

# NORMS ON RINGS AND THE HILBERT SCHEME OF POINTS ON THE LINE

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## Abstract

We generalize the algebraic results of Laksov and Skjelnes (2001) and Skjelnes (2002), and obtain easy and transparent proofs of the representability of the Hilbert functor of points on the affine scheme whose coordinate ring is any localization of the polynomial ring in one variable over an arbitrary base ring. The coordinate ring of the Hilbert scheme is determined. We also make explicit the relation between our methods and the beautiful treatment of the Hilbert scheme of curves via norms, indicated by Grothendieck (1995), and performed by Deligne (1973).

## 0. Introduction

In a previous article we proved the existence of an affine Hilbert scheme that parametrizes finite closed subschemes of length  $n$  of the affine scheme whose coordinate ring is the local ring of the affine line at the origin, and we gave an explicit description of the Hilbert scheme. The results on the representation and structure of the Hilbert scheme were generalized in [17] to the case of finite length subschemes of the affine scheme whose coordinate ring is any localization of the polynomial ring  $k[x]$  in a variable  $x$  over any base ring  $k$ . Instead of generalizing the special algebraic tools used in [10], the article [17] used the Spectral Mapping Theorem. Further explorations of the Spectral Mapping Theorem led to a short and simple proof of a generalized version of this result, valid for norms on algebras; see [12]. In the present article we generalize the algebraic results of [10], [17]. Our point of view as well as the algebraic techniques that we use are completely different from those of the articles [10], [17]. The methods are more general and allow us to obtain shorter and more transparent proofs of the results of these articles and to minimize the role of norms, and the use of the Generalized Spectral Mapping Theorem of [12].

We also give the precise relation between our work and the beautiful approach to the Hilbert scheme of curves via the theory of norms on algebras that was indicated by Grothendieck [5] and developed by Deligne [2, §5.5, pp. 120–124, §6.3, pp. 180–192]. Grothendieck and Deligne use their theory of norms to a morphism of finite type of locally noetherian schemes  $X \rightarrow S$ ,

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and obtain a natural transformation from the Hilbert functor  $\mathcal{H}ilb_{X/S}^n$  to the point functor of the  $n$ th symmetric product  $\text{Sym}_S^n(X)$ , when this product exists as a scheme. The norm was also constructed by Iversen [9] using coordinates, and used to study the same map. When  $X \rightarrow S$  is a family of smooth curves the Hilbert functor  $\mathcal{H}ilb_{X/S}^n$  is the same as the functor  $\text{Div}_{X/S}^n$  of relative divisors of degree  $n$  of  $X$  over  $S$ , and when  $X \rightarrow S$  is in addition quasi-projective, they prove that the morphism  $\text{Div}_{X/S}^n \rightarrow \text{Sym}_S^n(X)$  is an isomorphism [2, Proposition 6.3.9, p. 186]. In this article we use our algebraic results to construct explicitly an isomorphism  $\text{Hilb}_{X/S}^n \rightarrow \text{Sym}_S^n(X)$  when  $X = \text{Spec}(U^{-1}k[x])$  and  $S = \text{Spec}(k)$ , for any ring  $k$  and any multiplicative subset  $U$  of the polynomial ring  $k[x]$  in the variable  $x$  over  $k$ . Hence we generalize the result of Deligne and Grothendieck, and we make explicit their result when  $X$  is an open subscheme of  $\text{Spec}(k[x])$ . In particular we generalize to the important cases when  $X$  is the spectrum of the local ring  $k[x]_{(x)}$ , and when  $X$  is the spectrum of the function field  $k(x)$ .

