : F \to E$ by *restriction of scalars*, and conversely, u is determined by v by *extension* of scalars.

Finally, an A-module homomorphism $v : F \to E$ and a B-module structure $\varphi : B \to \mathrm{End}_A(E)$ on E makes the composite map $B \otimes_A F \xrightarrow{\varphi \otimes_A v} \mathrm{End}_A(E) \otimes_A E \xrightarrow{\mathrm{ev}} E$ into a B-module homomorphism since ev is an $\mathrm{End}_A(E)$-module homomorphism and $\varphi : B \to \mathrm{End}_A(E)$ is a ring homomorphism.

1.4 Corollary. *The bijection of the proposition induces a bijection between the following two sets*

(1) *B-module structures on E together with a homomorphism of B-modules $u : B \otimes_A F \to E$ such that*

$$E \xrightarrow{s} B \otimes_A F \xrightarrow{u} E$$

is the identity.
(2) *A-algebra homomorphisms $\varphi : B \to \mathrm{End}_A(E)$ together with A-module homomorphisms $v : F \to E$ such that*

$$E \xrightarrow{s} B \otimes_A F \xrightarrow{\varphi \otimes_A v} \mathrm{End}_A(E) \otimes_A E \xrightarrow{\mathrm{ev}} E$$

is the identity.

Proof. The corollary immediately follows from the proposition since surjectivity of u and $\mathrm{ev}(\varphi \otimes_A v)$ is automatic.

2. The local Hilbert scheme

The material of this section will basically give a construction of Hilbert schemes of points (see [GLS]), however the presentation here is different from that of [GLS]. We use that the A-algebra $\mathrm{Sym}_A(G \otimes_A \mathrm{End}_A(E)\check{})$ parametrizes module homomorphisms $u : G \to \mathrm{End}_A(E)$. Then we explicitely construct a residue algebra H of $\mathrm{Sym}_A(G \otimes_A \mathrm{End}_A(E)\check{})$ that parametrizes those u such that the elements of the image commute. Then H also parametrizes A-algebra homomorphisms $\mathrm{Sym}_A(G) \to \mathrm{End}_A(E)$.

2.1 Notation. Let G be an A-module and K a free A-module of finite rank. We denote by

$$u_K : \mathrm{Sym}_A(G \otimes_A K\check{}) \otimes_A G \to \mathrm{Sym}_A(G \otimes_A K\check{}) \otimes_A K$$

the $\mathrm{Sym}_A(G \otimes_A K\check{})$-module homomorphism defined by $u_K(1 \otimes_A x) = x \otimes_A \mathrm{tr}(1_A)$ for all $x \in G$, where $G \otimes_A K\check{}$ is considered as a submodule of $\mathrm{Sym}_A(G \otimes_A K\check{})$.

2.2. Lemma. *For every A-algebra homomorphism $A \to A'$ there is a natural bijection between the following two sets*

(1) *A'-module homomorphisms $A' \otimes_A G \to A' \otimes_A K$.*
(2) *A-algebra homomorphisms $\mathrm{Sym}_A(G \otimes_A K\check{}) \to A'$.*

The bijection maps $\varphi : \mathrm{Sym}_A(G \otimes_A K\check{}) \to A'$ to the homomorphism u defined by $u(1_{A'} \otimes_A x) = (\varphi \otimes_A \mathrm{id}_K)u_K(1 \otimes_A x)$ for all $x \in G$.

Proof. The set (2) is mapped to the set (1) via the following three isomorphisms:

(1) $\mathrm{Hom}_{A\text{-alg}}(\mathrm{Sym}_A(G \otimes_A K\check{}), A') \to \mathrm{Hom}_A(G \otimes_A K\check{}, A')$ that follows from the definition of the symmetric algebra.
(2) $\mathrm{Hom}_A(G \otimes_A K\check{}, A') \to \mathrm{Hom}_A(G, A' \otimes_A K)$, that is a canonical standard isomorphism, when K is free, that maps $u : G \otimes_A K\check{} \to A'$ to the compsoite homomorphism $G \xrightarrow{\mathrm{id}_G \otimes_A \mathrm{tr}} G \otimes_A K\check{} \otimes_A K \xrightarrow{u \otimes_A \mathrm{id}_K} A' \otimes_A K$.
(3) $\mathrm{Hom}_A(G, A' \otimes_A K) \to \mathrm{Hom}_{A'}(A' \otimes_A G, A' \otimes_A K)$, that is the standard isomorphism obtained by *extension of scalars.*

2.3 Notation. Let $u : G \to K$ be a homomorphism of A-modules with K free of finite rank. We denote by $\mathfrak{I}_Z(u)$ the image of the composite homomorphism

$$G \otimes_A K\check{} \xrightarrow{u \otimes_A \mathrm{id}\, K\check{}} K \otimes_A K\check{} \xrightarrow{\mathrm{ev}} A.$$

2.4 Lemma. *For every A-algebra $A \to A'$ the A'-module homomorphism*

$$A' \otimes_A G \xrightarrow{\mathrm{id}_{A'} \otimes_A u} A' \otimes_A K$$

is zero if and only if the homomorphism $A \to A'$ factors via the residue homomorphism $A \to A/\mathfrak{I}_Z(u)$.

