THE HILBERT SCHEME PARAMETERIZING FINITE LENGTH SUBSCHEMES OF THE LINE WITH SUPPORT AT THE ORIGIN

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Abstract. We introduce symmetrizing operators of the polynomial ring $A[x]$ in the variable x over a ring A. When A is an algebra over a field k these operators are used to characterize the monic polynomials $F(x)$ of degree n in $A[x]$ such that $A \otimes_k k[x]_{(x)}/(F(x))$ is a free A–module of rank n. We use the characterization to determine the Hilbert scheme parameterizing subschemes of length n of $k[x]_{(x)}$.

Introduction. We shall study the Hilbert scheme parameterizing finite length subschemes of the local ring $k[x]_{(x)}$ of the line at the origin. The Hilbert schemes parameterizing finite length subschemes of local rings have mostly been studied for local rings at smooth points on surfaces (see e.g. [B], [BI], [C], [G], [I1], [I2], [I3], [P]). The focus has been on the rational points of the Hilbert schemes rather than on the schemes themselves.

The purpose of the following work is to point out that we loose essential information about the Hilbert schemes parameterizing finite length subschemes of a local ring by considering rational points instead of families. Indeed, there is only one rational point $k[x]/(x^n)$ of the Hilbert scheme parameterizing subschemes of $k[x]_{(x)}$ of length n whereas, as we shall show in this article, the Hilbert scheme is affine of dimension n. The coordinate ring is equal to the localization of the symmetric polynomials of the ring $k[t_1, \dots, t_n]$ in n variables, in the multiplicatively closed subset consisting of the products $g(t_1) \cdots g(t_n)$ for all polynomials $g(x)$ in one variable over k such that $g(0) \neq 0$.

In forthcoming work [S1] the second author will use the techniques and results of the present article to show that the *functor of families with support at the origin*, in contrast to the Hilbert functor, is not even representable. The functor of families with support at the origin is frequently used by some authors because it has the same rational points as the Hilbert scheme. In [S2] the second author shows how the techniques of the present article can be used on any localization of $k[x]$, over any ring k, and gives the relation to the well known result that the Hilbert scheme of the projective line is given by the symmetric product.

An easy and fundamental result in commutative algebra states that the residue ring $A[x]/(F(x))$ of the polynomial ring in a variable x over a ring A by the ideal generated by a monic polynomial $F(x)$ of degree n, is a free A–module of rank n.

The key to the study of the Hilbert scheme parameterizing finite length subschemes of $k[x]_{(x)}$ is to determine when the ring $A \otimes_k k[x]_{(x)}/(F(x))$ is a free A–module of rank n. Easy examples show that the A–module $A \otimes_k k[x]_{(x)}/(F(x))$ neither has to be of rank n nor to be finitely generated. Indeed, we have that $A \otimes_k k[x]_{(x)}/(x-1) = 0$ for all A, and $k[u] \otimes_k k[x]_{(x)}/(x-u) = k[u]_{(u)}$ when $A = k[u]$ is a polynomial ring over k in the variable u.

Theorem (2.3), which is the main result of the article, characterizes the monic polynomials $F(x)$ in $A[x]$ of degree n such that $A \otimes_k k[x]_{(x)}/(F(x))$ is a free A–module of rank n. The essential technical tool used in the proof is the introduction of symmetrizing operators on the polynomials in the ring $A[x]$ associated to $F(x)$. The method is introduced in Section 1 and is the main technical novelty of the article.

Theorem (2.3) is used in Section 3, where we describe the ideals I in $A \otimes_k k[x]_{(x)}$ such that $A \otimes_k k[x]_{(x)}/I$ is a free A–module of rank n. We then proceed in Section 4 to determine the Hilbert scheme parameterizing length n subschemes of $k[x]_{(x)}$.

1. Notation and the symmetrizing operators.

1.1. Notation. Let A be a commutative ring. Denote by $A[x,t] = A[x, t_1, \dots, t_n]$ the polynomial ring over A in the variables $x, t_1, \dots, t_n$. Let $\varphi\colon A \to K$ be a homomorphism of commutative rings. We shall consider K as an A–algebra via this homomorphism and write $K[x,t] = K \otimes_A A[x,t]$. Let $G(x) = g_0 x^m + \dots + g_m$ be a polynomial in $A[x]$. We write $G^\varphi(x) = \varphi(g_0)x^m + \dots + \varphi(g_m)$ in $K[x]$.

We denote by $s_i(t)$ the i'th elementary symmetric function in the variables $t_1, \dots, t_n$.

