ON THE REPRESENTABILITY OF $\mathcal{H}ilb^n k[x]_{(x)}$

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Abstract. Let $k[x]_{(x)}$ be the polynomial ring $k[x]$ localized in the maximal ideal $(x) \subset k[x]$. We study the Hilbert functor parameterizing ideals of colength n in this ring *having support at the origin*. The main result of this article is that this functor is not representable. We also give a complete description of the functor as a limit of representable functors.

1. Introduction.

Let k be a field. Let R be a local noetherian k-algebra with maximal ideal P. The Hilbert functor of n-points on $\mathrm{Spec}(R)$, denoted by $\mathcal{H}ilb^n_R$, is determined by sending a scheme T to the set

$$\mathcal{H}ilb^n_R(T) = \left\{ \begin{array}{l} \text{Closed subschemes } Z \subseteq T \times_k \mathrm{Spec}(R) \text{ such that} \\ \text{the projection } Z \to T \text{ is flat, and the } \kappa(y)\text{-vector} \\ \text{space of global sections of the fibre } Z_y \text{ has} \\ \text{dimension } n \text{ for all points } y \in T. \end{array} \right\}.$$

We let $\mathcal{H}ilb^n R(T) \subseteq \mathcal{H}ilb^n_R(T)$ be the set of T-valued points Z of $\mathcal{H}ilb^n_R$ such that $Z_{red} \subseteq T \times_k \mathrm{Spec}(R/P)$. Here Z_{red} is the reduced scheme associated to Z. The assignment sending a k-scheme T to the set $\mathcal{H}ilb^n R(T)$ determines a contravariant functor from the category of noetherian k-schemes to sets. The functor $\mathcal{H}ilb^n R$ is *different* from the Hilbert functor $\mathcal{H}ilb^n_R$.

The functor $\mathcal{H}ilb^n R$ with $R = \mathbf{C}\{x, y\}$, the ring of convergent power series in two variables, was introduced by J. Briançon in [1], and its set of $\mathbf{C}$-rational points were described. The motivation behind the present paper was to understand the universal properties of $\mathcal{H}ilb^n \mathbf{C}\{x, y\}$.

Instead of analytic spaces, as considered in [1], we work in the category of noetherian k-schemes. Primarily our interest was in the representability of the functor $\mathcal{H}ilb^n k[[x, y]]$. However, we realized that the problems we faced were present for $\mathcal{H}ilb^n k[x]_{(x)}$, where $k[x]_{(x)}$ is the local ring of the line at the origin. To illustrate the difficulties of the representability of $\mathcal{H}ilb^n k[[x, y]]$ we will in this paper focus on $\mathcal{H}ilb^n k[x]_{(x)}$, the functor parameterizing colength n ideals in $k[x]_{(x)}$, having support in (x).

The scheme $\mathrm{Spec}(k[x]/(x^n))$ is the only closed subscheme of $\mathrm{Spec}(k[x]_{(x)})$ whose coordinate ring is of dimension n as a k-vector space. It follows that the functor $\mathcal{H}ilb^n k[x]_{(x)}$ has only one k-valued point. Thus in a naive geometric sense the functor $\mathcal{H}ilb^n k[x]_{(x)}$ is trivial. We shall see, however, that the functor $\mathcal{H}ilb^n k[x]_{(x)}$ is not

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representable! In fact we show in Theorem (4.8) that $\mathcal{H}ilb^n R$ is not representable when R is the local ring of a regular point on a variety.

In addition to Theorem (4.8) which is our main result, we show in Theorem (5.5) that the non-representable functor $\mathcal{H}ilb^n k[x]_{(x)}$ is pro-represented by $k[[s_1, \dots, s_n]]$, the formal power series ring in n-variables. In Theorem (6.8) we show that there exist a natural filtration of $\mathcal{H}ilb^n k[x]_{(x)}$ by representable subfunctors $\{\mathcal{H}^{n,m}\}_{m \geq 0}$, where $\mathcal{H}^{n,m}$ is a closed subfunctor of $\mathcal{H}^{n,m+1}$.

The three theorems (4.8), (5.5) and (6.8) completely describe $\mathcal{H}ilb^n k[x]_{(x)}$. The three mentioned results are more or less explicit applications of Theorem (3.5), which describes the set of elements in $\mathcal{H}ilb^n k[x]_{(x)}(\mathrm{Spec}(A))$ for arbitrary k-algebras A.

The paper is organized as follows: In Section (2) we recall some results from [3]. In Section (3) we establish Theorem (3.5). The sections (4), (5) and (6) are applications of Theorem (3.5). In Section (4) we show that $\mathcal{H}ilb^n k[x]_{(x)}$ is not representable. We pro-represent $\mathcal{H}ilb^n k[x]_{(x)}$ in Section (5). We give a filtration of $\mathcal{H}ilb^n k[x]_{(x)}$ by representable subfunctors in Section (6).

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2. Preliminaries.

2.1. Notation. Let k be a field. Let $k[x]$ be the ring of polynomials in one variable over k. The polynomials $f(x)$ in $k[x]$ such that $f(0) \neq 0$ form a multiplicatively closed subset S in $k[x]$. We write the fraction ring $k[x]_S = k[x]_{(x)}$. For every k-algebra A we write $A \otimes_k k[x] = A[x]$. The localization of the $k[x]$-algebra $A[x]$ in the multiplicatively closed set $S \subset k[x]$ is $A \otimes_k k[x]_{(x)}$. If I is an ideal in a ring A we let $\mathfrak{R}(I)$ denote its radical, and if P is a prime ideal we let $\kappa(P) = A_P/PA_P$ be its residue field.

Lemma 2.2. *Let A be a k-algebra. Let $I \subseteq A \otimes_k k[x]_{(x)}$ be an ideal such that $A \otimes_k k[x]_{(x)}/I$ is a free A-module of rank n. Then the following two assertions hold:*

(1) *The classes of $1, x, \dots, x^{n-1}$ form an A-basis for $A \otimes_k k[x]_{(x)}/I$.*
(2) *The ideal I is generated by a unique $F(x) = x^n - u_1 x^{n-1} + \cdots + (-1)^n u_n$ in $A[x]$.*

Proof. See [3], Lemma (3.2) for a proof of the first assertion. The second assertion follows from [3], Theorem (3.3). □

Proposition 2.3. *Let A be a k-algebra. Let $I \subseteq A \otimes_k k[x]_{(x)}$ be an ideal with residue ring $M = A \otimes_k k[x]_{(x)}/I$. Assume that*

(1) *There is an inclusion of ideals $(x) \subseteq \mathfrak{R}(I)$ in $A \otimes_k k[x]_{(x)}$.*
(2) *The A-module $M = A \otimes_k k[x]_{(x)}/I$ is flat.*
(3) *For every prime ideal P in A we have that $M \otimes_A \kappa(P)$ is of dimension n as a $\kappa(P)$-vector space.*

Then M is a free A-module of rank n.

Proof. We first show that $M \otimes_A A_P$ is free for every prime ideal P in A. Thus we assume that A is a local k-algebra. Assumption (1) is equivalent to the existence of an integer N such that we have an inclusion of ideals $(x^N) \subseteq I$ in $A \otimes_k k[x]_{(x)}$. Consequently we have a surjection

$$A \otimes_k k[x]_{(x)}/(x^N) \to M = A \otimes_k k[x]_{(x)}/I. \tag{2.3.1}$$

We have that $A \otimes_k k[x]_{(x)}/(x^N) = A[x]/(x^N)$. It follows from the surjection (2.3.1) that M is generated by the classes of $1, x, \dots, x^{N-1}$. In particular M is finitely generated. A flat and finitely generated module over a local ring is free, see [4] Theorem (7.10). Hence by Assumption (2) we have that M is a free A-module. By Assumption (3) we have that the rank of M is n.

