An introduction to bounded cohomology

Michelle Bucher

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Abstract

Lecture notes for a 4 x 1 hour mini-course on bounded cohomology given in April 2008, at KTH. It covers elementary definitions, low degree computations, actions on the circle, Milnor-Wood inequalities, and (non)existence of affine structures.

Lecture I

1. Definition

Let Γ be a (discrete) group, and A be an abelian group. Typically, $A = \mathbb{R}, \mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$. Set

$$C^n(\Gamma, A) = \left\{ f : \Gamma^{n+1} \to A \right\} \quad \text{and}$$
$$C_b^n(\Gamma, A) = \left\{ f : \Gamma^{n+1} \to A \mid f \text{ is bounded} \right\}.$$

(If $A = \mathbb{R}/\mathbb{Z}$ the condition to be bounded is void.) The coboundary operator $\delta : C^n(\Gamma, A) \to C^{n+1}(\Gamma, A)$ is defined as

$$\delta f(\gamma_0, ..., \gamma_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(\gamma_0, ..., \widehat{\gamma_i}, ..., \gamma_{n+1}),$$

for $f \in C^n(\Gamma, A)$ and $(\gamma_0, ..., \gamma_{n+1}) \in \Gamma^{n+1}$.

Exercise Check that $\delta^2 = 0$.

Terminology: Elements in $C^n(\Gamma, A)$, respectively $C_b^n(\Gamma, A)$, are called *cochains*, resp. *bounded cochains*. A cochain f is a *cocycle* if $\delta f = 0$ and a *coboundary* if there exists h with $\delta h = f$. In particular, a coboundary is always a cocycle.

Lemma The cocomplex

$$0 \to A \to C_b^0(\Gamma, A) \to C_b^1(\Gamma, A) \to C_b^2(\Gamma, A) \to ...$$

is exact.

Proof Exactness in A: Note that as usual, the augmentation $\varepsilon : A \to C_b^0(\Gamma, A)$ is defined by sending an element $a \in A$ to the constant function $f : \Gamma \to A$ with value a. This is clearly an injective map.
Exactness in $C_b^0(\Gamma, A)$: By definition, the image of ε consists of constant functions $\Gamma \to A$. This is precisely the kernel of $\delta : C^0(\Gamma, A) \to C^1(\Gamma, A)$ since if $\delta f = 0$, for $f : \Gamma \to A$, then $\delta(f)(\gamma_0, \gamma_1) = f(\gamma_1) - f(\gamma_0) = 0$ for every $\gamma_0, \gamma_1 \in \Gamma$.

Exactness in $C_b^n(\Gamma, A)$, $n > 0$: Since $\delta^2 = 0$ it remains to prove that the kernel of δ is included in the image of δ. Let $f \in C_b^n(\Gamma, \mathbb{R})$ with $\delta f = 0$. Define $h : \Gamma^n \to \mathbb{R}$ by

$$h(\gamma_1, ..., \gamma_n) = f(1, \gamma_1, ..., \gamma_n),$$

for every $(\gamma_1, ..., \gamma_n) \in \Gamma^n$, where 1 denotes the identity element in Γ. For $(\gamma_0, ..., \gamma_n) \in \Gamma^{n+1}$, we have

$$\begin{aligned}\delta h(\gamma_0, ..., \gamma_n) &= \sum_{i=0}^{n}(-1)^i h(\gamma_0, ..., \widehat{\gamma_i}, ..., \gamma_n) = \sum_{i=0}^{n}(-1)^i f(1, \gamma_0, ..., \widehat{\gamma_i}, ..., \gamma_n)) \\ &= f(\gamma_0, ..., \gamma_n),\end{aligned}$$

since $\delta f(1, \gamma_0, ..., \gamma_n) = 0$, and hence $f = \delta h$ is in the image of δ.

Observe that we have not used the fact that the cochains are bounded, and the lemma is also true for the unbounded cocomplex.

Denote by $C^n(\Gamma, A)^{\mathbf{\Gamma}}$, respectively $C_b^n(\Gamma, A)^{\mathbf{\Gamma}}$, the subspaces of $C^n(\Gamma, A)$, consisting of Γ-invariant (bounded) cochains, that is, cochains $f : \Gamma^{n+1} \to A$ for which $\gamma \cdot f = f$, for every $\gamma \in \Gamma$, where $\gamma \cdot f : \Gamma^{n+1} \to A$ is defined as

$$\gamma \cdot f(\gamma_0, ..., \gamma_n) = f(\gamma^{-1}\gamma_0, ..., \gamma^{-1}\gamma_n),$$

for every $(\gamma_0, ..., \gamma_n) \in \Gamma^{n+1}, \gamma \in \Gamma$.