1.2. The main construction. Let $G(x)$ be a polynomial in $A[x]$. We write

$$s_i(G(t)) = s_i(G(t_1), \dots, G(t_n)).$$

The polynomial $s_i(G(t))$ is symmetric in the variables $t_1, \dots, t_n$. We note that the symmetric function $s_i(x(t))$ associated to the polynomial $G(x) = x$ is equal to the elementary symmetric function $s_i(t)$ so there is no confusion of notation. We write

$$\begin{aligned}\Delta(G,t) &= \prod_{i=1}^{n} (G(x) - G(t_i)) \\ &= G(x)^n - s_1(G(t))G(x)^{n-1} + \dots + (-1)^n s_n(G(t)) \qquad (1.2.1)\end{aligned}$$

in $A[x,t]$. The polynomial $\Delta(G,t)$ is symmetric in the variables $t_1, \dots, t_n$ and $\Delta(x,t) = \prod_{i=1}^n (x - t_i) = x^n - s_1(t)x^{n-1} + \dots + (-1)^n s_n(t)$. Since $G(x) - G(t_i)$ is divisible by $x - t_i$ we obtain that

$$\Delta(G,t) = H(x,t)\Delta(x,t) \qquad (1.2.2)$$

in $A[x,t]$.

Fix a polynomial

$$F(x) = x^n - u_1 x^{n-1} + \dots + (-1)^n u_n$$

in $A[x]$. There is a unique A–algebra homomorphism

$$u\colon A[s_1(t), \dots, s_n(t)] \to A$$

determined by $u(s_i(t)) = u_i$ for $i = 1, \dots, n$. We have that $\Delta^u(x,t) = F(x)$. Write $s_{F,i}(G(t)) = u(s_i(G(t)))$. It follows from the formulas (1.2.1) and (1.2.2) that

$$G(x)^n - s_{F,1}(G(t))G(x)^{n-1} + \cdots + (-1)^n s_{F,n}(G(t)) = H^u(x)F(x). \tag{1.2.3}$$

in $A[x]$.

1.3. Lemma. *Let $G(x)$ be a polynomial in $A[x]$. If $s_{F,n}(G(t))$ is invertible in A we have that the class of $G(x)$ in $A[x]/(F(x))$ is invertible.*

Proof. When $s_{F,n}(G(t))$ is invertible we obtain from formula (1.2.3) the formula

$$\begin{aligned}(-1)^{n+1}s_{F,n}(G(t))^{-1}G(x)\,\big[G(x)^{n-1} - s_{F,1}(G(t))G(x)^{n-2} + \cdots \\ +(-1)^{n-1}s_{F,n-1}(G(t))\big] = 1 + (-1)^{n+1}s_{F,n}(G(t))^{-1}H^u(x)F(x).\end{aligned}$$

The Lemma follows immediately from the latter formula.

1.4. Lemma. *Let $\varphi\colon A \to K$ be a ring homomorphism of the ring A into a field K. Let $\alpha_1, \dots, \alpha_n$ be the roots of the polynomial*

$$F^\varphi(x) = x^n - \varphi(u_1)x^{n-1} + \cdots + (-1)^n\varphi(u_n)$$

in the algebraic closure of K. Then we have that

$$\varphi(s_{F,n}(G(t))) = G^\varphi(\alpha_1)\cdots G^\varphi(\alpha_n)$$

in K.

In particular, if k is a field and φ is a homomorphism of k–algebras, we have for each polynomial $g(x)$ in $k[x]$ that

$$\varphi\left(s_{F,n}(g(t))\right) = g(\alpha_1)\cdots g(\alpha_n)$$

in K.

Proof. From the construction of Section (1.2) for the ring K we obtain an expression

$$\Delta(G^\varphi, t) = G^\varphi(x)^n - s_1(G^\varphi(t))G^\varphi(x)^{n-1} + \cdots + (-1)^n s_n(G^\varphi(t))$$

in $K[x,t]$. Moreover, from the polynomial $F^\varphi(x)$ we obtain a unique K–algebra homomorphism

$$\varphi u\colon K[s_1(t), \dots, s_n(t)] \to K$$

determined by $(\varphi u)(s_i(t)) = \varphi(u_i)$. It follows from the construction of Section (1.2) applied to A and to K, that we have

$$\varphi(s_{F,i}(G(t))) = s_{F^\varphi,i}(G^\varphi(t))$$

for $i = 1, \dots, n$. Denote by $\overline{K}$ the algebraic closure of K. The K–algebra homomorphism

$$\alpha\colon K[t_1, \dots, t_n] \to \overline{K}$$

determined by $\alpha(t_i) = \alpha_i$ extends the homomorphism φu because $\alpha(s_i(t)) = s_i(\alpha_1, \dots, \alpha_n) = \varphi(u_i) = (\varphi u)(s_i(t))$. We have that $s_n(G^\varphi(t)) = G^\varphi(t_1)\cdots G^\varphi(t_n)$ in $K[t]$. Hence

$$\varphi\left(s_{F,n}(G(t))\right) = s_{F^\varphi,n}(G^\varphi(t)) = \alpha(G^\varphi(t_1)\cdots G^\varphi(t_n)) = G^\varphi(\alpha_1)\cdots G^\varphi(\alpha_n).$$

which is the formula of the first part of the Lemma.