Proof. The lemma is an immediate consequence of the definition of $\mathfrak{I}_Z(u)$.

2.5 Notation. Let

$$v : \mathrm{Sym}_A(G \otimes_A \mathrm{End}_A(E)\check{}) \otimes_A G \otimes_A G \to \mathrm{Sym}_A(G \otimes_A \mathrm{End}_A(E)\check{}) \otimes_A \mathrm{End}_A(E)$$

be the $\mathrm{Sym}_A(G \otimes_A \mathrm{End}_A(E)\check{})$-module homomorphism defined by $v(1 \otimes_A x \otimes_A y) = u_{\mathrm{End}_A(E)}(1 \otimes_A x)u_{\mathrm{End}_A(E)}(1 \otimes_A y) - u_{\mathrm{End}_A(E)}(1 \otimes_A y)u_{\mathrm{End}_A(E)}(1 \otimes_A x)$, where $u_{\mathrm{End}_A(E)}$ is defined in 2.1. We denote by H the residue algebra of $\mathrm{Sym}_A(G \otimes_A \mathrm{End}_A(E)\check{})$ modulo the ideal $\mathfrak{I}_Z(v)$ and let

$$\rho_H : \mathrm{Sym}_A(G \otimes_A \mathrm{End}_A(E)\check{}) \to H$$

be the residue homomorphism. From the $\mathrm{Sym}_A(G \otimes_A \mathrm{End}_A(E)\check{})$-module homomorphism $u_{\mathrm{End}_A(E)}$ and the A-algebra homomorphism ρ_H we obtain an H-module homomorphism $w : H \otimes_A G \to H \otimes_A \mathrm{End}_A(E)$ defined by $w(1 \otimes_A x) = (\rho_H \otimes_A 1_{\mathrm{End}_A(E)})u_{\mathrm{End}_A(E)}(1 \otimes_A x)$ for all $x \in G$. It follows from the definition of $\mathfrak{I}_Z(w)$ and H that the elements of the image of w commute. Consequently w gives a unique H-algebra homomorphism

$$\mu_H : H \otimes_A \mathrm{Sym}_A(G) \to H \otimes_A \mathrm{End}_A(E)$$

such that $\mu_H(1_H \otimes_A x) = w(1_H \otimes_A x)$ for all $x \in G$.

2.6 Lemma. *Let $A \to A'$ be an A-algebra. We have a bijection between the following two sets*

(1) *A'-algebra homomorphisms $A' \otimes_A \mathrm{Sym}_A(G) \to A' \otimes_A \mathrm{End}_A(E)$.*
(2) *A-algebra homomorphisms $H \to A'$.*

The bijection maps $\varphi : H \to A'$ to the homomorphism u defined by $u(1_{A'} \otimes_A x) = (\varphi \otimes_A \mathrm{id}_{\mathrm{End}_A(E)})\mu_H(1_H \otimes_A x)$ for all $x \in G$.

Proof. It follows from Lemma 2.2 that there is a bijection between A'-module homomorphisms $u : A' \otimes_A G \to A' \otimes_A \mathrm{End}_A(E)$ and A-algebra homomorphisms $\varphi : \mathrm{Sym}_A(G \otimes \mathrm{End}_A(E)\check{}) \to A'$. By the definition of $\mathfrak{I}_Z(w)$ and Lemma 2.4 the homomorphism φ factors via a homomorphism $\chi : H \to A'$ if and only if the elements $u(1_{A'} \otimes_A x)$ commute for all $x \in X$. However, the maps $u : A' \otimes_A G \to A' \otimes_A \mathrm{End}_A(E)$ such that the elements of its image commute correspond to A'-algebra homomorphisms $\psi : A' \otimes_A \mathrm{Sym}_A(G) \to A' \otimes_A \mathrm{End}_A(E)$ where $\psi(1_{A'} \otimes_A x) = u(1_{A'} \otimes_A x)$ for all $x \in G$.

2.7 Notation. Let G be an A-module and $\mathfrak{I}$ an ideal in $\mathrm{Sym}_A(G)$. Denote by $\iota : \mathfrak{I} \to \mathrm{Sym}_A(G)$ the inclusion map, let $B = \mathrm{Sym}_A(G)/\mathfrak{I}$, and let $\rho_B : \mathrm{Sym}_A(G) \to B$ be the residue homomorphism. We denote by v the composite H-module homomorphism

$$H \otimes_A \mathfrak{I} \xrightarrow{\mathrm{id}_H \otimes_A \iota} H \otimes_A \mathrm{Sym}_A(G) \xrightarrow{\mu_H} H \otimes_A \mathrm{End}(E). \tag{2.7.1}$$