Definition The usual *Eilenberg-MacLane cohomology* $H^*(\Gamma, A)$ of Γ is the cohomology of the cocomplex

$$0 \to C^0(\Gamma, A)^\Gamma \to C^1(\Gamma, A)^\Gamma \to C^2(\Gamma, A)^\Gamma \to ...$$

The *bounded cohomology* $H_b^*(\Gamma, A)$ of Γ is the cohomology of the cocomplex

$$0 \to C_b^0(\Gamma, A)^\Gamma \to C_b^1(\Gamma, A)^\Gamma \to C_b^2(\Gamma, A)^\Gamma \to ...$$

The cohomology groups $H^*(\Gamma, A)$ and $H_b^*(\Gamma, A)$ can thus be written as the quotients

$$H^n(\Gamma, A) \cong \ker\left(\delta : C^n(\Gamma, A) \to C^{n+1}(\Gamma, A)\right)/\mathrm{im}\left(\delta : C^{n-1}(\Gamma, A) \to C^n(\Gamma, A)\right),$$

$$H_b^n(\Gamma, A) \cong \ker\left(\delta : C_b^n(\Gamma, A) \to C_b^{n+1}(\Gamma, A)\right)/\mathrm{im}\left(\delta : C_b^{n-1}(\Gamma, A) \to C_b^n(\Gamma, A)\right).$$

The inclusion of cocomplexes $C_b^*(\Gamma, A) \subset C^*(\Gamma, A)$ induces a *comparison map*

$$c : H_b^*(\Gamma, A) \longrightarrow H^*(\Gamma, A)$$

on the cohomology groups.

2. Amenable groups

Definition A (discrete) group Γ is said to be *amenable* if there exists a left invariant mean on the Banach space $B(\Gamma)$ of real valued bounded functions on Γ equipped with the norm

$$\|f\|_\infty = \sup\{|f(\gamma| : \gamma \in \Gamma\},$$

i.e. there exists a linear functional $m : B(\Gamma) \to \mathbb{R}$ such that

1. $m(1_\Gamma) = 1$, where 1_Γ denotes the constant function with value 1,
2. $m(f) \geq 0$ whenever $f \geq 0$,
3. $m(L_\gamma f) = m(f)$, for every $f \in B(\Gamma)$ and $\gamma \in \Gamma$, where $L_\gamma : B(\Gamma) \to B(\Gamma)$ denotes the left shift operator defined as $L_\gamma f(\delta) = f(\gamma\delta)$, for every $\delta \in \Gamma$.

Examples: 1) Finite groups are amenable. Indeed, if Γ is finite, then

$$\begin{array}{rlcl} m: & B(\Gamma) & \longrightarrow & \mathbb{R} \\ & f & \longmapsto & \frac{1}{|\Gamma|}\Sigma_{\gamma\in\Gamma} f(\gamma) \end{array}$$

is a left invariant mean on $B(\Gamma)$.

2) Abelian groups are amenable.

3) Since extensions of amenable groups by amenable groups are amenable, it follows that solvable groups are amenable.

Proposition (Trauber) If Γ is an amenable group, then

$$H_b^*(\Gamma, \mathbb{R}) = 0, \text{ for } * > 0.$$

Proof Let $n > 0$ and $f \in C_b^n(\Gamma, \mathbb{R})^\Gamma$ with $\delta f = 0$. We need to show that there exists $h \in C_b^{n-1}(\Gamma, \mathbb{R})^\Gamma$ with $\delta h = f$. Let $m : B(\Gamma) \to \mathbb{R}$ be a left invariant mean. Define $h : \Gamma^n \to \mathbb{R}$ by

$$h(\gamma_1, ..., \gamma_n) = m(f(*, \gamma_1, ..., \gamma_n)),$$

for every $(\gamma_1, ..., \gamma_n) \in \Gamma^n$. Here, $f(*, \gamma_1, ..., \gamma_n)$ denotes the bounded function $\Gamma \to \mathbb{R}$ given by evaluating the first coordinate. We check that, for $(\gamma_0, ..., \gamma_n) \in \Gamma^{n+1}$, we have