Thus we have proven that $M \otimes_A A_P$ is free of rank n for every prime ideal P in A. It then follows by Assertion (1) of Lemma (2.2) that $M \otimes_A A_P$ has a basis given by the classes of $1, x, \dots, x^{n-1}$. Since the classes of $1, x, \dots, x^{n-1}$ form a basis for $M \otimes_A A_P$ for every prime ideal P of A, it follows that $1, x, \dots, x^{n-1}$ form a basis for M. $\square$

Theorem 2.4. *Let A be a k-algebra and let $F(x)$ in $A[x]$ be a polynomial where $F(x) = x^n - u_1x^{n-1} + \dots + (-1)^n u_n$. The following three assertions are equivalent.*

(1) *For all maximal ideals P of A with residue map $\varphi : A \to A/P$, the roots of $F^\varphi(x) = x^n - \varphi(u_1)x^{n-1} + \dots + (-1)^n\varphi(u_n)$ in the algebraic closure of A/P are zero or transcendental over k.*
(2) *The ring $A \otimes_k k[x]_{(x)}/(F(x))$ is canonically isomorphic to $A[x]/(F(x))$.*
(3) *The A-module $A \otimes_k k[x]_{(x)}/(F(x))$ is free of rank n with a basis consisting of the classes of $1, x, \dots, x^{n-1}$.*

Proof. See [3], Assertions (1), (4) and (5) of Theorem (2.3). $\square$

Corollary 2.5. *Let $F(x) = x^n - u_1x^{n-1} + \dots + (-1)^n u_n$ be an element of $A[x]$. Assume that the coefficients $u_1, \dots, u_n$ are in the Jacobson radical of A. Then we have that $M = A \otimes_k k[x]_{(x)}/(F(x))$ is canonically isomorphic to $A[x]/(F(x))$. In particular we have a canonical isomorphism $M = A[x]/(F(x))$ when A is local and the coefficients $u_1, \dots, u_n$ of $F(x)$ are in the maximal ideal of A.*

Proof. Let P be a maximal ideal, and let $\varphi : A \to A/P$ be the residue map. We have that $F^\varphi(x) = x^n$ since the coefficients $u_1, \dots, u_n$ of $F(x)$ are in the Jacobson radical of A. Consequently the roots of $F^\varphi(x)$ are zero, and the Assertion (1) of the Theorem is satisfied. $\square$

Corollary 2.6. *Assume that $F(x)$ in $A[x]$ is such that the assertions of the Theorem are satisfied. Then an inclusion of ideals $(x^N) \subseteq (F(x))$ in $A \otimes_k k[x]_{(x)}$ is equivalent to an inclusion of ideals $(x^N) \subseteq (F(x))$ in $A[x]$.*

Proof. Obviously an inclusion of ideals in $A[x]$ extends to an inclusion of ideals in the fraction ring $A \otimes_k k[x]_{(x)}$. Consequently it suffices to show that an inclusion $(x^N) \subseteq (F(x))$ in $A \otimes_k k[x]_{(x)}$ gives an inclusion $(x^N) \subseteq (F(x))$ in $A[x]$. Assume that we have an inclusion of ideals $(x^N) \subseteq (F(x))$ in $A \otimes_k k[x]_{(x)}$, or equivalently a surjection

$$A \otimes_k k[x]_{(x)}/(x^N) \to A \otimes_k k[x]_{(x)}/(F(x)). \tag{2.6.1}$$

We have that $F(x)$ in $A[x]$ satisfies the conditions in the Theorem. Hence we have a canonical isomorphism $A \otimes_k k[x]_{(x)}/(F(x)) = A[x]/(F(x))$. Then the surjection (2.6.1) gives a surjection $A[x]/(x^N) \to A[x]/(F(x))$ which is equivalent to an inclusion of ideals $(x^N) \subseteq (F(x))$ in $A[x]$. $\square$

3. Polynomials with nilpotent coefficients.

The purpose of this section is to establish Theorem (3.5). Applications of Theorem (3.5) are given in Sections (4), (5) and (6).

3.1. Set up and Notation. We will study ideals generated by monic polynomials with nilpotent coefficients. For this purpose we introduce the following terminology; Let A be a commutative ring, and let $A[t_1,\dots,t_n]$ be the polynomial ring over A in the variables $t_1,\dots,t_n$. Let $s_i(t)=s_i(t_1,\dots,t_n)$ be the i'th elementary symmetric function in the variables $t_1,\dots t_n$. The elementary symmetric functions $s_i(t)$ are homogeneous in the variables $t_1,\dots,t_n$, having degree $\deg(s_i(t))=i$. We let $A_0=A$ and consider the ring of symmetric functions $A[s_1(t),\dots,s_n(t)]=\oplus_{i\geq 0}A_i$ as graded in $t_1,\dots,t_n$. For every positive integer d we have the ideal $\oplus_{i\geq d}A_i\subseteq A[s_1(t),\dots,s_n(t)]$. We denote the residue ring by

$$Q_d := A[s_1(t),\dots,s_n(t)]/\oplus_{i\geq d}A_i. \tag{3.1.1}$$

Lemma 3.2. *Let $u_1,\dots,u_n$ be nilpotent elements in a ring A. Then the homomorphism $u: A[s_1(t),\dots,s_n(t)]\to A$, determined by $u(s_i)=u_i$ for $i=1,\dots,n$, factors through Q_d for some integer d.*

Proof. The coefficients $u_1,\dots,u_n$ are nilpotent by assumption. Hence there exist integers n_i such that $u_i^{n_i}=0$ for every $i=1,\dots,n$. Let $\tau=\max\{n_i\}$, and let $d=\tau+2\tau+\dots+n\tau$. We claim that $u: A[s_1(t),\dots,s_n(t)]\to A$ maps $\oplus_{i\geq d}A_i$ to zero. It is enough to show that monomials $m(s_1(t),\dots,s_n(t))$ of degree $\geq d$ are mapped to zero. We have that $m(s_1(t),\dots,s_n(t))=s_1(t)^{e_1}s_2(t)^{e_2}\dots s_n(t)^{e_n}$ where $e_1+2e_2+\dots+ne_n=\deg(m(s_1(t),\dots,s_n(t)))$. It follows that at least one $e_j\geq\tau$, and consequently $u_j^{e_j}=0$. Thus we have that $u(s_1(t)^{e_1}s_2(t)^{e_2}\dots s_n(t)^{e_n})=u_1^{e_1}u_2^{e_2}\dots u_n^{e_n}=0$. □

3.3. Polynomials with nilpotent coefficients. For every monic polynomial $F(x)=x^n-u_1x^{n-1}+\dots+(-1)^nu_n$ in $A[x]$ we let $u_F: A[s_1(t),\dots,s_n(t)]\to A$ be the A-algebra homomorphism determined by $u_F(s_i(t))=u_i$ for $i=1,\dots,n$. Let

$$\Delta(t,x)=\prod_{i=1}^{n}(x-t_i)=x^n-s_1(t)x^{n-1}+\dots+(-1)^ns_n(t). \tag{3.3.1}$$

If $D(t,x)=D(t_1,\dots,t_n,x)$ is symmetric in the variables $t_1,\dots,t_n$, we let $D^{u_F}(x)$ in $A[x]$ be the image of $D(t,x)$ by the map $u_F\otimes 1: A[s_1(t),\dots,s_n(t)][x]\to A[x]$. In particular we have that $\Delta^{u_F}(x)=F(x)$.