Moreover we let H_B be the residue algebra of H modulo the ideal $\mathfrak{I}_Z(v)$ and we denote the residue homomorphism by

$$\rho_{H_B} : H \to H_B.$$

We tensorize the modules of (2.7.1) by H_B over H and obtain a homomorphism of H_B-modules

$$H_B \otimes_A \mathfrak{I} \xrightarrow{\mathrm{id}_{H_B} \otimes_A \iota} H_B \otimes_A \mathrm{Sym}_A(G) \xrightarrow{\mathrm{id}_{H_B} \otimes_H \mu_H} H_B \otimes_A \mathrm{End}_A(E)$$

such that the composite homomorphism is zero by the definition of $\mathfrak{I}_Z(v)$. Consequently we obtain a homomorphism of H_B-algebras

$$\mu_{H_B} : H_B \otimes_A B \to H_B \otimes_A \mathrm{End}_A(E)$$

such that $\mu_{H_B}(1 \otimes_A \rho_B(f)) = (\rho_{H_B} \otimes_A \mathrm{id}_{\mathrm{End}_A(E)})\mu_H(1 \otimes_A f)$ for all $f \in \mathrm{Sym}_A(G)$.

The algebra H_B parametrizes all B-module structures on E, and μ_{H_B} is the *universal homomorphism*.

2.8 Proposition. *Let $A \to A'$ be an A-algebra. We have a bijection between the following two sets sets*

(1) *A'-algebra homomorphisms $A' \otimes_A B \to A' \otimes_A \mathrm{End}_A(E)$.*
(2) *A-algebra homomorphisms $H_B \to A'$.*

The bijection maps $\varphi : H_B \to A'$ to the homomorphisms u defined by $u(1_{A'} \otimes_A f) = (\varphi \otimes_A \mathrm{id}_{\mathrm{End}_A(E)})\mu_{H_B}(1_{H_B} \otimes_A f)$ for all $f \in B$.

Proof. It follows from Lemma 2.4 that an A-algebra homomorphism $H \to A'$ factors via $\rho_{H_B} : H \to H_B$ if and only if the composite homomorphism $A' \otimes_A \mathfrak{I} \xrightarrow{\mathrm{id}_{A'} \otimes_A \iota} A' \otimes_A \mathrm{Sym}_A(G) \xrightarrow{\mathrm{id}_{A'} \otimes_H \mu_H} A' \otimes_A \mathrm{End}_A(E)$ is zero. However the latter composite homomorphism is zero if and only if $\mathrm{id}_{A'} \otimes_H \mu_H$ factors via an A'-algebra homomorphism $A' \otimes_A B \to A' \otimes_A \mathrm{End}_A(E)$.

3. The local quot scheme

It follows from Lemma 2.2 that the A-algebra $\mathrm{Sym}_A(F \otimes_A E\check{})$ parametrizes homomorphisms $F \to E$. We use this, together with the properties of the A-algebra H_B, to construct a residue algebra Q^s of $\mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A H_B$ that parametrizes the *local quot functor* of Section 1.1.

3.1 Notation. We defined in 2.1 an A-module homomorphism

$$u_E : \mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A F \to \mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A E$$

such that $u_E(1 \otimes_A y) = y \otimes_A \mathrm{tr}(1)$ for all $y \in F$.

Let $A \to B$ be an A-algebra and fix a presentation $0 \to \mathfrak{I} \to \mathrm{Sym}_A(G) \to B \to 0$ of B as in section 2.7. Moreover, fix an A-module homomorphism

$$s : E \to B \otimes_A F$$

and let

$$v : \mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A H_B \otimes_A E \to \mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A H_B \otimes_A E$$

be the composite homomorphsim of the $\mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A H_B$-module homomorphisms

$$\begin{aligned}
\mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A H_B \otimes_A E &\xrightarrow{1 \otimes_A 1 \otimes_A s} \mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A H_B \otimes_A B \otimes_A F \\
&\xrightarrow{\sim} \mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A F \otimes_A H_B \otimes_A B \\
&\xrightarrow{u_E \otimes_A \mu_{H_B}} \mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A E \otimes_A H_B \otimes_A \mathrm{End}_A(E) \\
&\xrightarrow{\sim} \mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A H_B \otimes_A \mathrm{End}_A(E) \otimes_A E \\
&\xrightarrow{\mathrm{id} \otimes_A \mathrm{id} \otimes_A \mathrm{ev}} \mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A H_B \otimes_A E
\end{aligned}$$

where the isomorphisms without names are the appropriate permutations of the factors in the tensor products. Let Q^s be the residue algebra of $\mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A H_B$ modulo the ideal $\mathfrak{I}_Z(v - \mathrm{id})$ and let

$$\rho_{Q^s} : \mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A H_B \to Q^s$$

be the residue homomorphism.