One of the most interesting features of the results of [17, 10] is that they show that one should take extreme care in studying only the rational points of Hilbert schemes. Indeed, the Hilbert schemes that are described in [17, 10] are generally of high dimension but have extremely few, and sometimes no, rational points. For example will the Hilbert scheme parametrizing finite closed subschemes of length  $n$  of the scheme  $\text{Spec}(k[x]_{(x)})$ , when  $k$  is a field, have dimension  $n$ , but only one rational point, and for  $\text{Spec}(k(x))$  it will have dimension  $n - 1$  and no rational points. In contrast most of the results on the structure of the Hilbert schemes found in the literature concern the rational points of Hilbert schemes parametrizing subschemes of length  $n$  in a projective scheme  $X$  over a field  $K$ . To get an idea of the extremely complicated phenomena that can occur, it is natural to focus on the subschemes concentrated at a single rational point  $p$  of  $X$ . Such schemes are most naturally parametrized by the fibre over the cycle  $np$  of the Hilbert–Chow morphism  $\mathcal{H}ilb_{X/S}^n \rightarrow \text{Chow}_{X/S}^n$  from the Hilbert scheme parametrizing subschemes of length  $n$  in  $X$  to the Chow scheme parametrizing cycles of dimension zero and degree  $n$  in  $X$  [4]. The functor of this fibre, although representable, appears to be very hard to describe. It is therefore habitual to consider related functors that describe colength  $n$  ideals of the local ring  $\mathcal{O}_{X,p}$  of  $X$  at  $p$ , and which have the same rational points as the fibre of the Hilbert–Chow morphism. The case which is mostly studied is when  $p$  is a smooth point on a surface; see [6–8, 13] for references.

Let  $k$  be a ring and  $A$  a  $k$ -algebra, and let  $X = \text{Spec}(A)$  and  $S = \text{Spec}(k)$ . The Hilbert functor [5, p. 274] of finite closed subschemes of length  $n$  of  $X$  assigns to an  $S$ -scheme  $T$  the set

$$\mathcal{H}ilb_{X/S}^n(T) := \{\text{closed subschemes } Z \text{ of } T \times_S X \text{ such that the projection map } p : Z \rightarrow T \text{ is finite and such that the } \mathcal{O}_T\text{-module } p_*(\mathcal{O}_Z) \text{ is locally free of rank } n\}.$$

There are several other functors that appear to be natural candidates for the Hilbert functor. One that is frequently used assigns to a  $k$ -scheme  $T$  the set

$$\mathcal{H}_{X/S}^n(T) := \{\text{closed subschemes } Z \text{ of } T \times_S X \text{ such that the projection map } p : Z \rightarrow T \text{ is flat and such that the fibre } p^{-1}(t) \text{ is the spectrum of a } k(t)\text{-algebra of dimension } n \text{ as a vector space, for every point } t \text{ in } T\}.$$

When  $A$  is *essentially of finite type* over  $k$ , that is, a localization of a finitely generated  $k$ -algebra, the two functors coincide; see [11, Theorem 3.5 p. 5624].

Perhaps the most frequently studied functor for the colength  $n$  ideals with support in a maximal ideal  $Q$  in a local  $k$ -algebra  $A$  associates to a  $k$ -scheme  $T$  the set

$$\mathcal{N}_{X/S}^n(T) = \{Z \in \mathcal{H}_{X/S}^n(T) \mid Z \text{ has support on } T \times_k \text{Spec}(A/Q)\}.$$

Assume that  $k$  is a field. Then the three functors mentioned above have the same rational points. However, the functor  $\mathcal{N}_{X/S}^n$  is not representable by a noetherian scheme even when  $A$  is the local ring of a regular point on a variety of positive dimension [16].

Assume that  $A$  is a local  $k$ -algebra with maximal ideal  $Q$ . When  $B = A/Q^N$  and  $Y = \text{Spec}(B)$  then  $\mathcal{H}_{Y/S}^n$  is a subfunctor of  $\mathcal{N}_{X/S}^n$ . The functor  $\mathcal{H}_{Y/S}^n$  can be shown to be representable for a wide class of local rings  $A$ , but depends heavily on the power  $N$  of the maximal ideal [16].