$$\begin{aligned} \delta h(\gamma_0, ..., \gamma_n) &= \sum_{i=0}^{n} (-1)^i h(\gamma_0, ..., \widehat{\gamma_i}, ..., \gamma_n) = \sum_{i=0}^{n} (-1)^i m(f(*, \gamma_0, ..., \widehat{\gamma_i}, ..., \gamma_n)) \\ &= m(\sum_{i=0}^{n} (-1)^i f(*, \gamma_0, ..., \widehat{\gamma_i}, ..., \gamma_n)) = f(\gamma_0, ..., \gamma_n), \end{aligned}$$

since, in view of the cocycle relation $\delta f = 0$, the function $\Sigma_{i=0}^n (-1)^i f(*, \gamma_0, ..., \widehat{\gamma_i}, ..., \gamma_n)$ is the constant function with value $f(\gamma_0, ..., \gamma_n)$. It remains to check that h is Γ-invariant:

$$h(\gamma\gamma_1, ..., \gamma\gamma_n) = m(f(*, \gamma\gamma_1, ..., \gamma\gamma_n) = m(f(\gamma^{-1}*, \gamma_1, ..., \gamma_n)),$$

since f is Γ-invariant, and the latter expression is further equal to

$$m(\gamma \cdot f(*, \gamma_1, ..., \gamma_n)) = m(f(*, \gamma_1, ..., \gamma_n),$$

where the last equality follows from the fact that m is left invariant.

Of course, the analogous statement for usual group cohomology is false, since for example one has $H^1(\mathbb{Z}, \mathbb{R}) = \mathbb{R}$. The proof of the proposition does not carry through since the mean needs to be applied to bounded functions. However, for finite groups, any cocycle is obviously bounded, and the same proof shows that the (usual) real valued cohomology of a finite group is trivial.

3. Low degree

It is sometimes convenient, especially in low degree, to work with a different cocomplex computing the (bounded) cohomology of a group, namely the inhomogeneous cocomplex $\overline{C}^*(\Gamma, A)$: In degree n it consists of

$$\overline{C}^n(\Gamma, A) = \{h : \Gamma^n \longrightarrow A\} \text{ and}$$
$$\overline{C}_b^n(\Gamma, A) = \{h : \Gamma^n \longrightarrow A : h \text{ is bounded}\}.$$

The inhomogenous coboundary operator $d : \overline{C}^n(\Gamma, A) \to \overline{C}^{n+1}(\Gamma, A)$ is defined as

$$dh(\gamma_1, ..., \gamma_{n+1}) = h(\gamma_2, ..., \gamma_{n+1}) + \sum_{i=1}^{n} (-1)^i h(\gamma_1, ..., \gamma_i\gamma_{i+1}, ..., \gamma_n) + (-1)^{n+1} h(\gamma_1, ..., \gamma_n).$$

Proposition The cohomology groups $H^*(\Gamma, A)$ and $H_b^*(\Gamma, A)$ can be computed from the cocomplexes $(\overline{C}^*(\Gamma, A), d)$ and $(\overline{C}_b^*(\Gamma, A), d)$ respectively.

Proof The correspondences

$$\begin{array}{rcl} C^n(\Gamma, A)^\Gamma & \longleftrightarrow & \overline{C}^n(\Gamma, A) \\ f & \longmapsto & \{(\gamma_1, ..., \gamma_n) \longmapsto f(1, \gamma_1, ..., \gamma_n)\} \\ \{(\gamma_0, ..., \gamma_n) \longmapsto h(\gamma_0^{-1}\gamma_1, \gamma_1^{-1}\gamma_2, ..., \gamma_{n-1}^{-1}\gamma_n\} & \longleftarrow & h \end{array}$$

define cochain maps which are inverse to each other.

Thus we trade one superfluous variable for an assymmetric coboundary operator.

Degree 0

We have $\overline{C}^0(\Gamma, A) = \{h : \Gamma^0 \to A\} = A$, and for $h \in \overline{C}^0(\Gamma, A)$ and $\gamma \in \Gamma$,

$$dh(\gamma) = h - h = 0,$$

thus any cochain is a cocycle, and since there are no coboundaries, $H^0(\Gamma, A) = \overline{C}^0(\Gamma, A) = A$. Since $\overline{C}^0(\Gamma, A) = \overline{C}_b^0(\Gamma, A)$, the same holds for bounded cohomology: $H_b^0(\Gamma, A) = A$.