The second part of the Lemma follows from the first because $g^\varphi(x) = g(x)$ for all polynomials $g(x)$ in $k[x]$.

2. Roots of $F^\varphi(x)$ and invertible elements in $A[x]/(F(x))$.

2.1. Notation. We shall use the notation of Sections (1.1) and (1.2). Let A be a ring and let P be a prime ideal. We write $\kappa(P) = A_P/PA_P$ for the residue field. Let k be a field and assume that A is a k–algebra. Denote by $k[x]_{(x)}$ the localization of $k[x]$ in the multiplicatively closed subset $k[x] \setminus (x)$ of polynomials $g(x)$ in $k[x]$ such that $g(0) \neq 0$. We have that $A \otimes_k k[x]_{(x)}$ is the localization of the $k[x]$–algebra $A \otimes_k k[x]$ in the multiplicatively closed set $k[x] \setminus (x)$.

2.2 Lemma. *Let K be a field extension of k. Let $G(x)$ be a polynomial in $K[x]$ that has a non–zero root in the algebraic closure $\overline{K}$ of K which is algebraic over k. Then there is a polynomial $g(x)$ in $k[x]$ with $g(0) \neq 0$ and a factorization $I(x)g(x) = H(x)G(x)$ in $K[x]$, where $I(x)$ is a non–zero polynomial with $\deg(I) < \deg(G)$ whose roots in $\overline{K}$ are zero or transcendental over k.*

Proof. We shall prove the Lemma by induction on the degree m of $G(x)$.

Let α be a non–zero root of $G(x)$ in $\overline{K}$ which is algebraic over k. Denote by $g_1(x) \in k[x]$ and $G_1(x) \in K[x]$ the minimal polynomials of α over k respective K. Then we have that $g_1(0) \neq 0$ and $\deg(G_1) \geq 1$. Moreover we have factorizations $g_1(x) = H_1(x)G_1(x)$ and $G(x) = I_1(x)G_1(x)$ in $K[x]$, where $I_1(x)$ is non–zero and $\deg(I_1) < \deg(G)$. Consequently $I_1(x)g_1(x) = H_1(x)I_1(x)G_1(x) = H_1(x)G(x)$ in $K[x]$.

When $m = 1$ we have that $I_1(x)$ is a non–zero constant in K and the Lemma holds. If all the roots of $I_1(x)$ are zero or transcendental over k we have proved the Lemma.

Assume that the Lemma holds for all polynomials of degree less that m. It remains to prove the Lemma when $m > 1$ and $I_1(x)$ has a non–zero root in $\overline{K}$ that is algebraic over k. Since $\deg(I_1) = \deg(G) - \deg(G_1) < m$ it follows from the induction assumption that there is a polynomial $g_2(x) \in k[x]$ such that $g_2(0) \neq 0$, and a factorization $I(x)g_2(x) = H_2(x)I_1(x)$ in $K[x]$, where $I(x)$ is a non–zero polynomial such that $\deg(I) < \deg(I_1)$ whose roots are all zero or transcendental over k. We get that $I(x)g_1(x)g_2(x) = H_2(x)g_1(x)I_1(x) = H_1(x)H_2(x)G(x)$, and we have proved the Lemma.

2.3. Theorem. *Let A be a k–algebra, and let*

$$F(x) = x^n - u_1x^{n-1} + \cdots + (-1)^n u_n$$

be a polynomial in $A[x]$. The following six assertions are equivalent:

(1) *For all maximal ideals P of A with residue map $\varphi\colon A \to \kappa(P)$, the roots of $F^\varphi(x) = x^n - \varphi(u_1)x^{n-1} + \cdots + (-1)^n\varphi(u_n)$ in the algebraic closure of $\kappa(P)$ are zero or transcendental over k.*
(2) *For every polynomial $g(x)$ in $k[x]$ with $g(0) \neq 0$ we have that $s_{F,n}(g(t))$ is invertible in A.*
(3) *For every polynomial $g(x)$ in $k[x]$ with $g(0) \neq 0$ we have that the class of $g(x)$ in $A[x]/(F(x))$ is invertible.*
(4) *The canonical fraction map*

$$A[x]/(F(x)) \to A \otimes_k k[x]_{(x)}/(F(x))$$

is an isomorphism.