For every non-negative integer p we define $d_p(t_i,x)$ in $A[t_1,\dots,t_n,x]$ by

$$d_p(t_i,x)=(x+t_i)(x^2+t_i^2)\dots(x^{2^p}+t_i^{2^p}). \tag{3.3.2}$$

It follows by induction on p that $(x-t_i)d_p(t_i,x)=x^{2^{p+1}}-t_i^{2^{p+1}}$. We let

$$D_p(t,x)=\prod_{i=1}^{n}d_p(t_i,x). \tag{3.3.3}$$

For every non-negative integer N, we let $s_i(t^N)=s_i(t_1^N,,\dots,t_n^N)$, which is a homogeneous symmetric function in the variables $t_1,\dots,t_n$. We have that the degree of $s_i(t^N)$ is $\deg(s_i(t^N)) = iN$. Both $\Delta(t,x)$ and $D_p(t,x)$ are symmetric in the variables $t_1,\dots,t_n$. Their product is

$$\begin{aligned}\Delta(t,x)D_p(t,x) &= \prod_{i=1}^{n}(x^{2^{p+1}} - t_i^{2^{p+1}})\\ &= x^{2^{p+1}n} - s_1(t^{2^{p+1}})x^{2^{p+1}(n-1)} + \cdots + (-1)s_n(t^{2^{p+1}}).\end{aligned} \tag{3.3.4}$$

Proposition 3.4. *Let $F(x) = x^n - u_1x^{n-1} + \cdots + (-1)^n u_n$ be an element of $A[x]$. Then the coefficients $u_1,\dots,u_n$ are nilpotent if and only if we have an inclusion of ideals $(x) \subseteq \mathfrak{R}(F(x))$ in $A[x]$.*

Proof. Assume that the coefficients $u_1,\dots,u_n$ of $F(x)$ are nilpotent.We must show that $x \in \mathfrak{R}(F(x))$, or equivalently that $x^N \in (F(x))$ for some integer N. By Lemma (3.2) the map $u_F : A[s_1(t),\dots,s_n(t)] \to A$ determined by $u_F(s_i(t)) = u_i$, factors through Q_d for some integer d. Let p be an integer such that $2^{p+1} \geq d$. The function $D_p(t,x)$ (3.3.3) is symmetric in the variables $t_1,\dots,t_n$. We will show that $D_p^{u_F}(x)$ in $A[x]$ is such that $F(x)D_p^{u_F}(x) = x^N$. The product $\Delta(t,x)D_p(t,x)$ is given in (3.3.4), and the degree of the symmetric functions $s_i(t^{2^{p+1}}) = i2^{p+1} \geq d$. Consequently the class of $\Delta(t,x)D_p(t,x)$ in $Q_d[x]$ equals $x^{2^{p+1}n}$. We obtain that

$$x^{2^{p+1}n} = \Delta^{u_F}(x)D_p^{u_F}(x) = F(x)D_p^{u_F}(x), \tag{3.4.1}$$

in $A[x]$. Hence we have that $(x) \subseteq \mathfrak{R}(F(x))$.

Conversely, assume that we have an inclusion of ideals $(x) \subseteq \mathfrak{R}(F(x))$ in $A[x]$. Then there exists a $G(x)$ in $A[x]$ such that $x^N = F(x)G(x)$ for some integer N. Let P be a prime ideal of A, and let $\varphi : A \to \kappa(P) = K$ be the residue map. Let $F^\varphi(x)$ and $G^\varphi(x)$ be the classes of $F(x)$ and $G(x)$, respectively, in $K[x]$. We have

$$x^N = F^\varphi(x)G^\varphi(x) = (x^n - \varphi(u_1)x^{n-1} + \cdots + (-1)^n\varphi(u_n))G^\varphi(x), \tag{3.4.2}$$

in $K[x]$. The ring $K[x]$ is a unique factorization domain, hence $\varphi(u_i) = 0$ for $i = 1,\dots,n$. Therefore the classes of u_i are zero in A/P for all prime ideals P of A. This shows that $u_1,\dots,u_n$ are nilpotent. □

Theorem 3.5. *Let A be a k-algebra, and let $I \subseteq A\otimes_k k[x]_{(x)}$ be an ideal. Write the residue ring as $M = A\otimes_k k[x]_{(x)}/I$. The following two assertions are equivalent.*

1. *M is a flat A-module such that for every prime ideal P in A we have that $M \otimes_A \kappa(P)$ is of dimension n as a $\kappa(P)$-vector space, and we have an inclusion of ideals $(x) \subseteq \mathfrak{R}(I)$ in $A \otimes_k k[x]_{(x)}$.*
2. *The ideal I is generated by an element $F(x)$ in $A[x]$, of the form $F(x) = x^n - u_1x^{n-1} + \cdots + (-1)^n u_n$, where the coefficients $u_1,\dots,u_n$ are nilpotent.*

Proof. Assume that Assertion (1) holds. By Proposition (2.3) we have that M is a free A-module of rank n. It follows from Lemma (2.2) that the ideal I is generated by a unique $F(x) = x^n - u_1x^{n-1} + \cdots + (-1)^n u_n$ in $A[x]$, and that the classes of $1, x, \dots, x^{n-1}$ form a basis for M. Consequently $F(x)$ in $A[x]$ is such that the assertions of Theorem (2.4) hold. By assumption there is an inclusion of ideals $(x) \subseteq \mathfrak{R}(F(x))$ in $A \otimes_k k[x]_{(x)}$. Equivalently $(x^N) \subseteq (F(x))$ in $A \otimes_k k[x]_{(x)}$ for some integer N. By Corollary (2.6) we get an inclusion of ideals $(x^N) \subseteq (F(x))$ in $A[x]$. It follows from Proposition (3.4) that the coefficients $u_1, \dots, u_n$ of $F(x)$ are nilpotent.

Conversely, assume that Assertion (2) holds. Since the coefficients $u_1, \dots, u_n$ of $F(x)$ are nilpotent, we get by Corollary (2.5) that $F(x)$ is such that the assertions of Theorem (2.4) are satisfied. Thus $M = A \otimes_k k[x]_{(x)}/(F(x))$ is a free A-module of rank n. In particular we have that M is a flat A-module such that $M \otimes_A \kappa(P)$ is of rank n, for every prime ideal P in A. What is left to prove is the inclusion of ideals $(x) \subseteq \mathfrak{R}(F(x))$ in $A \otimes_k k[x]_{(x)}$, which is an immediate consequence of Proposition (3.4). $\square$

4. The non-representability of $\mathcal{H}ilb^n k[x]_{(x)}$.

In this section we define for every local noetherian k-algebra R, the functor $\mathcal{H}ilb^n R$. We will show in Theorem (4.8) that the functor $\mathcal{H}ilb^n R$ is not representable when R is the local ring of a regular point on a variety.

4.1. Notation. If Z is a scheme, we let Z_{red} be the associated reduced scheme. Given a morphism of schemes $Z \to T$, we write $Z_y = Z \times_T \operatorname{Spec}(\kappa(y))$ for the fibre of this morphism over $y \in T$, where $\kappa(y)$ is the residue field of the point $y \in T$.

Lemma 4.2. *Let I and J be two ideals in a ring A. Assume that I is finitely generated. Then an inclusion $I \subseteq \mathfrak{R}(J)$ is equivalent to the existence of an integer N such that $I^N \subseteq J$.*

Proof. Let $x_1, \dots, x_m$ be a set of generators for the ideal I. Assume that we have an inclusion of ideals $I \subseteq \mathfrak{R}(J)$. It follows that there exists integers n_i such that $x_i^{n_i} \in J$, for $i = 1, \dots, m$. Thus we have that $I^N \subseteq J$, when $N \geq \sum_{i=1}^m (n_i - 1) + 1$. The converse is immediate, and we have proven the Lemma. $\square$

Lemma 4.3. *Let I be an ideal in a noetherian k-algebra R. Let T be a noetherian k-scheme. Suppose that $Z \subseteq T \times_k \operatorname{Spec}(R)$ is a closed subscheme. Then $Z_{red} \subseteq T \times_k \operatorname{Spec}(R/I)$ if and only if there exists an integer $N = N(Z)$ such that $Z \subseteq T \times_k \operatorname{Spec}(R/I^N)$.*

Proof. The scheme T is noetherian and we can find a finite affine open cover $\{U_i\}$ of T. Thus $\{U_i \times_k \operatorname{Spec}(R)\}$ is a finite affine open cover of $T \times_k \operatorname{Spec}(R)$. It follows from the finite covering of $T \times_k \operatorname{Spec}(R)$ that it is enough to prove the statement for each $U_i \times_k \operatorname{Spec}(R)$. Hence we may assume that T is affine.