In the literature several other functors are used to parametrize the ideals of colength  $n$  in a local ring  $A$ , most notably  $\mathcal{H}_{\text{Spec}(\hat{A})/\text{Spec}(k)}^n$ , where  $\hat{A}$  is the  $Q$ -adic completion of  $A$ . Although many results are known about the  $k$ -rational points, very little is known about the representability, even when  $\hat{A} = k[[x]]$  is the formal power series ring in one variable.

The above remarks illustrate that the  $k$ -rational points reveal very little about the representability of the Hilbert functors, or about the representing scheme when the functor is representable, even when the schemes are defined over an algebraically closed field. Particular examples when the local ring  $A$  is equal to  $k[x]_{(x)}$  can be found in [10, 16]. Most authors are, however, only interested in the rational points and it is not always clear what functor they consider. Hence it is unclear what the parameter scheme is, or whether the parameter scheme exists at all.

### 1. The basic algebraic results

In this section we shall prove the results from algebra that are needed in our study of the Hilbert scheme of ideals in the localizations of a polynomial ring in one variable. We generalize the results of [10, 17]. The point of view, as well as the methods, are completely different from those of [10, 17].

Throughout we shall work with commutative algebras over a fixed (commutative) base ring  $k$ . We fix a multiplicative subset  $U$  of the polynomial ring  $k[x]$  in one variable over  $k$ .

**LEMMA 1.1** *Let  $A$  be a commutative ring, let  $S$  be a multiplicative subset of  $A$ , and let  $I$  be an ideal of  $S^{-1}A$ . Assume that the quotient  $B := S^{-1}A/I$  is integral over  $A$ . Then the natural map  $A \rightarrow B$  is surjective.*

*Proof.* It suffices to show, for every element  $s$  in  $S$ , that the inverse of the image of  $s$  in  $B$ , denoted by  $1/s$ , is in the image of  $A \rightarrow B$ . Since  $B$  is integral over  $A$  we have an equation  $(1/s)^n + a_1(1/s)^{n-1} + \dots + a_n = 0$  in  $B$  with  $a_1, a_2, \dots, a_n$  in  $A$ . By multiplying the equation by  $s^{n-1}$  we see that  $1/s$  is in the image of  $A \rightarrow B$ .

**DEFINITION 1.2** Let  $Q$  be a  $k[x]$ -algebra such that  $Q$  as a  $k$ -module is locally free of rank  $n$ . Then the *characteristic polynomial* of  $Q$  is the characteristic polynomial of multiplication by  $x$  on  $Q$ ; it is denoted  $P_Q$ . By the Cayley–Hamilton theorem, the polynomial  $P_Q$  is in the kernel of the structure map  $k[x] \rightarrow Q$ . In particular, the structure map induces a map of  $k$ -algebras  $k[x]/(P_Q) \rightarrow Q$ .

**THEOREM 1.3** *Let  $Q$  be a quotient of  $U^{-1}k[x]$  such that  $Q$  as a  $k$ -module is locally free of rank  $n$ . Then the two maps of  $k$ -algebras*

$$k[x]/(P_Q) \xrightarrow{\varphi} U^{-1}k[x]/(P_Q) \xrightarrow{\psi} Q,$$

*of which the first is the localization map and the second is the induced map of Definition 1.3, are isomorphisms.*

*In particular we have that  $U^{-1}k[x]/(P_Q)$  and  $Q$  are free  $k$ -modules of rank  $n$  with a basis given by the images of  $1, x, \dots, x^{n-1}$ .*

*Proof.* It follows from Lemma 1.1, applied with  $A := k[x]/(P_Q)$ , that  $\psi\varphi$  is surjective. Since both the source and target of  $\psi\varphi$  are locally free of rank  $n$  we have that  $\psi\varphi$  is an isomorphism. For each  $F$  in  $U$ , the image under  $\psi\varphi$  of the class of  $F$  in  $k[x]/(P_Q)$  is invertible in  $Q$ . Consequently the class of  $F$  in  $k[x]/(P_Q)$  is invertible. It follows that  $\varphi$  is an isomorphism, and consequently that  $\psi$  is an isomorphism.