(5) *The A–module $A \otimes_k k[x]_{(x)}/(F(x))$ is free of rank n with a basis consisting of the classes of $1, x, \dots, x^{n-1}$.*
(6) *For all maximal ideals P of A with residue map $\varphi\colon A \to \kappa(P)$, the $\kappa(P)$–vectorspace $\kappa(P) \otimes_k k[x]_{(x)}/(F^\varphi(x))$ is n–dimensional with a basis consisting of the classes of $1, x, \dots, x^{n-1}$.*

Proof. Assume that assertion (1) holds. Let $\varphi\colon A \to K$ be the residue map associated to a maximal ideal of A. It follows from Lemma (1.4) that we have $\varphi(s_{F,n}(g(t))) = g(\alpha_1)\cdots g(\alpha_n)$ in K, where $\alpha_1, \dots, \alpha_n$ are the roots of $F^\varphi(x)$ in the algebraic closure of K. Since $g(0) \neq 0$ we have that $g(\alpha_i) \neq 0$ both when α_i is zero and when α_i is transcendental over k. Hence $\varphi(s_{F,n}(g(t))) \neq 0$. Since this holds for all maximal ideals of A we have that $s_{F,n}(g(t))$ is not contained in any maximal ideal of A and thus is invertible in A. Consequently assertion (2) holds.

It follows from Lemma (1.3) that assertion (2) implies assertion (3).

The canonical map of assertion (4) is the fraction map of the $k[x]$–algebra $A[x]$ by the multiplicatively closed subset $k[x] \setminus (x)$. Thus the fraction map is an isomorphism if and only if the classes of the elements of $k[x] \setminus (x)$ in $A[x]/(F(x))$ are invertible. Consequently assertions (3) and (4) are equivalent.

Since the A–module $A[x]/(F(x))$ is free of rank n with a basis consisting of the classes $1, x, \dots, x^{n-1}$, we have that assertions (4) and (5) are equivalent.

It is evident that assertion (5) implies assertion (6).

We shall prove that assertion (6) implies assertion (1). Assume that assertion (1) does not hold. Then there exists a maximal ideal in A with residue map $\varphi\colon A \to K$ such that $F^\varphi(x)$ has a non–zero root in the algebraic closure of K that is algebraic over k. It follows from Lemma (2.2) that there is a polynomial $g(x) \in k[x]$ with $g(0) \neq 0$, and a factorization $I(x)g(x) = H(x)F^\varphi(x)$ in $K[x]$ where $I(x)$ is a non–zero polynomial with $\deg(I) < \deg(F^\varphi) = n$, such that the roots of $I(x)$ are zero or transcendental over k. Since $g(x)$ is invertible in $K \otimes_k k[x]_{(x)}$ we obtain that $(I(x)) \subseteq (F^\varphi(x))$ in $K \otimes_k k[x]_{(x)}$ and thus a surjection

$$K \otimes_k k[x]_{(x)}/(I(x)) \to K \otimes_k k[x]_{(x)}/(F^\varphi(x)). \tag{2.3.1}$$

We already proved that assertion (1) implies assertion (6). Hence we conclude that $K \otimes_k k[x]_{(x)}/(I(x))$ is a vectorspace of dimension $\deg(I)$. Since we have the surjection (2.3.1) the dimension of the K–vectorspace $K \otimes_k k[x]_{(x)}/(F^\varphi(x))$ is at most equal to $\deg(I)$ and thus strictly smaller than n. Hence assertion (6) does not hold. We have thus proved that assertion (6) implies assertion (1).

2.4. Corollary. *Assume that $A = K$ is a field. Let $G(x)$ be a polynomial in $K[x]$ of degree n. Then the K–vectorspace $K \otimes_k k[x]_{(x)}/(G(x))$ is generated by the classes of $1, x, \dots, x^{n-1}$.*

Proof. If the roots of $G(x)$ in the algebraic closure $\overline{K}$ of K are zero or transcendental over k, the Corollary follows from the Theorem.