Let $T = \operatorname{Spec}(A)$, and let the closed subscheme Z be given by the ideal $J \subseteq A \otimes_k R$. We write I_A for the image of the natural map $A \otimes_k I \to A \otimes_k R$. The ring R is noetherian, hence $I \subseteq R$ is finitely generated. Consequently the ideal $I_A \subseteq A \otimes_k R$ is finitely generated. It follows from Lemma (4.2) that $I_A \subseteq \mathfrak{R}(J)$ if and only if $I_A^N \subseteq J$ for some N. $\square$

Definition 4.4. Let R be a local noetherian k-algebra. Let P be the maximal ideal of R. Let n be a fixed positive integer. We define for any k-scheme T the set

$$\mathcal{H}ilb^n R(T) = \left\{ \begin{array}{l} \text{Closed subschemes } Z \subseteq T \times_k \operatorname{Spec}(R), \text{ where the} \\ \text{projection } Z \to T \text{ is flat, and the } \kappa(y)\text{-vector space} \\ \text{of global sections of the fibre } Z_y \text{ has dimension } n \\ \text{for all } y \in T, \text{ and such that } Z_{\mathrm{red}} \subseteq T \times_k \operatorname{Spec}(R/P). \end{array} \right\}.$$

Lemma 4.5. *The assignment sending a k-scheme T to the set $\mathcal{H}ilb^n R(T)$, determines a contravariant functor from the category of noetherian k-schemes to sets.*

Proof. Let $U \to T$ be a morphism of noetherian k-schemes. If Z is a T-valued point of $\mathcal{H}ilb^n R$ we must show that $Z_U = U \times_T Z$ is an element of $\mathcal{H}ilb^n R(U)$. The only non-trivial part of the claim is to show that Z_U is supported at $U \times_k \operatorname{Spec}(R/P)$.

Since Z is supported at $T \times_k \operatorname{Spec}(R/P)$ there exists by Lemma (4.3) an integer N such that $Z \subseteq T \times_k \operatorname{Spec}(R/P^N)$. It follows that $Z_U \subseteq U \times_k \operatorname{Spec}(R/P^N)$. Hence by Lemma (4.3) we have that Z_U is supported at $U \times_k \operatorname{Spec}(R/P)$. □

Remark. Note that we restrict ourselves to noetherian k-schemes. It is not clear whether $\mathcal{H}ilb^n R$ is a presheaf of sets on the category of k-schemes.

Remark. When $R = \mathbf{C}\{x, y\}$, the ring of convergent power series in two variables, the Definition (4.4) gives the functor of J. Briançon [1].

Lemma 4.6. *Let R be a local noetherian k-algebra. Let P be the maximal ideal of R, and let $\hat{R}$ be the P-adic completion of R. Then $\mathcal{H}ilb^n R$ is canonically isomorphic to $\mathcal{H}ilb^n \hat{R}$.*

Proof. We have that $\hat{R}$ is a local ring with maximal ideal $\hat{P} = P \otimes_R \hat{R}$. Furthermore we have for any positive integer N that $R/P^N = \hat{R}/\hat{P}^N$. It follows that for any k-scheme T we have that

$$T \times_k \operatorname{Spec}(R/P^N) = T \times_k \operatorname{Spec}(\hat{R}/\hat{P}^N). \tag{4.6.1}$$

Thus if Z is an element of $\mathcal{H}ilb^n R(T)$ there is by Lemma (4.3) an integer N such that Z is a closed subscheme of $T \times_k \operatorname{Spec}(R/P^N)$. By (4.6.1) it follows that Z is a closed subscheme of $T \times_k \operatorname{Spec}(\hat{R})$ having support in $T \times_k \operatorname{Spec}(\hat{R}/\hat{P})$. We get that Z is an element of $\mathcal{H}ilb^n \hat{R}(T)$. It is clear that a similar argument shows that the converse also holds; any element $Z \in \mathcal{H}ilb^n \hat{R}(T)$ is naturally identified as an element of $\mathcal{H}ilb^n R(T)$. □

Lemma 4.7. *Let A be a k-algebra. Let ϵ be a nilpotent element in A such that the smallest integer j where $e^j = 0$ is $j = 2^{m+1}$. Then the smallest integer N such that we have an inclusion of ideals $(x^N) \subseteq (x^n - \epsilon x^{n-1})$ in $A \otimes_k k[x]_{(x)}$ is $N = 2^{(m+1)} + n - 1$.*

Proof. We first show that we have an inclusion $(x^N) \subseteq (x^n - \epsilon x^{n-1})$ in $A \otimes_k k[x]_{(x)}$, with $N = 2^{m+1} + n - 1$. For every non-negative integer p we let

$$d_p(\epsilon, x) = (x + \epsilon)(x^2 + \epsilon^2) \ldots (x^{2^p} + \epsilon^{2^p}) \quad \text{in} \quad A[x]. \tag{4.7.1}$$

We have that $(x - \epsilon)d_p(\epsilon, x) = x^{2^{(p+1)}} - \epsilon^{2^{(p+1)}}$ in $A[x]$. Thus when $p \geq m$, we have that $(x - \epsilon)d_p(\epsilon, x) = x^{2^{p+1}}$ in $A[x]$. It follows that there is an inclusion of ideals

$$(x^{2^{m+1}+n-1}) \subseteq (x^n - \epsilon x^{n-1}) \quad \text{in} \quad A \otimes_k k[x]_{(x)}. \tag{4.7.2}$$

We need to show that $2^{m+1} + n - 1$ is the smallest integer such that the inclusion (4.7.2) in $A \otimes_k k[x]_{(x)}$ holds.

Let $N + r = 2^{m+1} + n - 1$, where r is a non-negative integer. Assume that we have an inclusion of ideals $(x^N) \subseteq (x^n - \epsilon x^{n-1})$ in $A \otimes_k k[x]_{(x)}$. The element

$\epsilon \in A$ is nilpotent, hence by Corollary (2.5) we have that $F(x) = x^n - \epsilon x^{n-1}$ is such that the assertions of Theorem (2.4) are satisfied. It follows by Corollary (2.6) that an inclusion of ideals $(x^N) \subseteq (F(x))$ in $A \otimes_k k[x]_{(x)}$ is equivalent to an inclusion of ideals $(x^N) \subseteq (F(x))$ in $A[x]$. Consequently there exists a $G(x)$ in $A[x]$ such that $x^N = (x^n - \epsilon x^{n-1})G(x)$. Let $d_m(\epsilon, x)$ in $A[x]$ be the polynomial as defined in (4.7.1). We have that $(x - \epsilon)d_m(\epsilon, x) = x^{2^{m+1}}$. Hence we get the following identity in $A[x]$;

$$(x^n - \epsilon x^{n-1})d_m(\epsilon, x) = x^{2^{m+1}+n-1} = x^N x^r = (x^n - \epsilon x^{n-1})G(x)x^r. \qquad (4.7.3)$$

The element x^n is not a zero divisor in the ring $A[x]$. It follows that the element $(x^n - \epsilon x^{n-1})$ is not a zero divisor in $A[x]$. From the identity in (4.7.3) we obtain the identity $(x^n - \epsilon x^{n-1})(d_m(\epsilon, x) - G(x)x^r) = 0$ in $A[x]$, which implies that $d_m(\epsilon, x) = G(x)x^r$ in $A[x]$. The polynomial $d_m(\epsilon, x)$ (4.7.1) has a constant term $\epsilon^{2^{(m+1)}-1} \neq 0$. Consequently x does not divide $d_m(\epsilon, x)$. Therefore $r = 0$, and $N = 2^{m+1} + n - 1$ is the smallest integer such that we have an inclusion of ideals $(x^N) \subseteq (x^n - \epsilon x^{n-1})$ in $A \otimes_k k[x]_{(x)}$. $\square$

Remark. When $\epsilon(m) = \epsilon$ is as in Lemma (4.7), we have that the closed subscheme $Z_m = \mathrm{Spec}(A \otimes_k k[x]_{(x)}/(x^n - \epsilon x^{n-1})) \subseteq \mathrm{Spec}(A \otimes_k k[x]_{(x)})$ is a subscheme of $\mathrm{Spec}(A) \times_k \mathrm{Spec}(k[x]/(x^N))$ if and only if $N \geq 2^{m+1} + n - 1$.