**COROLLARY 1.4** *The correspondence of Definition 1.2, under which a  $k[x]$ -algebra  $Q$  is mapped to the characteristic polynomial  $P_Q$ , defines a bijection between the set of  $k$ -algebra quotients  $U^{-1}k[x] \rightarrow Q$  such that  $Q$  is a locally free  $k$ -module of rank  $n$  and the set of monic polynomials  $P$  in  $k[x]$  of degree  $n$  such that, for every  $F$  in  $U$ , the class of  $F$  in  $k[x]/(P)$  is invertible.*

*Proof.* We shall give the inverse to the correspondence of Definition 1.2. For each monic polynomial  $P$  in  $k[x]$  of degree  $n$  such that for all polynomials  $F$  in  $U$  the class of  $F$  in  $k[x]/(P)$  is invertible, the map  $k[x]/(P) \rightarrow U^{-1}k[x]/(P)$  is an isomorphism. Hence we have a quotient map  $U^{-1}k[x] \rightarrow U^{-1}k[x]/(P)$  to a free  $k$ -module of rank  $n$ . It follows from the theorem that we have obtained the inverse to the correspondence of Definition 1.3.

## 2. Representation of the Hilbert functor on algebras

We shall in this section show that the Hilbert functor is representable by an affine scheme and give an explicit description of the coordinate ring of the Hilbert scheme.

Let  $k[x_1, x_2, \dots, x_n]$  be the polynomial ring in  $n$  independent variables  $x_1, x_2, \dots, x_n$  over  $k$  and let  $k[s_1, s_2, \dots, s_n]$  be the subalgebra generated by the elementary symmetric functions  $s_1, s_2, \dots, s_n$  in  $x_1, x_2, \dots, x_n$ . Let

$$U_n := \{F(x_1)F(x_2) \cdots F(x_n) : F \in U\}.$$

Then  $U_n$  is a multiplicative subset of  $k[s_1, s_2, \dots, s_n]$ .

Let  $R$  be a ring and  $P$  a monic polynomial of degree  $n$  in the polynomial ring  $R[t]$  in the variable  $t$  over  $R$ . For every polynomial  $F$  in  $R[t]$  we denote by  $N_P(F)$  the determinant of the endomorphism on  $R[t]/(P)$  given by multiplication by the class of  $F$ . In [12] there are given several descriptions of the map  $N_P : R[t] \rightarrow R$  (see in particular the *Spectral Mapping Theorem*, [12, Corollary (7.2), p. 356]). Clearly we have that the class of  $F$  in  $R[t]/(P)$  is invertible if and only if  $N_P(F)$  is invertible in  $R$ . Moreover, we have that  $P$  is the characteristic polynomial of the endomorphism of  $R[t]/(P)$  given by multiplication by the class of  $t$ .

Consider in  $k[s_1, s_2, \dots, s_n][t]$  the following polynomial:

$$P_n(t) = \prod_{i=1}^n (t - x_i) = t^n - s_1 t^{n-1} + \dots + (-1)^n s_n. \tag{2.0.1}$$

As the  $s_1, \dots, s_n$  are algebraically independent over  $k$ , the polynomial  $P_n$  is the *universal polynomial* over  $k$ . It follows from the Spectral Mapping Theorem of [12, Corollary (7.2), p. 356] that the norm  $N_{P_n} : k[s_1, s_2, \dots, s_n][t] \rightarrow k[s_1, s_2, \dots, s_n]$  is given by

$$N_{P_n}(F) = \prod_{i=1}^n F(x_i).$$

In particular we have that  $N_{P_n}(U) = U_n$ .