Assume that $G(x)$ has a non–zero root in $\overline{K}$ that is algebraic over k. It follows from Lemma (2.2) that there is a polynomial $g(x)$ in $k[x]$ with $g(0) \neq 0$, and a factorization $I(x)g(x) = H(x)G(x)$ in $K[x]$ where $I(x)$ is a non–zero polynomial with $\deg(I) < \deg(G)$ whose roots in $\overline{K}$ are zero or transcendental over k. We obtain that $(I(x)) \subseteq (G(x))$ in $K \otimes_k k[x]_{(x)}$ and thus a surjection $K \otimes_k k[x]_{(x)}/(I(x)) \to$

$K \otimes_k k[x]_{(x)}/(G(x))$. By Theorem (2.3) we have that $K \otimes_k k[x]_{(x)}/(I(x))$ is generated by the classes of $1, x, \dots, x^{m-1}$ with $m = \deg(I) < \deg(G) = n$. Thus $K \otimes_k k[x]_{(x)}/(G(x))$ is generated by the classes of $1, x, \dots, x^{n-1}$.

3. Ideals in $A \otimes_k k[x]_{(x)}$ whose residue rings are free A–modules.

3.1. Notation. Let k be a field and let A be a k–algebra. We denote by $k[x]$ the polynomial ring in a variable x over k and write $A[x] = A \otimes_k k[x]$. We denote by $k[x]_{(x)}$ the localization of $k[x]$ in the multiplicatively closed set $k[x] \setminus (x)$ consisting of polynomials $g(x)$ of $k[x]$ with $g(0) \neq 0$.

3.2. Lemma. *Given an ideal I of $A \otimes_k k[x]_{(x)}$ such that the residue ring $A \otimes_k k[x]_{(x)}/I$ is a free A–module of rank n. Then the A–module $A \otimes_k k[x]_{(x)}/I$ has a basis consisting of the classes of the elements $1, x, \dots, x^{n-1}$.*

Proof. We must prove that the A–module homomorphism

$$A^n \to A \otimes_k k[x]_{(x)}/I \tag{3.2.1}$$

which sends the coordinates of A^n to the classes of $1, x, \dots, x^{n-1}$ is an isomorphism. It suffices to prove that the localization of the map (3.2.1) in each prime ideal of A is an isomorphism. Hence we may assume that A is a local k–algebra. In fact, it suffices to prove that the map (3.2.1) is surjective since any set of n generators of a free module of rank n form a basis.

Assume that A is local and let K be the residue field of A. We denote by I_K the image of the ideal I by the residue map $A \otimes_k k[x]_{(x)} \to K \otimes_k k[x]_{(x)}$. From the map (3.2.1) we obtain a homomorphism

$$K^n \to K \otimes_k k[x]_{(x)}/I_K \tag{3.2.2}$$

of K–vectorspaces which sends the coordinates of K^n to the classes of $1, x, \dots, x^{n-1}$. If the map (3.2.2) is injective then it is surjective because $\dim_K(K \otimes_k k[x]_{(x)}/I_K) = n$ by assumption. On the other hand, if (3.2.2) is not injective, there is a polynomial $G(x) = x^m + g_1 x^{m-1} + \dots + g_m$ in $K[x]$ of degree $m \leq n$ such that $(G(x)) \subseteq I_K$ in $K \otimes_k k[x]_{(x)}$. Hence we have a surjection $K \otimes_k k[x]_{(x)}/(G(x)) \to K \otimes_k k[x]_{(x)}/I_K$. It follows from Corollary (2.4) that $K \otimes_k k[x]_{(x)}/(G(x))$ is generated by the classes of $1, x, \dots, x^{m-1}$. Thus the K–vectorspace $K \otimes_k k[x]_{(x)}/I_K$ is generated by the classes of $1, x, \dots, x^{n-1}$, and the map (3.2.2) is surjective even when it is not injective.

By assumption the A–module $A \otimes_k k[x]_{(x)}/I$ is finitely generated. Hence it follows from Nakayama's Lemma that the A–module homomorphism (3.2.1) is surjective.

3.3. Theorem. *Given an ideal I in $A \otimes_k k[x]_{(x)}$ such that $A \otimes_k k[x]_{(x)}/I$ is a free A–module of rank n. Then the ideal I is generated by a unique monic polynomial*

$$F(x) = x^n - u_1 x^{n-1} + \dots + (-1)^n u_n$$

in $A[x]$.

Proof. Since the A–module $A \otimes_k k[x]_{(x)}/I$ is free of rank n by assumption, it follows from Lemma (3.2) that the A–module $A \otimes_k k[x]_{(x)}/I$ has a basis consisting of the classes of the elements $1, x, \dots, x^{n-1}$. Hence the class of x^n in $A \otimes_k k[x]_{(x)}/I$ can be written as a unique A–linear combination of the classes of $1, x, \dots, x^{n-1}$. It follows that there is a unique monic polynomial $F(x) = x^n - u_1x^{n-1} + \cdots + (-1)^n u_n$ in $A[x]$ whose image is contained in I.