Theorem 4.8. *Let R be a local noetherian k-algebra with maximal ideal P. Assume that the P-adic completion of R is $\hat{R} = k[[x_1, \dots, x_r]]$, the formal power series ring in $r > 0$ variables. Then the functor $\mathcal{H}ilb^n R$ is not representable in the category of noetherian k-schemes.*

Proof. Write $x = x_1, \dots, x_r$, and set $k[x]_{(x)} = k[x_1, \dots, x_r]_{(x_1, \dots, x_r)}$ the localization of the polynomial ring $k[x_1, \dots, x_r]$ in the maximal ideal $(x_1, \dots, x_r)$. By Lemma (4.6) it suffices to show that $\mathcal{H}ilb^n k[x]_{(x)}$ is not representable.

Assume that $\mathcal{H}ilb^n k[x]_{(x)}$ is representable. Let H be the noetherian k-scheme representing the functor $\mathcal{H}ilb^n k[x]_{(x)}$. Let $U \in \mathcal{H}ilb^n k[x]_{(x)}(H)$ be the universal family. Then in particular we have that $U_{red} \subseteq H \times_k \mathrm{Spec}(k)$. Hence, by Lemma (4.3) there exists an integer N such that we have an closed immersion

$$U \subseteq H \times_k \mathrm{Spec}(k[x_1, \dots, x_r]/(x_1, \dots, x_r)^N). \qquad (4.8.1)$$

We let m be an integer such that $2^{m+1} + n - 1 > N$. Write $A_m = k[u]/(u^{2^{(m+1)}})$. Let $Z_m = \mathrm{Spec}(A_m \otimes_k k[x]_{(x)}/(x_1^n - \epsilon x_1^{n-1}, x_2, \dots, x_r)) \subseteq \mathrm{Spec}(A_m \otimes_k k[x]_{(x)})$, where $\epsilon \in A_m$ is the class of u in A_m. We have that $Z_m = \mathrm{Spec}(A_m \otimes_k k[x_1]_{(x_1)}/(x_1^n - \epsilon x_1^{n-1}))$. It follows from Theorem (3.5) that Z_m is an A_m-valued point of the functor $\mathcal{H}ilb^n k[x]_{(x)}$.

By the universality of the pair (H, U) there exists a morphism $\mathrm{Spec}(A_m) \to H$ such that $Z_m = \mathrm{Spec}(A_m) \times_H U$. It then follows from the closed immersion in (4.8.1) that $Z_m \subseteq \mathrm{Spec}(A_m) \times_k \mathrm{Spec}(k[x]/(x_1, \dots, x_r)^N)$. However, since $2^{m+1} + n - 1 > N$ we have by the remark following Lemma (4.7), that $\mathrm{Spec}(A_m \otimes_k k[x_1]_{(x_1)}/(x_1^n - \epsilon x_1^{n-1}))$ is *not* a subscheme of $\mathrm{Spec}(A_m) \times_k \mathrm{Spec}(k[x_1]/(x_1^N))$. Hence we get that Z_m can not be a closed subscheme of $\mathrm{Spec}(A_m) \times_k \mathrm{Spec}(k[x_1]/(x_1, \dots, x_r)^N)$. We have thus reached a contradiction and proven the Theorem. $\square$

5. Pro-representing $\mathcal{H}ilb^n k[x]_{(x)}$.

5.1. Set up. Let $s_1, \dots, s_n$ be independent variables over the field k. We write $R_n = k[[s_1, \dots, s_n]]$ for the completion of the polynomial ring $k[s_1, \dots, s_n]$ in the maximal ideal $(s_1, \dots, s_n)$. We will show that R_n pro-represents the functor $\mathrm{Hilb}^n\, k[x]_{(x)}$. We recall the basic notions from [5].

5.2. Notation. Let $\mathbf{C}_k$ be the category where the objects are local artinian k-algebras with residue field k, and where the morphisms are (local) k-algebra homomorphism. If A is an object of $\mathbf{C}_k$ we say that A is an *artin* ring.

We write H_n for the restriction of the functor $\mathcal{H}ilb^n k[x]_{(x)}$ to the category $\mathbf{C}_k$. Note that an artin ring A, that is an element of the category $\mathbf{C}_k$ has only one prime ideal. The residue field of the only prime ideal of A is k. The ideal (x^n) is the only ideal I of $k[x]_{(x)}$ such that the residue ring $k[x]_{(x)}/I$ has dimension n as a k-vector space. It follows that the covariant functor H_n from the category $\mathbf{C}_k$ to sets, maps an artin ring A to the set

$$H_n(A) = \left\{ \begin{array}{l} \text{Ideals } I \subseteq A \otimes_k k[x]_{(x)} \text{ such that the residue ring} \\ M = A \otimes_k k[x]_{(x)}/I \text{ is a flat } A\text{-module, where} \\ M \otimes_A k = k[x]/(x^n), \text{ and such that there is an} \\ \text{inclusion of ideals } (x) \subseteq \mathfrak{R}(I) \text{ in } A \otimes_k k[x]_{(x)}. \end{array} \right\}. \tag{5.2.1}$$

Remark. Let $\mathcal{H}ilb^n_{k[x]_{(x)}}$ denote the usual Hilbert functor and consider its restriction to the category $\mathbf{C}_k$. Thus an A-valued point of $\mathcal{H}ilb^n_{k[x]_{(x)}}$ is an ideal $I \subseteq A \otimes_k k[x]_{(x)}$ such that the residue ring $M = A \otimes_k k[x]_{(x)}/I$ is flat over A, and such that $M \otimes_A k = k[x]/(x^n)$. We shall show that the restriction of the Hilbert functor $\mathcal{H}ilb^n_{k[x]_{(x)}}$ to the category $\mathbf{C}_k$ coincides with the functor H^n.

Note that an A-valued point $M = A \otimes_k k[x]_{(x)}/I$ of $\mathcal{H}ilb^n_{k[x]_{(x)}}$ is not *a priori* finitely generated as an module over A. However we have the following general result ([2], Theorem (2.4)).

Let A be a local ring with nilpotent radical. Let M be a flat A-module, and denote the maximal ideal of A by P. If $\dim_{\kappa(P)}(M \otimes_A \kappa(P)) = \dim_{\kappa(Q)}(M \otimes_A \kappa(Q)) = n$, for all minimal prime ideals Q in A, then M is a free A-module of rank n.

It follows that when A is an artin ring, and $M = A \otimes_k k[x]_{(x)}/I$ is an A-valued point of $\mathcal{H}ilb^n_{k[x]_{(x)}}$, then M is free and of rank n as an A-module. It then follows by Lemma (2.2) that the ideal I is generated by a monic polynomial $F(x) = x^n - u_1 x^{n-1} + \cdots + (-1)^n u_n$ in $A[x]$. Since we have that $M \otimes_A k = k[x]/(x^n)$ we get that the coefficients $u_1, \dots, u_n$ of $F(x)$ are nilpotent. Hence by Theorem (3.5) we have that $(x) \subseteq \mathfrak{R}(I)$ in $A \otimes_k k[x]_{(x)}$. We have shown that the two functors $\mathcal{H}ilb^n k[x]_{(x)}$ and $\mathcal{H}ilb^n_{k[x]_{(x)}}$ coincide when restricted to the category $\mathbf{C}_k$ of artin rings.

Lemma 5.3. *Let A be an artin ring. Let $\psi : R_n = k[[s_1, \dots, s_n]] \to A$ be a local k-algebra homomorphism. Let $F_n^{\psi}(x) = x^n - \psi(s_n)x^{n-1} + \cdots + (-1)\psi(s_n)$. Then $(F_n^{\psi}(x)) \subseteq A \otimes_k k[x]_{(x)}$ is an A-valued point of H_n.*

Proof. Since the map ψ is local we have that $\psi(s_i)$ is in the maximal ideal $\mathfrak{m}_A$ of A, for each $i = 1, \dots, n$. The ring A is artin. Consequently $\mathfrak{m}_A^q = 0$ for some integer q.