Denote by $\rho_1 : \mathrm{Sym}_A(F \otimes_A E\check{}) \to Q^s$ and $\rho_2 : H_B \to Q^s$ the A-algebra homomorphisms that determine ρ_{Q^s}, that is, $\rho_1(f) = \rho_{Q^s}(f \otimes_A 1)$ and $\rho_2(g) = \rho_{Q^s}(1 \otimes g)$ for all $f \in \mathrm{Sym}_A(F \otimes_A E\check{})$ and $g \in H_B$. We obtain a universal Q^s-algebra homomorphism

$$\mu_{Q^s} : Q^s \otimes_A B \to Q^s \otimes_A \mathrm{End}_A(E)$$

defined by $\mu_{Q^s}(1_{Q^s} \otimes_A f) = (\rho_2 \otimes_A \mathrm{id}_{\mathrm{End}_A(E)})\mu_{H_B}(1 \otimes_A f)$ for all $f \in B$ and a Q^s-module homomorphism $u'_{Q^s} : Q^s \otimes_A F \to Q^s \otimes_A E$ defined by $u'_{Q^s}(1_{Q^s} \otimes_A y) = (\rho_1 \otimes_A \mathrm{id}_E)u_E(1 \otimes_A y)$ for all $y \in F$. When we give $Q^s \otimes_A E$ the $Q^s \otimes_A B$-module structure given by μ_{Q^s} we denote by

$$u_{Q^s} : Q^s \otimes_A B \otimes_A F \to Q^s \otimes_A E$$

the $Q^s \otimes_A B$-module homomorphism obtained from u'_{Q^s} by extension of scalars. When we identify $Q^s \otimes_A B \otimes_A F$ with $(Q^s \otimes_A B) \otimes_{Q^s} (Q^s \otimes_A F)$ and $Q^s \otimes_A \mathrm{End}_A(E) \otimes_A E$ with $(Q^s \otimes_A \mathrm{End}_A(E)) \otimes_{Q^s} (Q^s \otimes_A E)$ we obtain a composite homomorphism

$$Q^s \otimes_A E \xrightarrow{\mathrm{id}_{Q^s} \otimes_A s} Q^s \otimes_A B \otimes_A F \xrightarrow{\mu_{Q^s} \otimes_A u'_{Q^s}}$$
$$Q^s \otimes_A \mathrm{End}_A(E) \otimes_A E \xrightarrow{\mathrm{id}_{Q^s} \otimes_A \mathrm{ev}} Q^s \otimes_A E$$

that is the identity by the definition of $\mathfrak{I}_Z(v - \mathrm{id})$ and Q^s.

3.2 Theorem. *The A-algebra Q^s represents the local quot functor defined in section 1.1. The universal homomorphisms are $\mu_{Q^s} : Q^s \otimes_A B \to Q^s \otimes_A \mathrm{End}_A(E)$ and $u_{Q^s} : Q^s \otimes_A B \otimes_A F \to Q^s \otimes_A E$.*

More precisely, let $A \to A'$ be an A-algebra. We have a bijection between the following three sets

(1) *$A' \otimes_A B$-module structures on $A' \otimes_A E$ and $A' \otimes_A B$-linear homomorphisms $u : A' \otimes_A B \otimes_A F \to A' \otimes_A E$ for the $A' \otimes_A B$-module structure such that*

$$A' \otimes_A E \xrightarrow{1 \otimes_A s} A' \otimes_A B \otimes_A F \xrightarrow{u} A' \otimes_A E$$

is the identity.

(2) *A'-module homomorphisms $v : A' \otimes_A F \to A' \otimes_A E$ and A'-algebra homomorphism $\varphi : A' \otimes_A B \to A' \otimes_A \mathrm{End}(E)$ such that the composite homomorphism*

$$A' \otimes_A E \xrightarrow{\mathrm{id}_{A'} \otimes s} A' \otimes_A B \otimes_A F \xrightarrow{\varphi \otimes_A v} A' \otimes_A \mathrm{End}_A(E) \otimes_A E \xrightarrow{\mathrm{id}_{A'} \otimes_A \mathrm{ev}} A' \otimes_A E \tag{3.2.1}$$

is the identity.

(3) *A-algebra homomorphisms $Q^s \to A'$.*

The bijection from the set (2) to the set (1) is described in Proposition 1.3 and the bijection from the set (3) to the set (2) is defined as follows:

Let $\varphi : Q^s \to A'$ be an A-algebra homomorphism. The homomorphism $\varphi \rho_{Q^s}$: $\mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A H_B \to A'$ is determined by A-algebra homomorphisms ρ_1 : $\mathrm{Sym}_A(F \otimes_A E\check{}) \to A'$ and $\rho_2 : H_B \to A'$. The homomorphism ρ_2 defines an A'-algebra homomorphism $\psi : A' \otimes_A B \to A' \otimes_A \mathrm{End}_A(E)$ by Proposition 2.8, and ρ_1 defines, by Lemma 2.2 with $E = K$ and $G = F$, an A'-module homomorphism $v : A' \otimes_A F \to A' \otimes_A E$. We extend v to an $A' \otimes_A B$-module homomorphism $A' \otimes_A B \otimes_A F \to A' \otimes_A E$, when $A' \otimes_A E$ has the $A' \otimes_A B$-module structure given by ψ.