To show that $F(x)$ generates I we must prove that the surjective residue map

$$A \otimes k[x]_{(x)}/(F(x)) \to A \otimes_k k[x]_{(x)}/I \tag{3.3.1}$$

is injective. It suffices to prove that the localization of (3.3.1) at every prime ideal of A is injective. Hence we may assume that A is local. Denote by $\varphi \colon A \to K$ the residue map, and let I_K be the image of the ideal I by the residue map $A \otimes_k k[x]_{(x)} \to K \otimes_k k[x]_{(x)}$. From the map (3.3.1) we obtain the K–linear residue map

$$K \otimes_k k[x]_{(x)}/(F^\varphi(x)) \to K \otimes_k k[x]_{(x)}/I_K. \tag{3.3.2}$$

By assumption we have that the A–module $A \otimes_k k[x]_{(x)}/I$ is free of rank n. Consequently we have that $K \otimes_k k[x]_{(x)}/I_K$ is an n–dimensional K–vectorspace. It follows from Lemma (3.2) that the classes of $1, x, \dots, x^{n-1}$ in $K \otimes_k k[x]_{(x)}/I_K$ form a K–basis. From Corollary (2.4) it follows that the classes of $1, x, \dots, x^{n-1}$ in the K–vectorspace $K \otimes_k k[x]_{(x)}/(F^\varphi(x))$ are generators. The existence of the surjection (3.3.2) therefore shows that the classes of $1, x, \dots, x^{n-1}$ in $K \otimes_k k[x]_{(x)}/(F^\varphi(x))$ form a K–basis. Hence it follows from Theorem (2.3) that the roots of $F^\varphi(x)$ in the algebraic closure of K are zero or transcendental over k. Consequently it follows from Theorem (2.3) that the classes of $1, x, \dots, x^{n-1}$ in the A–module $A \otimes_k k[x]_{(x)}/(F(x))$ form a basis. On the other hand it follows from Lemma (3.2) that the classes of $1, x, \dots, x^{n-1}$ in the A–module $A \otimes_k k[x]_{(x)}/I$ form a basis. It follows that the map (3.3.1) is injective. We have proved the Theorem.

4. The coordinate ring of the Hilbert scheme.

4.1. Notation. Let k be a field. Denote by $k[x, t] = k[x, t_1, \dots, t_n]$ the polynomial ring over k in the variables $x, t_1, \dots, t_n$. For every k–algebra A we write $A[x, t] = A \otimes_k k[x, t]$. We denote by $s_i(t)$ the i'th elementary symmetric polynomial in the variables $t_1, \dots, t_n$. For every polynomial $g(x)$ in $k[x]$ we form, as in the construction of Section (1.2), the symmetric polynomial $s_n(g(t)) = g(t_1) \cdots g(t_n)$ in $k[t_1, \dots, t_n]$. The set

$$U = \{s_n(g(t)) \colon g(x) \in k[x] \text{ and } g(0) \neq 0\}$$

form a multiplicatively closed subset of $k[s_1(t), \dots, s_n(t)]$. We write

$$H_n = U^{-1}k[s_1(t), \dots, s_n(t)]$$

and

$$F_n(x) = x^n - s_1(t)x^{n-1} + \cdots + (-1)^n s_n(t)$$

in $H_n[x]$.

4.2 Proposition. *Let $\psi \colon H_n = U^{-1}k[s_1(t), \dots, s_n(t)] \to A$ be a k–algebra homomorphism. Let $\psi(s_i(t)) = u_i$ for $i = 1, \dots, n$ and write*

$$F_n^\psi(x) = x^n - u_1x^{n-1} + \cdots + (-1)^n u_n$$

in $A[x]$. Then the A–module

$$A \otimes_k k[x]_{(x)}/(F_n^{\psi}(x))$$

is free of rank n with a basis consisting of the classes of the elements $1, x, \dots, x^{n-1}$.

In particular the H_n module $H_n \otimes_k k[x]_{(x)}/(F_n(x))$ is free of rank n with a basis consisting of the classes of the elements $1, x, \dots, x^{n-1}$.

Proof. The fraction map $\xi\colon k[s_1(t), \dots, s_n(t)] \to H_n$ composed with ψ defines a map $\zeta\colon k[s_1(t), \dots, s_n(t)] \to A$. We have that ζ is the restriction to $k[s_1(t), \dots, s_n(t)]$ of the map u of the construction of Section (1.2) for the polynomial $F_n^{\psi}(x)$. Hence we have that

$$\psi(s_n(g(t))) = \zeta\,(s_n(g(t))) = s_{F^{\psi},n}(g(t))$$

for all polynomials $g(x)$ of $k[x]$. Given a polynomial $g(x)$ of $k[x]$ with $g(0) \neq 0$. Since ζ factors via the fraction map ξ and $s_n(g(t))$ is invertible in H_n by definition, we have that $\zeta(s_n(g(t))) = s_{F^{\psi},n}(g(t))$ is invertible in A. Consequently it follows from Theorem (2.3) that $A \otimes_k k[x]_{(x)}/(F_n^{\psi}(x))$ is a free A–module with a basis consisting of the classes of the elements $1, x, \dots, x^{n-1}$. We have proved the Proposition.