It follows that the coefficients $\psi(s_1),\dots,\psi(s_n)$ of $F_n^{\psi}(x)$ are nilpotent. By Theorem (3.5) we have an inclusion of ideals $(x)\subseteq \mathfrak{R}(F_n^{\psi}(x))$ in $A\otimes_k k[x]_{(x)}$ and the residue ring $M=A\otimes_k k[x]_{(x)}/(F_n^{\psi}(x))$ is a flat A-module such that $M\otimes_A k$ is of dimension n as a k-vector space. Thus we have proven that the ideal $(F_n^{\psi}(x))\subseteq A\otimes_k k[x]_{(x)}$ is an element of $H_n(A)$. $\square$

5.4. The pro-couple (R_n,ξ). Let $\mathfrak{m}$ be the maximal ideal of $R_n=k[[s_1,\dots,s_n]]$. For every positive integer q we let $s_{q,1},\dots,s_{q,n}$ be the classes of $s_1,\dots,s_n$ in $R_n/\mathfrak{m}^q$. It follows from Lemma (5.3) that the ideal generated by $F_n^q(x)=x^n-s_{q,1}x^{n-1}+\cdots+(-1)^n s_{q,n}$ in $R/\mathfrak{m}^q[x]$ generates an $R_n/\mathfrak{m}^q$-point of H_n. We get a sequence

$$\xi=\{(F_n^q(x))\}_{q\geq 0}, \tag{5.4.1}$$

where $(F_n^q(x))$ is an $R_n/\mathfrak{m}^q$-point for every non-negative integer q. Clearly ξ defines a point in the projective limit $\varprojlim_q\{H^n(R_n/\mathfrak{m}^q)\}$. Thus we have that (R_n,ξ) is a *pro-couple* of H_n.

We let h_R be the covariant functor from $\mathbf{C}_k$ to sets, which sends an artin ring A to the set of local k-algebra homomorphisms $\mathrm{Hom}_{k\text{-loc}}(R_n,A)$. We note that a local k-algebra homomorphism $\psi:R_n\to A$ factors through $R_n/\mathfrak{m}^q$ for high enough q. We get that the pro-couple (R_n,ξ) induces a morphism of functors $F_\xi:h_R\to H_n$ which for any artin ring A, maps an element $\psi\in h_R(A)$ to the element $(F_n^{\psi}(x))$ in $H_n(A)$. Here $F_n^{\psi}(x)$ is as in Lemma (5.3).

Theorem 5.5. *Let $R_n=k[[s_1,\dots,s_n]]$, and let ξ be as in* (5.4.1)*. The morphism of functors $F_\xi:h_R\to H_n$ induced by the pro-couple (R_n,ξ), is an isomorphism.*

Proof. We must construct an inverse to the morphism $F_\xi:h_R\to H_n$. Let A be an artin ring, and let $I\subseteq A\otimes_k k[x]_{(x)}$ be an ideal satisfying the properties of (5.2.1). We have that Assertion (1) of Theorem (3.5) holds. Consequently the ideal $I\subseteq A\otimes_k k[x]_{(x)}$ is generated by a unique $F(x)=x^n-u_1x^{n-1}+\cdots+(-1)u_n$ in $A[x]$, where $u_1,\dots,u_n$ are nilpotent. The coefficients $u_1,\dots,u_n$ of $F(x)$ are in the maximal ideal of A, hence the map $\psi:k[[s_1,\dots,s_n]]\to A$ sending s_i to u_i, determines a local k-algebra homomorphism. We have thus constructed a morphism of functors $G:H_n\to h_R$. It is clear that G is the inverse of F_ξ. $\square$

6. A filtration of $\mathcal{H}ilb^n k[x]_{(x)}$ by schemes.

We will show in this section that there is a natural filtration of $\mathcal{H}ilb^n k[x]_{(x)}$ by representable functors $\{\mathcal{H}^{n,m}\}_{m\geq 0}$, where $\mathcal{H}^{n,m}$ is a closed subfunctor of $\mathcal{H}^{n,m+1}$ for all m. The functors $\mathcal{H}^{n,m}$ are the Hilbert functors parameterizing closed subschemes of length n of $\mathrm{Spec}(k[x]/(x^{n+m}))$.

An outline of Section (6) is as follows. We will define the functors $\mathcal{H}^{n,m}$ from the category of k-schemes, not necessarily noetherian schemes, to sets. We then construct schemes $\mathrm{Spec}(H_{n,m})$ which we show represent $\mathcal{H}^{n,m}$. Thereafter we restrict $\mathcal{H}^{n,m}$ to the category of noetherian k-schemes, and show that we get a filtration of $\mathcal{H}ilb^n k[x]_{(x)}$.

Definition 6.1. Let $n>0, m\geq 0$ be integers. In the polynomial ring $k[x]$ we have the ideal (x^{n+m}) and we denote the residue ring by $R=k[x]/(x^{n+m})=k[x]_{(x)}/(x^{n+m})$. We denote by $\mathcal{H}^{n,m}=\mathcal{H}ilb^n_R$ the local Hilbert functor of n-points on $\mathrm{Spec}\,R$. Thus $\mathcal{H}^{n,m}$ is the contravariant functor from the category of k-schemes to sets, determined by sending a k-scheme T to the set

$$\mathcal{H}^{n,m}(T) = \left\{ \begin{array}{l} \text{Closed subschemes } Z \subseteq T \times_k \operatorname{Spec}(k[x]/(x^{n+m})), \\ \text{such that the projection } Z \to T \text{ is flat, and the} \\ \kappa(y)\text{-vector space of global sections of the fibre} \\ Z_y \text{ has dimension } n \text{ for all points } y \in T. \end{array} \right\}.$$

6.2. Construction of the rings $H_{n,m}$. Let $P_n = k[s_1, \dots, s_n]$ be the polynomial ring in the variables $s_1 \dots, s_n$ over k. Let m be a fixed non-negative integer, and let $y_1, \dots, y_m, x$ be algebraically independent variables over P_n. We define $F_n(x) = x^n - s_1x^{n-1} + \cdots + (-1)^n s_n$ in $P_n[x]$, and we let $Y_m(x) = x^m + y_1x^{m-1} + \cdots + y_m$. The product $F_n(x)Y_m(x)$ is

$$F_n(x)Y_m(x) = x^{n+m} + C_{m,1}(y)x^{n+m-1} + \cdots + C_{m,n+m}(y). \tag{6.2.1}$$

As a convention we let $s_0 = y_0 = 1$, and $y_j = 0$ for negative values of j. The coefficient $C_{m,i}(y)$ is the sum of products $(-1)^j s_j y_{i-j}$, where $j = 0, \dots, n$, and $i - j = 0, 1, \dots, m$. We have

$$\begin{aligned} C_{m,i}(y) &= y_i - s_1 y_{i-1} + \cdots + (-1)^n s_n y_{i-n} && \text{when } i = 1, \dots, m. \\ C_{m,m+j}(y) &= (-1)^j y_m s_j + \cdots + (-1)^n y_{m+j-n} s_n && \text{when } j = 1, \dots, n. \end{aligned} \tag{6.2.2}$$

For every non-negative integer m we let $I_m \subseteq P_n[y_1, \dots, y_m]$ be the ideal generated by the coefficients $C_{m,1}(y), \dots, C_{m,m+n}(y)$. We write

$$H_{n,m} = P_n[y_1, \dots, y_m]/I_m = P_n[y_1, \dots, y_m]/(C_{m,1}(y), \dots, C_{m,m+n}). \tag{6.2.3}$$

Using (6.2.2) we note that $C_{m,m}(y) = y_m + C_{m-1,m}(y)$. For every positive integer m we define the P_n-algebra homomorphism

$$c_m : P_n[y_1, \dots, y_m] \to P_n[y_1, \dots, y_{m-1}] \tag{6.2.4}$$

by sending y_i to y_i when $i = 1, \dots, m-1$, and y_m to $-C_{m-1,m}(y)$.