### 2.1. Representation of the Hilbert functor

For each  $k$ -algebra  $A$  we denote by  $\mathcal{H}ilb_{A/k}^n$  the Hilbert functor on  $k$ -algebras that we naturally obtain from the functor  $\mathcal{H}ilb_{\text{Spec}(A)/\text{Spec}(k)}^n$  of the Introduction. Thus, for a  $k$ -algebra  $R$  we have that

$$\mathcal{H}ilb_{A/k}^n(R) := \{\text{ideals } I \text{ in } R \otimes_k A \text{ such that the } R\text{-module } (R \otimes_k A)/I \text{ is locally free of rank } n\}.$$

It follows from Corollary 1.4 and the discussion above that  $\mathcal{H}ilb_{U^{-1}k[x]/k}^n(R)$  corresponds to the set of all monic polynomials  $P$  of degree  $n$  in  $R[t]$  such that  $N_P(F)$  is invertible in  $R$  for all  $F$  in  $U$ . A monic polynomial  $P$  of degree  $n$  in  $R[t]$  is uniquely given by the map of  $k$ -algebras  $\varphi : k[s_1, s_2, \dots, s_n] \rightarrow R$  under which the coefficients of  $P_n(t)$  are mapped to the corresponding coefficients of  $P(t)$ . Clearly, for any polynomial  $F$  in  $k[s_1, s_2, \dots, s_n][t]$  we have that the image of  $N_{P_n}(F)$  by  $\varphi$  is  $N_P(F^\varphi)$ , where  $F^\varphi$  is obtained by applying  $\varphi$ ; to the coefficients of  $F$ . Consequently we have that  $N_P(F^\varphi)$  is invertible in  $R$  for all  $F$  in  $U$  if and only if  $k[s_1, s_2, \dots, s_n] \rightarrow R$  maps  $U_n$  into the set of invertible elements in  $R$ , that is, if and only if  $k[s_1, s_2, \dots, s_n] \rightarrow R$  extends to a  $k$ -algebra map

$$U_n^{-1}k[s_1, s_2, \dots, s_n] \rightarrow R.$$

We consequently have a bijection

$$\varepsilon : \mathcal{H}ilb_{U^{-1}k[x]/k}^n(R) \rightarrow \text{Hom}_{k\text{-alg}}(U_n^{-1}k[s_1, s_2, \dots, s_n], R) \tag{2.1.1}$$

which clearly is functorial in  $R$ . We rephrase the existence of the functorial bijection  $\varepsilon$  as the following theorem.

**THEOREM 2.2** *The  $k$ -algebra  $U_n^{-1}k[s_1, s_2, \dots, s_n]$  represents the Hilbert functor  $\mathcal{H}ilb_{U^{-1}k[x]/k}^n$ . The ideal generated by  $P_n(x)$  in  $U_n^{-1}k[s_1, s_2, \dots, s_n] \otimes_k U^{-1}k[x]$  is the universal element.*

**COROLLARY 2.3** *The Hilbert functor  $\mathcal{H}ilb_{\text{Spec}(U^{-1}k[x])/ \text{Spec}(k)}^n$ , on the category of  $\text{Spec}(k)$ -schemes, is represented by the affine scheme  $\text{Spec}(U_n^{-1}k[s_1, s_2, \dots, s_n])$ .*

*Proof.* The assertion follows from the theorem because the Hilbert functor on schemes is a sheaf for the Zariski topology.

### 2.4. Localization

Let  $V_n$  be the multiplicative subset of  $k[x_1, x_2, \dots, x_n]$  consisting of all products  $F_1(x_1)F_2(x_2)\cdots F_n(x_n)$  with  $F_i$  in  $U$ . We have that  $U_n \subseteq V_n$ , and from the equation  $F_1(x_1)F_2(x_2)\cdots F_n(x_n)\prod_{i \neq j} F_i(x_j) = \prod_{i=1}^n \prod_{j=1}^n F_j(x_i)$  it follows that for any  $v \in V_n$  there exists a polynomial  $w$  such that  $vw \in U_n$ . Consequently, the canonical  $k$ -algebra homomorphism

$$U_n^{-1}k[x_1, x_2, \dots, x_n] \rightarrow V_n^{-1}k[x_1, x_2, \dots, x_n] \quad (2.4.1)$$

is an isomorphism.