4.3. Notation. We denote by $\mathcal{H}ilb_a^n$ the *affine Hilbert functor* from the category of k–algebras to sets that sends a k–algebra A to

$$\begin{aligned}\mathcal{H}ilb_a^n(A) =\{&\text{Ideals } I \text{ in } A \otimes_k k[x]_{(x)} \text{ such that}\\ &A \otimes_k k[x]_{(x)}/I \text{ is a free } A\text{–module of rank } n\}.\end{aligned}$$

It follows from Proposition (4.2) that we, for every k–algebra A, obtain a natural map

$$\mathcal{F}(A)\colon \mathrm{Hom}_k(H_n, A) \to \mathcal{H}ilb_a^n(A)$$

which sends a k–algebra homomorphism $\psi\colon H_n \to A$ to the ideal $(F_n^{\psi}(x))$ in $A \otimes_k k[x]_{(x)}$. Clearly $\mathcal{F}$ defines a morphism of functors from k–algebras to sets.

4.4. Theorem. *The morphism $\mathcal{F}$ is an isomorphism of functors. Equivalently, the k–algebra H_n represents the functor $\mathcal{H}ilb_a^n$.*

Proof. We shall construct an inverse to $\mathcal{F}$. Given a k–algebra A and an ideal I in $A \otimes_k k[x]_{(x)}$ such that the A–module $A \otimes_k k[x]_{(x)}/I$ is free of rank n. It follows from Theorem (3.3) that there is a unique polynomial $F(x) = x^n - u_1x^{n-1} + \cdots + (-1)^n u_n$ in $A[x]$ such that $(F(x)) = I$ in $A \otimes_k k[x]_{(x)}$. In particular we have that $A \otimes_k k[x]_{(x)}/(F(x))$ is a free A–module of rank n. Hence it follows from Theorem (2.3) that the element $s_{F,n}(g(t))$ is invertible in A for all polynomials $g(x)$ in $k[x]$ such that $g(0) \neq 0$.

We have that the k–algebra homomorphism $\zeta\colon k[s_1(t), \dots, s_n(t)] \to A$ determined by $\zeta(s_i(t)) = u_i$ for $i = 1, \dots, n$ coincides with the restriction to $k[s_1(t), \dots, s_n(t)]$ of the homomorphism u of Section (1.2). It follows that $\zeta(s_i(g(t))) = s_{F,i}(g(t))$ for $i = 1, \dots, n$ and for all polynomials $g(x)$ in $k[x]$. In particular the elements $s_n(g(t))$ of U are mapped to the invertible elements $s_{F,n}(g(t))$ in A. Consequently the k–algebra homomorphism ζ factors through a unique k–algebra homomorphism $\psi\colon H_n = U^{-1}k[s_1(t), \dots, s_n(t)] \to A$. Thus we have constructed a map $\mathcal{H}ilb_a^n(A) \to \mathrm{Hom}_k(H_n, A)$. It is easy to check that this map is the inverse to $\mathcal{F}(A)$.

4.5. Definition. We define the *Hilbert functor $\mathcal{H}ilb^n$ of families of length n subschemes of* $\operatorname{Spec} k[x]_{(x)}$ as the contravariant functor from schemes over k to sets, that to a k–scheme T associates the set

$$\begin{aligned}\mathcal{H}ilb^n(T) = \{&\text{Closed subschemes } Z \text{ in } T \times_{\operatorname{Spec} k} \operatorname{Spec} k[x]_{(x)} \text{ such that}\\ &p_*\mathcal{O}_Z \text{ is a locally free } \mathcal{O}_T\text{–module of rank } n, \text{ where}\\ &p\colon T \times_{\operatorname{Spec} k} \operatorname{Spec} k[x]_{(x)} \to T \text{ is the projection}\}.\end{aligned}$$

Equivalently the set $\mathcal{H}ilb^n(T)$ consists of the quasi–coherent ideals $\mathcal{I}$ in $\mathcal{O}_T \otimes_{\mathcal{O}_{\operatorname{Spec} k}} \mathcal{O}_{\operatorname{Spec} k[x]_{(x)}}$ such that $\left(\mathcal{O}_T \otimes_{\mathcal{O}_{\operatorname{Spec} k}} \mathcal{O}_{\operatorname{Spec} k[x]_{(x)}}\right)/\mathcal{I}$ is locally free considered as an $\mathcal{O}_T$–module via p.