Lemma 6.3. *The natural map $P_n \to P_n[y_1, \dots, y_m]/(C_{m,1}(y), \dots, C_{m,m}(y))$ is an isomorphism for each non-negative integer m. In particular the map $P_n \to H_{n,m}$ is surjective.*

Proof. Consider the homomorphism c_m as defined in (6.2.4). It is clear that c_m is surjective and that we get an induced isomorphism

$$P_n[y_1, \dots, y_m]/(C_{m,m}(y)) \backsimeq P_n[y_1, \dots, y_{m-1}]. \tag{6.3.1}$$

When $i \leq m$ we have that $C_{m,i}(y)$ is a function in the variables $y_1, \dots, y_i$. Hence when $i = 1, \dots, m-1$ the elements $C_{m,i}(y)$ are invariant under the action of c_m. From (6.2.2) we get that $C_{m,i}(y) = C_{m-1,i}(y)$ when $i = 1, \dots, m-1$. It follows by successive use of (6.3.1) that we get an induced isomorphism

$$P_n[y_1, \dots, y_m]/(C_{m,1}(y), \dots, C_{m,m}(y)) \backsimeq P_n. \tag{6.3.2}$$

It is easy to see that the map (6.3.2) composed with the natural map induced by $P_n \to P_n[y_1, \dots, y_m]$, is the identity map on P_n. $\square$

Lemma 6.4. *For every positive integer m, the P_n-algebra homomorphism c_m (6.2.4) induces a surjective map $H_{n,m} \to H_{n,m-1}$.*

Proof. Let $\hat{c}_m$ be the composite of the residue map $P_n[y_1, \dots, y_{m-1}] \to H_{n,m-1}$ and c_m. We first show that we get an induced map $H_{n,m} \to H_{n,m-1}$. That is, we show that the ideal $I_m \subseteq P_n[y_1, \dots, y_m]$ defining $H_{n,m}$, is in the kernel of $\hat{c}_m$.

The ideal I_m is generated by $C_{m,1}(y), \dots, C_{m,m+n}(y)$. As noted in the proof of Lemma (6.3) the elements $C_{m,i}(y)$ are mapped to $C_{m-1,i}(y)$ when $i = 1, \dots, m-1$, whereas $C_{m,m}(y)$ is in the kernel of c_m. Consequently we need to show that the elements $C_{m,m+j}(y)$ are mapped to zero by $\hat{c}_m$. Using (6.2.2) we get that

$$\begin{aligned} C_{m,m+j}(y) &= (-1)^j y_m s_j + (-1)^{j+1} y_{m-1} s_{j+1} \cdots + (-1)^n y_{m+j-n} s_n \\ &= (-1)^n y_m s_j + C_{m-1,m+j}(y) \quad \text{when } j \leq n-1. \end{aligned} \tag{6.4.1}$$

It follows that $C_{m,m+j}(y)$, for $j = 1, \dots, n-1$ are mapped to zero by $\hat{c}_m$. The last generator of I_m is $C_{m,m+n}(y) = (-1)^n y_m s_n$, clearly in the kernel of $\hat{c}_m$. Thus we have proven that the ideal I_m is in the kernel of $\hat{c}_m : P_n[y_1, \dots, y_m] \to H_{n,m-1}$.

We need to show that the induced map $H_{n,m} \to H_{n,m-1}$ is surjective. From Lemma (6.3) we have that the natural map $P_n \to H_{n,m}$ is surjective for all m. Since the map c_m is P_n-linear, it follows that the induced map $H_{n,m} \to H_{n,m-1}$ is P_n-linear and the result follows. □

Definition 6.5. The natural map $P_n = k[s_1, \dots, s_n] \to H_{n,m}$ is surjective by Lemma (6.3), for all m. We let $s_{m,i}$ be the class of s_i in $H_{n,m}$, for $i = 1, \dots, n$. Define

$$F_{n,m}(x) = x^n - s_{m,1} x^{n-1} + \cdots + (-1)^n s_{m,n} \quad \text{in } H_{n,m}[x]. \tag{6.5.1}$$

Lemma 6.6. *Let A be a k-algebra. Let I be an ideal in $A \otimes_k k[x]_{(x)}$ such that the residue ring $A \otimes_k k[x]_{(x)}/I$ is a free A-module of rank n, and such that there is an inclusion of ideals $(x^{n+m}) \subseteq I$ in $A \otimes_k k[x]_{(x)}$. Then there is a unique k-algebra homomorphism $\psi : H_{n,m} \to A$ such that*

$$F^{\psi}_{n,m}(x) = x^n - \psi(s_{m,1}) x^{n-1} + \cdots + (-1)^n \psi(s_{m,n})$$

in $A[x]$ generates I.

Proof. It follows by Assertion (2) of Lemma (2.2) that I is generated by a unique $F(x) = x^n - u_1 x^{n-1} + \cdots + (-1)^n u_n$ in $A[x]$. By Assertion (1) of Lemma (2.2) the classes of $1, x, \dots, x^{n-1}$ form a basis for M. Consequently $F(x)$ in $A[x]$ satisfies the assertions of Theorem (2.4). By Corollary (2.6) the inclusion of ideals $(x^{n+m}) \subseteq (F(x))$ in $A \otimes_k k[x]_{(x)}$ is equivalent to the existence of $G(x)$ in $A[x]$ such that $x^{n+m} = F(x)G(x)$. Let $G(x) = x^m + g_1 x^{m-1} + \cdots + g_m$ in $A[x]$. The coefficients $g_1, \dots, g_m$ are uniquely determined by $G(x)$, hence uniquely determined by the ideal I. Let $y_1, \dots, y_m$ be independent variables over k. We get a well-defined k-algebra homomorphism $\theta : k[s_1, \dots, s_n, y_1, \dots, y_m] \to A$ determined by $\theta(s_i) = u_i$ where $i = 1, \dots, n$, and $\theta(y_j) = g_j$ where $j = 1, \dots, m$. We have thus constructed a k-algebra homomorphism $\theta : P_n[y_1, \dots, y_m] \to A$. We will next show that the map θ factors through $H_{n,m}$. We have that

$$\begin{aligned} x^{n+m} &= F(x)G(x) \\ &= (x^n - u_1 x^{n-1} + \cdots + (-1)^n u_n)(x^m + g_1 x^{m-1} + \cdots + g_m) \\ &= x^{n+m} + c_1 x^{n+m-1} + \cdots + c_{n+m} \end{aligned} \tag{6.6.1}$$

in $A[x]$. It follows that the coefficients c_j where $j = 1, \dots, m+n$ are zero in A. The homomorphism θ induces a map $P_n[y_1, \dots, y_m][x] \to A[x]$ which sends $F_{n,m}(x)$ to $F(x)$ and $Y_m(x) = x^m + y_1 x^{m-1} + \cdots + y_m$ to $G(x)$. It follows that the coefficient equations $C_{m,j}(y)$ (6.2.1) where $j = 1, \dots, m+n$, are mapped to $c_j = 0$. Hence the homomorphism $\theta : P_n[y_1, \dots, y_m] \to A$ factors through $H_{n,m}$. Let $\psi : H_{n,m} \to A$ be the induced map. We have for each $i = 1, \dots, n$ that $\psi(s_{m,i}) = \theta(s_i) = u_i$. Consequently we get that $F^{\psi}_{n,m}(x) = F(x)$. We have thus proven the existence of a map $\psi : H_{n,m} \to A$ such that $F^{\psi}_{n,m}(x)$ generates the ideal I in $A \otimes_k k[x]_{(x)}$.