Proof. The bijection between (1) and (2) is given in Corollary 1.4 when we use the canonical isomorphism of A'-module $A' \otimes_A \mathrm{End}_A(E) \to \mathrm{End}_{A'}(A' \otimes_A E)$.

From the description of the map from the set in (3) to the set in (2) given in the theorem it follows that the map is a bijection, since an A-algebra homomorphism $\mathrm{Sym}_A(F \otimes_A E\check{}) \otimes_A H_B \to A'$ factors via ρ_{Q^s} if and only if the composite homomorphism (3.2.1) is the identity.

That the map from the set (3) to the set (1) is functorial follows from the explicit description of the maps in the theorem.

4. The quot functor

4.1 Definition. Let $f : X \to S$ be a morphism of schemes. For every morphism $g : S' \to S$ we write

$$\begin{array}{ccc} X' = X \times_S S' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ S' & \xrightarrow[g]{} & S. \end{array}$$

Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module that is flat over S and such that $\operatorname{Supp} \mathcal{M}$ is a scheme that is finite over S. Here $\operatorname{Supp} \mathcal{M}$ is the subscheme of X defined by the *annihilator* of $\mathcal{M}$. We say that $\mathcal{M}$ is of *relative rank* n over S if $f_*\mathcal{M}$ is a locally free $\mathcal{O}_S$-module of rank n. The latter condition is equivalent to the condition that $\dim_{\boldsymbol{\kappa}(s)}(f_*\mathcal{M} \otimes_{\mathcal{O}_X} \boldsymbol{\kappa}(s)) = n$ for all points $s \in S$ (see [LPS]).

Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. We denote by $\mathcal{Q}uot^n_{\mathcal{F}/X/S}$ the functor from S-schemes to sets that to a morphism $g : S' \to S$ associates the set $\mathcal{Q}uot^n_{\mathcal{F}/X/S}(S')$ of classes of surjections $g'^*\mathcal{F} \to \mathcal{M}$ of $\mathcal{O}_{X'}$-modules, where $\mathcal{M}$ is a coherent $\mathcal{O}_{X'}$-module which is flat over S' with support that is a finite scheme over S' with *relative rank* n (see [G], or [D]).

Let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-module of rank n and fix an $\mathcal{O}_S$-module homomorphism $s : \mathcal{E} \to f_*\mathcal{F}$. We denote by $\mathcal{Q}uot^s_{\mathcal{F}/X/S}$ the subfunctor of $\mathcal{Q}uot^n_{\mathcal{F}/X/S}$ that to an S-scheme S' associates the set $\mathcal{Q}uot^s_{\mathcal{F}/X/S}(S')$ of all equivalence classes of surjections $g'^*\mathcal{F} \to \mathcal{M}$ such that the composite homomorphism

$$g^*\mathcal{E} \xrightarrow{g^*s} g^*f_*\mathcal{F} \to f'_*g'^*\mathcal{F} \to f'_*\mathcal{M}$$

is surjective, that is, an isomorphism.

When convenient we write $\mathcal{Q}uot^n_{\mathcal{F}}$ and $\mathcal{Q}uot^s_{\mathcal{F}}$ for the functors $\mathcal{Q}uot^n_{\mathcal{F}/X/S}$, respectively $\mathcal{Q}uot^s_{\mathcal{F}/X/S}$, and indicate in the text that the functors are taken relative to the homomorphism $f : X \to S$.

4.2 Lemma. *Let $A \to B$ be an A-algebra and let M be a B-module that is free of rank n as an A-module. Then $B/\operatorname{Ann}_B(M)$ is integral over A.*

Proof. Since M is finitely generated as an A-module it is finitely generated as a B-module. Let $x_1, \dots, x_n$ be generators as a B-module. We obtain a B-module homomorphism $B \to \prod_{i=1}^n M$ that maps b to $(bx_1, \dots, bx_n)$. This gives an injection $B/\operatorname{Ann}_B(M) \to \prod_{i=1}^n M$ of B-modules. In particular we have that $\prod_{i=1}^n M$ is a faithful $B/\operatorname{Ann}_B(M)$-module. Moreover, the composite homomorphism $A \to B/\operatorname{Ann}_B(M) \to \prod_{i=1}^n M$ is injective since M is free as an A-module. For every element $f \in B/\operatorname{Ann}_B(M)$ we have that $A[f]$ is contained in the finitely generated A-module $\prod_{i=1}^n M$ that is faithful over $A[f]$. Hence f is integral over A ([L] Chapter VII, §1 INT3).