4.6. Theorem. *We have that the scheme* $\operatorname{Spec} H_n$ *with the universal family* $\operatorname{Spec}\left(H_n \otimes_k k[x]_{(x)}/(F_n(x))\right)$ *represents* $\mathcal{H}ilb^n$.

Proof. For every k–scheme T and every point Z in $\mathcal{H}ilb^n(T)$ we cover T with affine open subsets $\operatorname{Spec} A$ such that $p_*\mathcal{O}_Z|\operatorname{Spec} A$ is a free $\mathcal{O}_{\operatorname{Spec} A}$–module. It follows from Theorem (4.4) that we have a unique morphism $\operatorname{Spec} A \to \operatorname{Spec} H_n$ such that the family $\operatorname{Spec}\left(H_n \otimes_k k[x]_{(x)}/(F_n(x))\right)$ pulls back to the family $Z \cap p^{-1}(\operatorname{Spec} A)$ over $\operatorname{Spec} A$. Since the map is unique the morphisms for the affine subsets covering T glue together to a morphism $T \to \operatorname{Spec} H_n$ such that the family $\operatorname{Spec}\left(H_n \otimes_k k[x]_{(x)}/(F_n(x))\right)$ pulls back to Z.

4.7. Note. The point $s_1(t) = \cdots = s_n(t) = 0$ is, as expected, the only k–rational point of $\operatorname{Spec} H_n$. Indeed, let $(u_1, \dots, u_n)$ be point different from the origin of the affine n–dimensional space $\mathbf{A}_k^n$ over k. To show that this point does not lie in $\operatorname{Spec} H_n$ we chose roots $\alpha_1, \dots, \alpha_n$ in $\overline{k}$ of the polynomial $x^n - u_1x^{n-1} + \cdots + (-1)^n u_n$ in $k[x]$. Since all the u_i are not zero there is at least one root α_i which is not zero. Let $g(x) \in k[x]$ be the minimal polynomial of such a root. Then $g(0) \neq 0$ and thus $s_n(g(t)) \in U$. The map $\alpha\colon k[t_1, \dots, t_n] \to \overline{k}$ which sends t_i to α_i for $i = 1, \dots, n$ iduces the map $u\colon k[s_1(t), \dots, s_n(t)] \to k$ which sends $s_i(t)$ to u_i for $i = 1, \dots, n$. However, the map u does not factor through the fraction map $k[s_1(t), \dots, s_n(t)] \to H_n$ because $s_n(g(t)) = g_1(t) \cdots g_n(t)$ is mapped to the element $g(\alpha_1) \cdots g(\alpha_n) = 0$ in k. Hence the point $(u_1, \dots, u_n)$ is not in $\operatorname{Spec} H_n$.

When $n = 1$ we have that $H_1 = k[x]_{(x)}$ is a local ring. However, when $n \geq 2$ the ring H_n is not local. To see this we first prove that the ideal $(F(t))$ in $k[s_1(t), \dots, s_n(t)]$ generated by a non–constant, symmetric polynomial $F(t)$ which is irreducible in $k[t_1, \dots, t_n]$ does not intersect the multiplicatively closed subset U of $k[s_1(t), \dots, s_n(t)]$. Assume that $(F(t))$ intersects U. Then we have that $F(t)G(t) = s_n(f(t)) = f(t_1) \cdots f(t_n)$ in $k[s_1(t), \dots, s_n(t)]$ for a polynomial $f(x) \in k[x]$ with $f(0) \neq 0$. Then $F(t)$ divides one of the polynomials $f(t_1), \dots, f(t_n)$ in $k[t_1, \dots, t_n]$. Hence $F(t)$ is a polynomial in one of the variables t_i. Hence, when $n \geq 2$ it can not be symmetric, contrary to our assumption. Thus $(F(t))$ does not intersect U.

For each non–constant, symmetric, irreducible polynomial $F(t)$ in $k[t_1, \dots, t_n]$ we can choose an ideal P_F which is maximal among the ideals in $k[s_1(t), \dots, s_n(t)]$ that contain $F(t)$ and do not intersect U. Then P_F is a prime ideal and the ideal $P_F H_n$ is maximal in H_n. In this way we can construct an abundance of maximal ideals in H_n when $n \geq 2$. For example we can choose $F_v(t) = v + s_1(t)$ with $v \in k$. It is clear that the maximal ideals $P_{F_v} H_n$ for different v are all different.

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