We have the obvious identifications

$$(k[x])^{\otimes_{k^n}} = k[x_1, x_2, \dots, x_n] \quad \text{and} \quad (U^{-1}k[x])^{\otimes_{k^n}} = V_n^{-1}k[x_1, x_2, \dots, x_n].$$

Hence we obtain an identification

$$(U^{-1}k[x])^{\otimes_{k^n}} = U_n^{-1}k[x_1, x_2, \dots, x_n]. \quad (2.4.2)$$

The symmetric group  $\mathfrak{S}_n$  operates on the left of (2.4.2) by permutation of the factors of tensor product. The subalgebra of invariants is the algebra  $TS_k^n(U^{-1}k[x])$  of symmetric tensors. Under the identification (2.4.2), the action of  $\mathfrak{S}_n$  on the right is given by permutation of the variables  $x_1, x_2, \dots, x_n$ . The ring of invariants of the right-hand side is  $U_n^{-1}k[s_1, s_2, \dots, s_n]$ . Indeed, if an element  $w/u$  in  $U_n^{-1}k[x_1, x_2, \dots, x_n]$  is invariant under  $\mathfrak{S}_n$ , then there is, for every element  $\sigma$  in  $\mathfrak{S}_n$ , an element  $u_\sigma$  in  $U_n$  such that  $u_\sigma w = u_\sigma \sigma w$ . Let  $v := \prod_{\sigma \in \mathfrak{S}_n} u_\sigma$ . Then  $v \in U_n$ , and  $vw$  is invariant under  $\mathfrak{S}_n$  and thus is in  $k[s_1, s_2, \dots, s_n]$ . Hence  $w/u = vw/vu$  is in  $U_n^{-1}k[s_1, s_2, \dots, s_n]$  as we wanted to prove. Therefore, by taking invariants under  $\mathfrak{S}_n$  of each side of the identity (2.5.2), we obtain the identification

$$TS_k^n(U^{-1}k[x]) = U_n^{-1}k[s_1, s_2, \dots, s_n]. \quad (2.4.3)$$

### 3. The Grothendieck-Deligne approach via norms

We sketch the approach of Grothendieck and Deligne to the representation of the Hilbert scheme of families of smooth curves, and show that the norm they construct, with the appropriate identifications, gives the same map as the map  $\varepsilon$  of Section 2.

Let  $A$  be a  $k$ -algebra, and let  $TS_k^n A$  be the  $k$ -algebra of symmetric tensors, that is, the subalgebra of  $A^{\otimes_{k^n}}$  invariant under the symmetric group  $\mathfrak{S}_n$  [14, III §5, p. 251; 1, §2, AIV.40]. Grothendieck and Deligne defined a natural map, functorial in the  $k$ -algebra  $R$  [6, 6.3.8, p. 186],

$$\delta : \mathcal{Hilb}_{A/k}^n(R) \rightarrow \text{Hom}_{k\text{-alg}}(TS_k^n A, R). \quad (3.0.1)$$

They proved that the map is a bijection when  $\text{Spec}(A) \rightarrow \text{Spec}(k)$  is smooth of relative dimension 1, and that for a quasi-projective smooth family of curves  $X \rightarrow S$  the maps in (3.0.1) globalize and give a representation of the Hilbert functor [2, Proposition 6.3.9, p. 186]

$$\text{Hilb}_{X/S}^n = \text{Sym}_S^n(X).$$

We shall below give the construction, via norms, of the map (3.0.1), and we show that when  $A = U^{-1}k[x]$  the map  $\delta$  is equal to the bijection  $\varepsilon$  described in section 2.1.