4.3 Proposition. *Let $f : X \to S$ be a morphism of affine schemes. Assume that $\mathcal{E}$ is a free $\mathcal{O}_S$-module and that $\mathcal{F} = f^*\mathcal{F}_0$ where $\mathcal{F}_0$ is an $\mathcal{O}_S$-module. Then the functor $\mathcal{Q}uot^s_{\mathcal{F}/X/S}$ is representable by Q^s for $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, $E = \Gamma(S, \mathcal{E})$, $F = \Gamma(S, \mathcal{F}_0)$ and $s : E \to B \otimes_A F$ corresponds to the homomorphism s.*

Proof. In the correspondence between affine schemes over S and A-algebras we see that, in order to represent $\mathcal{Q}uot^s_{\mathcal{F}/X/S}$, we must represent the functor that to A' associates the $A' \otimes_A B$-module homomorphisms $u : A' \otimes_A B \otimes_A F \to M$ where M is a free A'-module of rank n such that $A' \otimes_A E \xrightarrow{\mathrm{id}_{A'} \otimes_A s} A' \otimes_A B \otimes_A F \xrightarrow{u} M$ is surjective. This functor is representable by Theorem 3.2, taken into account that the condition $\operatorname{Supp} M = A' \otimes_A B / \operatorname{Ann}_{A' \otimes_A B}(M)$ is finite over A' is automatically fulfilled by Lemma 4.2.

Like several of the reductions of this sections the following result is well known.

4.4 Lemma. *Let $f : X \to S$ be a homomorphism of affine schemes. Assume that $\mathcal{E}$ is a free $\mathcal{O}_S$-module. Then $\mathcal{Q}uot^s_{\mathcal{F}/X/S}$ is representable.*

More precisely, there is a free $\mathcal{O}_S$-module $\mathcal{F}_0$, a surjection $u : f^*\mathcal{F}_0 \to \mathcal{F}$ of $\mathcal{O}_X$-modules, and a homomorphism of $\mathcal{O}_S$-modules $s_0 : \mathcal{E} \to f_* f^* \mathcal{F}_0$ such that $s = (f_* u) s_0$. These homomorphisms make $\mathcal{Q}uot^s_{\mathcal{F}/X/S}$ into a closed subfunctor of $\mathcal{Q}uot^{s_0}_{f^*\mathcal{F}_0/X/S}$.

Proof. Let $F = \Gamma(X, \mathcal{F})$ and $E = \Gamma(S, \mathcal{E})$. Choose a surjection of A-modules $F_0 \to F$ with F_0 free. We then obtain, by extension of scalars, a surjection of B-modules $B \otimes_A F_0 \to F$, and consequently a surjection $u : f^*\mathcal{F}_0 \to \mathcal{F}$ of $\mathcal{O}_X$-modules with $\mathcal{F}_0 = \widetilde{F_0}$. We lift $s : E \to F$ to an A-module homomorphism $E \to B \otimes_A F_0$ via the surjection $B \otimes_A F_0 \to F$. The corresponding lifting $s_0 : \mathcal{E} \to f_* f^* \mathcal{F}_0$ has the property that $s = (f_* u) s_0$.

It is clear that for every A-algebra $A \to A'$ we have a map $\mathcal{Q}uot^n_{\mathcal{F}/X/S}(A') \to \mathcal{Q}uot^n_{f^*\mathcal{F}_0/X/S}(A')$ that takes $\mathcal{F} \xrightarrow{v} \mathcal{M}$ to the composite homomorphism $f^*\mathcal{F}_0 \xrightarrow{u} \mathcal{F} \xrightarrow{v} \mathcal{M}$, and that this map defines a closed immersion of functors $\mathcal{Q}uot^n_{\mathcal{F}/X/S} \to \mathcal{Q}uot^n_{f^*\mathcal{F}_0/X/S}$. Moreover the closed immersion maps $\mathcal{Q}uot^s_{\mathcal{F}/X/S}$ into $\mathrm{Quot}^{s_0}_{f^*\mathcal{F}_0/X/S}$ because $\mathcal{E} \xrightarrow{s} f_*\mathcal{F} \xrightarrow{v} \mathcal{M}$ is surjective if and only if $\mathcal{E} \xrightarrow{s_0} f_* f^* \mathcal{F}_0 \xrightarrow{f_* u} f_* \mathcal{F} \xrightarrow{v} \mathcal{M}$ is surjective.

The representability of $\mathrm{Quot}^{s_0}_{f^*\mathcal{F}_0/X/S}$ follows from Theorem 3.2 and Proposition 4.3.

4.5 Proposition. *Let $f : X \to S$ be a homomorphism of affine schemes. Then $\mathcal{Q}uot^n_{\mathcal{F}/X/S}$ is representable.*

More precisely, the functor $\mathcal{Q}uot^n_{\mathcal{F}/X/S}$ is covered by open affine subfunctors $\mathcal{Q}uot^s_{\mathcal{F}/X/S}$ for all $\mathcal{O}_S$-module homomorphisms $s : \mathcal{E} \to f_*\mathcal{F}$, where $\mathcal{E}$ is a free $\mathcal{O}_S$-module of rank n.