We need to show that the map ψ is the only map with the property that $(F^{\psi}_{n,m}(x)) = I$. Let $\psi' : H_{n,m} \to A$ be a k-algebra homomorphism such that $F^{\psi'}_{n,m}(x)$ generates the ideal I in $A \otimes_k k[x]_{(x)}$. By Assertion (2) of Lemma (2.2) the ideal $I \subseteq A \otimes_k k[x]_{(x)}$ is generated by a unique monic polynomial $F(x)$ in $A[x]$. It follows that we must have $F^{\psi'}_{n,m}(x) = F(x)$. Thus if $u_1, \dots, u_n$ are the coefficients of $F(x)$, we get that $\psi'(s_{m,i}) = u_i$. A k-algebra homomorphism $H_{n,m} \to A$ is determined by its action on $s_{m,1}, \dots, s_{m,n}$. Hence $\psi = \psi'$, and the map ψ is unique. □

Proposition 6.7. *The functor* $\mathcal{H}^{n,m}$ *is represented by* $\operatorname{Spec}(H_{n,m})$. *The universal family is given by* $\operatorname{Spec}(H_{n,m}[x]/(F_{n,m}(x)))$.

Proof. We first show that $\operatorname{Spec}(H_{n,m}[x]/(F_{n,m}(x)))$ is an $H_{n,m}$-valued point of $\mathcal{H}^{n,m}$. We have that $F_{n,m}(x) = x^n - s_{m,1}x^{n-1} + \cdots + (-1)^n s_{m,n}$ in $H_{n,m}[x]$. Since $F_{n,m}(x)$ is of degree n and has leading coefficient 1, we have that $H_{n,m}[x]/(F_{n,m}(x))$ is a free $H_{n,m}$-module of rank n. By the identity in (6.2.1) and the construction of $H_{n,m}$ we have an inclusion of ideals $(x^{n+m}) \subseteq (F_{n,m}(x))$ in $H_{n,m}[x]$. Thus we have that $H_{n,m}[x]/(F_{n,m}(x)) = H_{n,m} \otimes_k R/(F_{n,m}(x))$, where $R = k[x]/(x^{n+m})$, and consequently $\operatorname{Spec}(H_{n,m}[x]/(F_{n,m}(x)))$ is an $H_{n,m}$-valued point of $\mathcal{H}^{n,m}$.

We then have a morphism of functors $F : \operatorname{Hom}(-, \operatorname{Spec}(H_{n,m})) \to \mathcal{H}^{n,m}$, which we claim is an isomorphism.

Let T be a k-scheme and let Z be an T-valued point of $\mathcal{H}^{n,m}$. Let $p : T \times_k \operatorname{Spec}(k[x]/(x^{n+m})) \to T$ be the projection on the first factor. Let $\operatorname{Spec}(A) = U \subseteq T$ be an open affine subscheme and let the closed subscheme $Z \cap p^{-1}(U) \subseteq U \times_k \operatorname{Spec}(k[x]/(x^{n+m}))$ be given by the ideal $J \subseteq A \otimes_k k[x]/(x^{n+m})$. Let I be the inverse image of J under the residue map $A \otimes_k k[x]_{(x)} \to A \otimes_k k[x]/(x^{n+m})$.

It follows from the definition of the functor $\mathcal{H}^{n,m}$ that the ideal I satisfies the conditions of Proposition (2.3). Hence $A \otimes_k k[x]_{(x)}/I$ is a free A-module of rank n. We have by definition an inclusion of ideals $(x^{n+m}) \subseteq I$ in $A \otimes_k k[x]_{(x)}$. Consequently we get by Lemma (6.6) a unique map $f_U : U \to \operatorname{Spec}(H_{n,m})$ such that $Z \cap p^{-1}(U) = U \times_{H_{n,m}} \operatorname{Spec}(H_{n,m}[x]/(F_{n,m}(x)))$.

Thus, if $\{U_i\}$ is an open affine covering of T, we get maps $f_i : U_i \to \operatorname{Spec}(H_{n,m})$ with the property that

$$Z \cap p^{-1}(U_i) = U_i \times_{H_{n,m}} \operatorname{Spec}(H_{n,m}[x]/(F_{n,m}(x))). \tag{6.7.1}$$

The maps $f_i : U_i \to \operatorname{Spec}(H_{n,m})$ are unique with respect to the property (6.7.1). Hence the maps f_i glue together to a unique map $f_Z : T \to \operatorname{Spec}(H_{n,m})$ such that $Z = T \times_{H_{n,m}} \operatorname{Spec}(H_{n,m}[x]/(F_{n,m}(x)))$. It follows from the uniqueness of the map f_Z that the assignment sending a T-valued point Z to the morphism f_Z sets up an bijection between the set $\mathcal{H}^{n,m}(T)$ and the set $\operatorname{Hom}(T, \operatorname{Spec}(H_{n,m}))$. □

Theorem 6.8. *Let n be a fixed positive integer. There is a filtration of the functor $\mathcal{H}ilb^n k[x]_{(x)}$ by an ascending chain of representable functors*

$$\mathcal{H}^{n,0} \subseteq \mathcal{H}^{n,1} \subseteq \mathcal{H}^{n,2} \subseteq \dots,$$

where $\mathcal{H}^{n,m}$ is a closed subfunctor of $\mathcal{H}^{n,m+1}$, for every m.

Proof. By Proposition (6.7) the functors $\mathcal{H}^{n,m}$ are represented by $\mathrm{Spec}(H_{n,m})$ where the universal family is given by $U_{n,m} = \mathrm{Spec}(H_{n,m}[x]/(F_{n,m}(x))$. Let $c_{m+1} : H_{n,m+1} \to H_{n,m}$ be the surjective map of Lemma (6.3). It follows from the P_n-linearity of c_{m+1} that the induced map $H_{n,m+1}[x] \to H_{n,m}[x]$ maps $F_{n,m+1}(x)$ to $F_{n,m}(x)$. Consequently we have that $\mathrm{Spec}(H_{n,m})$ is a closed subscheme of $\mathrm{Spec}(H_{n,m+1})$ such that $U_{n,m+1} \times_{H_{n,m+1}} \mathrm{Spec}(H_{n,m}) = U_{n,m}$. Hence we have that $\mathcal{H}^{n,m}$ is a closed subfunctor of $\mathcal{H}^{n,m+1}$.

From the constructions (6.2.3) of the rings $H_{n,m}$ it is evident that they are noetherian. It follows that the restriction of the functor $\mathcal{H}^{n,m}$ to the category of *noetherian* k-schemes, is represented by $\mathrm{Spec}(H_{n,m})$.

That the functors $\{\mathcal{H}^{n,m}\}_{m \geq 0}$ give a filtration of the functor $\mathcal{H}ilb^n k[x]_{(x)}$, follows from Lemma (4.3). Indeed, let T be noetherian k-scheme and let Z be a T-valued point of $\mathcal{H}ilb^n k[x]_{(x)}$. Then there exists an integer $N = N(Z)$ such that $Z \subseteq T \times_k \mathrm{Spec}(k[x]/(x^N))$. Consequently the T-valued point Z of $\mathcal{H}ilb^n k[x]_{(x)}$ is a T-valued point of $\mathcal{H}^{n,N-n}$. $\square$

6.9. Examples of $H_{n,m}$. The rings $H_{n,m}$ are all of the form $k[s_1, \dots, s_n]/J_m$, where J_m is generated by n elements. With $n = 1$ it is not difficult to solve the equations (6.2.2). We get that $H_{1,m} = k[u]/(u^{m+1})$. Thus we have that the scheme $\mathrm{Spec}\, k[x]/(x^{m+1})$ itself represents the Hilbert functor $\mathcal{H}^{1,m}$ of 1-points on $\mathrm{Spec}(k[x]/(x^{m+1})$, for all non-negative integers m.

In general, with $n > 1$ a description of the generators of the ideal J_m is not known, even though they can be recursively solved. For instance, we have

$$\begin{aligned} H_{2,1} &= k[x,y]/(x^2, xy) \\ H_{2,2} &= k[x,y]/(x^3 - 2xy, x^2y - y^2) \\ H_{2,3} &= k[x,y]/(x^4 - 3x^2y + y^2, x^3y - 2xy^2). \end{aligned}$$

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