3.1. *The norm construction*

To define  $\delta$  we note that an element in the source  $\mathcal{H}ilb_{A/k}^n(R)$  is given by a quotient,

$$R \otimes_k A \rightarrow Q,$$

where  $Q$  as an  $R$ -module is a locally free of rank  $n$ . The determinant,

$$\det_Q : \text{End}_R(Q) \rightarrow R,$$

defines in a natural way a *multiplicative homogeneous polynomial law, of degree  $n$* , also called a *norm of degree  $n$* , from the  $R$ -algebra  $E = \text{End}_R(Q)$  to  $R$  (see [14, I §2, p. 219, §8, p. 266; 15, p. 870 1, Exercise 9, AIV.89–90] for polynomial laws and [3, 2.5 p. 14; 12, §5, pp. 352–353] for norms). Let  $\Gamma^n E$  be the degree- $n$  part of the algebra of divided powers of the  $R$ -module  $E$ . Then the norm defined by  $\det_Q$  determines a unique  $R$ -module map  $\Gamma^n E \rightarrow R$  that composed with  $\gamma^n : E \rightarrow \Gamma^n E$  is equal to  $\det_Q$  [14, IV.1, Proposition IV.1, p. 265, IV §2, Théorème IV.1, p. 266, 1; Exercise 10(a), (b), AIV.90]. For each element  $u$  in  $E$  the  $n$ th divided power  $\gamma^n(u)$  in  $\Gamma^n E$  is mapped to  $\det u$  in  $R$ . Since  $E$  is an  $R$ -algebra we have that  $\Gamma^n E$  is also an  $R$ -algebra. In fact the algebra  $\Gamma^n E$  is universal for norms of degree  $n$ . That is, there is a bijection between norms of degree  $n$  from  $E$  to an  $R$ -algebra  $A$ , and  $R$ -algebra homomorphisms  $\varphi : \Gamma^n E \rightarrow A$  such that  $\varphi \gamma^n$  corresponds to the norm from  $E$  to  $A$  [3, 2.4.2 p. 11; 15 Théorème p. 871].

Since  $E$  is a locally free  $R$ -module of finite rank, the natural map of  $R$ -modules,  $\Gamma^n E \rightarrow TS_R^n \circ(E)$ , under which  $\gamma^n(u)$  is mapped to  $u^{\otimes n} \in E^{\otimes n}$  for all  $u$  in  $E$ , is an isomorphism of  $R$ -modules [14, IV §5, Proposition IV.5, p. 272; 1, Exercise 8(a), AIV.89]. It is also an isomorphism of  $R$ -algebras. In fact  $\Gamma^n E$  commutes with base extension [14, III §8, Théorème III.3, p. 262], and consequently  $TS_R^n(E)$  commutes with base extension when  $E$  is locally free. It follows that the map  $u \rightarrow u^{\otimes n}$  determines a norm of degree  $n$  from  $E$  to  $TS_R^n(E)$ . Hence we obtain, from the quotient  $Q$ , a natural  $R$ -algebra map  $TS_R^n(E) \rightarrow R$  which maps  $u^{\otimes n}$  to  $\det u$ .

The quotient  $Q$  is also an  $A$ -module. Thus we have a  $k$ -algebra homomorphism  $A \rightarrow E$  under which  $a \in A$  is mapped to the multiplication  $a_Q$ , by  $a$  on  $Q$ , and consequently there is a  $k$ -algebra homomorphism  $TS_k^n A \rightarrow TS_k^n E$ . We also have a natural  $k$ -algebra homomorphism  $TS_k^n E \rightarrow TS_R^n E$  coming from the  $k$ -algebra structure on  $R$ . Composing the three maps  $TS_k^n A \rightarrow TS_k^n E$ ,  $TS_k^n E \rightarrow TS_R^n E$ , and  $TS_R^n E \rightarrow R$ , we obtain a  $k$ -algebra homomorphism,