Proof. It is clear that the subfunctors $\mathcal{Q}uot^s_{\mathcal{F}/X/S}$ of $\mathcal{Q}uot^n_{\mathcal{F}/X/S}$ are open. We have to show that $\mathcal{Q}uot^n_{\mathcal{F}/X/S}$ is covered by these functors. Let $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, $F = \Gamma(\operatorname{Spec}(B), \mathcal{F})$ and $E = \Gamma(\operatorname{Spec}(A), \mathcal{E})$. For every A-algebra $A \to A'$ we let the A'-module homomorphism $u : A' \otimes_A F \to M$ correspond to an element $g'^*\mathcal{F} \to \mathcal{M}$ in $\mathcal{Q}uot^n_{\mathcal{F}/X/S}(\operatorname{Spec}(A'))$. For every maximal ideal $\mathfrak{p}$ in A' we can, since M is a finitely generated A'-module, find an element $a' \in A' \setminus \mathfrak{p}$ and an A-module homomorphism $s_M : E \to F$ such that the composite $A'_{a'}$-module homomorphism $A'_{a'} \otimes_A E \xrightarrow{s_M} A'_{a'} \otimes_A F \to M_{a'}$ is surjective. This shows that the image $g'^*\mathcal{F}|f'^{-1}\operatorname{Spec}(A'_{a'}) \to \mathcal{M}|f'^{-1}\operatorname{Spec}(A'_{a'})$ of the element $g'^*\mathcal{F} \to \mathcal{M}$ by the map $\mathcal{Q}uot^n_{\mathcal{F}/X/S}(\operatorname{Spec}(A')) \to \mathcal{Q}uot^n_{\mathcal{F}/X/S}(\operatorname{Spec}(A'_{a'}))$ lies in $\mathcal{Q}uot^s_{\mathcal{F}/X/S}(\operatorname{Spec}(A'_{a'}))$ so the set $\mathcal{Q}uot^s_{\mathcal{F}/X/S}(\operatorname{Spec}(A'_{a'}))$ *covers* $g'^*\mathcal{F}|f'^{-1}\operatorname{Spec}(A'_{a'}) \to \mathcal{M}|f'^{-1}\operatorname{Spec}(A'_{a'})$ considered as an element in $\mathcal{Q}uot^n_{\mathcal{F}/X/S}(\operatorname{Spec}(A'_{a'}))$.

The representability of $\mathcal{Q}uot^s_{\mathcal{F}/X/S}$ follows from Lemma 4.4 and it follows that the Zariski sheaf $\mathcal{Q}uot^n_{\mathcal{F}/X/S}$ is representable.

4.6 Lemma. *Let R be a graded A-algebra. For every prime ideal $\mathfrak{p}$ of A write $\boldsymbol{\kappa}(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$.*

Let Z be a closed subscheme of $\operatorname{Proj}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R)$ that is finite over $\operatorname{Spec}(\boldsymbol{\kappa}(\mathfrak{p}))$. Then there is an element $a \in A$ not in $\mathfrak{p}$ and an element $f \in R_a$ such that Z is contained in the open subscheme $\operatorname{Spec}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_{A_a} (R_a)_{(f)}) = \operatorname{Spec}(\boldsymbol{\kappa}(\mathfrak{p})) \times_{\operatorname{Spec}(A_a)} \operatorname{Spec}((R_a)_{(f)})$ of $\operatorname{Proj}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_{A_a} R_a) = \operatorname{Spec}(\boldsymbol{\kappa}(\mathfrak{p})) \times_{\operatorname{Spec}(A_a)} \operatorname{Proj}(R_a)$.

Proof. Since Z is finite over $\operatorname{Spec}(\boldsymbol{\kappa}(\mathfrak{p}))$ the fiber of the induced morhpism $Z \to \operatorname{Spec}(\boldsymbol{\kappa}(\mathfrak{p}))$ consists of a finite number of points, corresponding to homogeneous prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_k$ in $\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R$ that do not contain the irrelevant ideal. Their union consequently do not contain the irrelevant ideal. Hence we can find a homogeneous element $g \in \boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R$ of positive degree that is not contained in any of the ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_k$. Thus Z is contained in the open subscheme $\operatorname{Spec}((\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R)_{(g)})$ of $\operatorname{Proj}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R)$.

Clearly we can find an element $a \in A$ not in $\mathfrak{p}$ and an element $f \in R_a$ such that $1_{\boldsymbol{\kappa}(\mathfrak{p})} \otimes_{A_a} f$ is the image of g by the natural isomorphism $\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R \to \boldsymbol{\kappa}(\mathfrak{p}) \otimes_{A_a} R_a$. However, then $\operatorname{Spec}((\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A R)_{(g)}) = \operatorname{Spec}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_{A_a} (R_a)_{(f)})$, and we have proved the lemma.