Degree 1

Since the coboundary map $\delta : \overline{C}^0(\Gamma, A) \to \overline{C}^1(\Gamma, A)$ is the zero map, there are no coboundaries in degree 1. To determine the cocycles, take $h \in \overline{C}^1(\Gamma, A)$ with $\delta h = 0$. In other words, h is a map $h : \Gamma \to A$ such that

$$dh(\gamma_1, \gamma_2) = h(\gamma_1) - h(\gamma_1\gamma_2) + h(\gamma_2) = 0$$

for every $\gamma_1, \gamma_2 \in \Gamma$, which precisely means that $h : \Gamma \to A$ is a homomorphism. Thus, we get $H^1(\Gamma, A) = Hom(\Gamma, A)$. Since there are no bounded homomorphisms from Γ to $\mathbb{R}$ or $\mathbb{Z}$, we have

$$H_b^1(\Gamma, \mathbb{R}) = H_b^1(\Gamma, \mathbb{Z}) = 0, \text{ for any group } \Gamma.$$

Degree 2: Quasimorphisms

It is classical that the cohomology group $H^2(\Gamma, A)$ is in one-to-one correspondence with isomorphism classes of central extensions of Γ by A (see [4] for this correspondence and (unbounded) group cohomology in general). An (inhomogeneous) cocycle can easily be constructed as the obstruction to the existence of a section of the projection from the central extension onto Γ. The corresponding cohomology class will be representable by a bounded cocycle if the section shows some boundedness properties (in an appropriate sense). For now, we restrict to the case $A = \mathbb{R}$ and concentrate on the kernel of the comparison map

$$H_b^2(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R}).$$

Definition Let Γ be a group. Its *space of quasimorphisms* is defined as

$$QM(\Gamma) = \{f : \Gamma \to \mathbb{R} : \exists C > 0 \text{ such that } |f(\gamma_1) + f(\gamma_2) - f(\gamma_1\gamma_2)| < C, \forall \gamma_1, \gamma_2 \in \Gamma\}.$$

Define a map

$$QM(\Gamma) \longrightarrow \ker(H_b^2(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R}))$$

by sending a quasimorphism $f \in QM(\Gamma)$ to the cohomology class $[\delta f]$. Note that this is well defined since δf, being a coboundary, is of course a cocycle. Also, it is mapped to zero in the Eilenberg Maclane cohomology of Γ, since it is by definition an (a priori unbounded) coboundary. In the kernel of the above map, one finds homomorphisms $f \in \mathrm{Hom}(\Gamma,\mathbb{R})$ since for those one has $\delta f = 0$, and bounded maps $f \in B(\Gamma, \mathbb{R})$ since in this case δf is a *bounded* coboundary. Furthermore, if for a quasimorphism $f \in QM(\Gamma)$, its image $[\delta f]$ is zero in $H_b^2(\Gamma, \mathbb{R})$, then there exists a bounded cochain h such that $\delta f = \delta h$, so that $f = (f - h) + h$ is the sum of a homomorphism and a bounded function. Note that the intersection of $\mathrm{Hom}(\Gamma,\mathbb{R})$ and $B(\Gamma, \mathbb{R})$ consists of the zero map. Finally note that the above map is surjective. We have thus proven:

Proposition Let Γ be a group. There is an isomorphism

$$QM(\Gamma)/(\mathrm{Hom}(\Gamma, \mathbb{R}) \oplus B(\Gamma, \mathbb{R})) \cong \ker(H_b^2(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})).$$

Note that the same results also holds for integral coefficients.