$$\delta_Q : TS_k^n A \rightarrow R. \tag{3.1.1}$$

If  $a \in A$ , then

$$\delta_Q(a^{\otimes n}) = \det a_Q. \tag{3.1.2}$$

Hence we have constructed a map  $\delta \mathcal{H}ilb_{A/k}^n(R) \rightarrow \text{Hom}_{k\text{-alg}}(TS_k^n A, R)$  that maps  $Q$  to  $\delta_Q$ . The map  $\delta$  is easily seen to be functorial with respect to the  $k$ -algebra  $R$ . In addition, the  $k$ -algebra homomorphism  $\delta_Q$  commutes with base change in the following sense. Let  $k \rightarrow k'$  be a map of rings. Form the  $k'$ -algebras  $A' := k' \otimes_k A$  and  $R' = k' \otimes_k R$ , and set  $Q' := R' \otimes_R Q = k' \otimes_k Q$ . Then, from the given quotient map  $R \otimes_k A \rightarrow Q$  we obtain by base change the quotient map



$R' \otimes_{k'} A' \rightarrow Q'$ , and hence a map of  $k'$ -algebras  $\delta_Q TS_k^n A' \rightarrow R'$ . In this notation, the composition of the map  $\delta_Q$  and the canonical map  $k' \otimes_k TS_k^n A \rightarrow TS_{k'}^n A'$  is equal to  $1 \otimes_k \delta_Q$ .

### 3.2. Comparison with the method of Grothendieck and Deligne

We shall prove, when  $A := U^{-1}k[x]$ , that  $\varepsilon : \mathcal{H}ilb_{A/k}^n(R) \rightarrow \text{Hom}_{k\text{-alg}}(U_n^{-1}k[s_1, s_2, \dots, s_n], R)$  of section 2.2 is the same as the map  $\delta : \mathcal{H}ilb_{A/k}^n(R) \rightarrow \text{Hom}_{k\text{-alg}}(TS_k^n A, R)$  of section 3.1 under the identification  $TS_k^n(U^{-1}k[x]) = U_n^{-1}k[s_1, s_2, \dots, s_n]$  of (2.4.3).

An element in  $\mathcal{H}ilb_{U^{-1}k[x]/k}^n(R)$  is given by a quotient  $R \otimes_k U^{-1}k[x] \rightarrow Q$  as in section 3.1. The images  $\delta_Q$  and  $\varepsilon_Q$  of the quotient  $Q$  under  $\delta$  and  $\varepsilon$  are  $k$ -algebra maps  $U_n^{-1}k[s_1, s_2, \dots, s_n] \rightarrow R$  and consequently determined by their values on  $s_1, \dots, s_n$ . Equivalently,  $\delta_Q$  and  $\varepsilon_Q$  are determined by the image of the polynomial  $P_n(t)$  (2.0.1) under the induced map  $k[s_1, s_2, \dots, s_n][t] \rightarrow R[t]$  of polynomial rings.

As noted in the construction of the map  $\varepsilon_Q$  in section 2.1, the polynomial  $P_n(t)$  is mapped by  $k[s_1, \dots, s_n][t] \rightarrow R[t]$  to the characteristic polynomial of  $x_Q$ : the endomorphism on  $Q$  given as multiplication by the class of the variable  $x$ .

To determine the image of  $P_n(t)$  under the map  $k[s_1, s_2, \dots, s_n][t] \rightarrow R[t]$  induced by  $\delta_Q$  we apply the observation at the end of section 3.1 with  $k' := k[t]$ . Then, under the canonical map  $k' \otimes_k TS_k^n A \rightarrow TS_{k'}^n A'$ , the polynomial  $P_n(t)$  is mapped to  $(t - x)^{\otimes n}$ . Therefore it follows from (3.1.2) that under the induced map, the polynomial  $P_n(t)$  is mapped to the characteristic polynomial of  $x_Q$ .

Hence we have proved that  $\delta_Q = \varepsilon_Q$ , and consequently that  $\delta = \varepsilon$ .

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