4.7 Theorem. *Let $\mathcal{R}$ be a quasi-coherent sheaf with is a graded $\mathcal{O}_S$-algebra, let $X = \operatorname{Proj}(\mathcal{R})$, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{Q}uot^n_{\mathcal{F}/X/S}$ is representable.*

More precisely, for every open affine subscheme $\operatorname{Spec}(A)$ of S we write $R = \Gamma(\operatorname{Spec}(A), \mathcal{R})$. Then $\mathcal{Q}uot^n_{\mathcal{F}/X/S}$ is covered by open subfunctors naturally isomorphic to $\mathcal{Q}uot^n_{\mathcal{F}|\operatorname{Spec}(R_{(r)})}$ relative to $\operatorname{Spec}(R_{(r)}) \to \operatorname{Spec}(A)$ for all $\operatorname{Spec}(A)$ in an open covering of S and all homogeneous elements r of positive degree in R.

Proof. For every affine open subset U of S we consider $\mathcal{Q}uot^n_{\mathcal{F}|f^{-1}(U)}$ relative to $f^{-1}(U) \to U$ as a subfunctor of $\mathcal{Q}uot_{\mathcal{F}/\operatorname{Proj}(\mathcal{R})/S}$ by letting $\mathcal{Q}uot^n_{\mathcal{F}|f^{-1}(U)}(S') = \emptyset$ when $g : S' \to S$ does not factor via U. It is clear that the subfunctors $\mathcal{Q}uot^n_{\mathcal{F}|f^{-1}(U)}$ are open and that they cover $\mathrm{Quot}^n_{\mathcal{F}/\operatorname{Proj}(\mathcal{R})/S}$. Consequently we can assume that $S = \operatorname{Spec}(A)$ is affine.

For every $a \in A$ and every $r \in R_a$ we can consider the functor $\mathcal{Q}uot_{a,r} = \mathcal{Q}uot^n_{\mathcal{F}|\operatorname{Spec}((R_a)_{(r)})}$ relative to $\operatorname{Spec}(R_a)_{(r)} \to \operatorname{Spec}(A_a)$ as a subfunctor of the functor $\mathcal{Q}uot_a = \mathcal{Q}uot^n_{\mathcal{F}|\operatorname{Proj}(R_a)}$ relative to $\operatorname{Proj}(R_a) \to \operatorname{Spec}(A_a)$. This is because, if $g : S' \to \operatorname{Spec}(A_a)$ is a morphism and $g'^*\mathcal{F}|f^{-1}(\operatorname{Spec}(R_a)_{(r)}) \to \mathcal{M}$ represents an element in $\mathcal{Q}uot_{a,r}(S')$, then $\operatorname{Supp}\mathcal{M} \subseteq f^{-1}(\operatorname{Spec}((R_a)_{(r)}))$ and $\operatorname{Supp}\mathcal{M}$ is finite over $\operatorname{Spec}(A_a)$, and $\operatorname{Spec}((R_a)_{(r)}) \to \operatorname{Spec}(A_a)$ is separated so $\operatorname{Supp}\mathcal{M}$ is closed in $\operatorname{Proj}(R_a) \times_{\operatorname{Spec}(A_a)} S'$. This means that we can extend $\mathcal{M}$ uniquely by zero to an $\mathcal{O}_{\operatorname{Proj}(R_a) \times_{\operatorname{Spec}(A_a)} S'}$-module $\mathcal{N}$ and the surjection $g'^*\mathcal{F}|f'^{-1}(\operatorname{Spec}((R_a)_{(r)})) \to \mathcal{M}$ can be extended to a surjection $g'^*\mathcal{F} \to \mathcal{N}$ that represent an element in $\mathcal{Q}uot_a(\operatorname{Spec}(A_a))$. It is clear that the subfunctors $\mathcal{Q}uot_{a,r}$ are open. It remains to show that they cover $\mathcal{Q}uot_{\mathcal{F}/\operatorname{Proj}(R)/\operatorname{Spec}(A)}$. For this it suffices to show that for every prime ideal $\mathfrak{p} \subset A$ and every surjection $g'^*\mathcal{F} \to \mathcal{M}$, with $g' : \operatorname{Spec}(\boldsymbol{\kappa}(\mathfrak{p})) \times_S$

$\mathrm{Proj}(R_a) \to S' = \mathrm{Spec}(\boldsymbol{\kappa}(\mathfrak{p}))$, where $\mathcal{M}$ has finite support over $\mathrm{Spec}(\boldsymbol{\kappa}(\mathfrak{p}))$ of relative rank n, there is an $a \in A \setminus \mathfrak{p}$ and a homogeneous element r in R_a such that the support of $\mathcal{M}$ is contained in the open subscheme $\mathrm{Spec}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_A (R_a)_{(r)}) = \mathrm{Spec}(\boldsymbol{\kappa}(\mathfrak{p})) \times_{\mathrm{Spec}(A_a)} \mathrm{Spec}((R_a)_{(r)})$ of $\mathrm{Proj}(\boldsymbol{\kappa}(\mathfrak{p}) \otimes_{A_a} R_a) = \mathrm{Spec}(\boldsymbol{\kappa}(\mathfrak{p})) \times_{\mathrm{Spec}(A_a)} \mathrm{Proj}(R_a)$. However this is the assertion of Lemma 4.6.

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