Lecture II

Example (Brooks [3]): Quasimorphisms in the free group F_2

Let $F_2 = < a, b >$ be the free group on the two generators a, b. Pick a word w in a, b, a^{-1}, b^{-1}. Define $f_w : F_2 \to \mathbb{R}$ as

$$f_w(\gamma) = \# \text{ of times } w \text{ occcurs in } \gamma - \# \text{ of times } w^{-1} \text{ occcurs in } \gamma,$$

for every γ in Γ. Note that if w is the empty word, or one of $a^{\pm 1}, b^{\pm 1}$, then f_w is a homomorphism. But if the length of w (i.e. the number of letters $a^{\pm 1}, b^{\pm 1}$ used to write w) is greater or equal to 2, then f_w cannot be a homomorphism (proof below). However, given γ_1, γ_2 in F_2, note that

$$|\delta f_w(\gamma_1, \gamma_2)| = |f_w(\gamma_1) + f_w(\gamma_2) - f_w(\gamma_1, \gamma_2)| \leq \text{length}(w),$$

since the only way an occurence of w in the product $\gamma_1\gamma_2$ is not counted in either γ_1 or γ_2 is if the word w overlaps both on γ_1 and γ_2 (i.e. the word w starts in the end of γ_1 and ends in the beginning of γ_2), and similarly for w^{-1}. Thus f_w is a quasimorphism. Let us now furthermore show that, if length$(w) \geq 2$, then f_w is a nontrivial quasimorphism, that is, it does not belong to $\mathrm{Hom}(F_2,\mathbb{R}) \oplus B(F_2,\mathbb{R})$. For this, note that $\mathrm{Hom}(F_2,\mathbb{R})$ is 2-dimensional, generated by f_a and f_b. If f_w were a sum of a homomorphism and a bounded function, there would exist real numbers $\alpha, \beta \in \mathbb{R}$ such that $f_w - \alpha f_a - \beta f_b$ is a bounded function on F_2. Suppose w is neither a power of a nor of b. Let k be an integer. We have

$$f_w(a^k) - \alpha f_a(a^k) - \beta f_b(a^k) = 0 - \alpha k - 0 = -\alpha k,$$

and the only way $-\alpha k$ can be bounded independently of k, is if $\alpha = 0$. By symmetry, one obtains $\beta = 0$. Thus, f_w has to be bounded. If $w = a^m$, with $|m| \geq 2$, then one can show as above that $\beta = 0$, but then, for any integer k, we have

$$f_{a^m}((ab)^k) - \alpha f_a((ab)^k) = 0 - \alpha k,$$

which again forces $\alpha = 0$, so that we conclude that f_{a^m} has to be bounded. By symmetry, the same conclusion holds for $w = b^m$, when $|m| \geq 2$. It remains to show that f_w is unbounded (for w different from the empty word). We start by observing that the word w^{-1} does not occur in w^2 or any other power w^k. Indeed, if w^{-1} did occur in w^2, then w would have to have, as a reduced word, the form $w = vz$, with $w^{-1} = zv$. But this would force $v = v^{-1}$ and $z = z^{-1}$, so that $v = z = w = id$. If w is cyclically reduced, then

$$\left|f_w(w^k)\right| \geq |k|\,,$$

for any integer k, and in particular f_w is not bounded. If w is not cyclically reduced then it has, as a reduced word, one of the form aza^{-1}, bzb^{-1}, $a^{-1}za$ or $b^{-1}zb$, for some reduced word z. Without loss of generality, suppose $w = aza^{-1}$. Then we have $\left|f_w((wb)^k)\right| \geq |k|$, and we can again conclude that f_w is unbounded.

In conclusion, we have just shown that

$$H_b^2(F_2,\mathbb{R}) \cong \ker(H_b^2(F_2,\mathbb{R}) \to H^2(F_2,\mathbb{R})) \cong QM(F_2)/(\mathrm{Hom}(F_2,\mathbb{R}) \oplus B(F_2,\mathbb{R})) \neq 0.$$

While Brooks gave this first example of a nontrivial quasimorphism, he further wrongly claimed that the f_w´s, where length$(w) \geq 2$, form a free basis of $H_b^2(F_2,\mathbb{R})$. This is false, since as observed by Grigorchuk [13],

$$f_{ab} + f_{a^{-1}b} + f_{ab^{-1}} + f_{a^{-1}b^{-1}}$$

is a bounded function. Grigorchuk also gave an infinite dimensional basis of $H_b^2(F_2,\mathbb{R})$.

This example naturally generalizes to torus knot groups and surface groups (Grigorchuk, [13]), nonelementary word hyperbolic groups (Epstein and Fujiwara, [8]), and further to groups acting on Gromov hyperbolic spaces (Fujiwara, [9]).

Uniform perfection

Definition A group Γ is *perfect* if $\Gamma = [\Gamma,\Gamma]$, where $[\Gamma,\Gamma]$ denotes the commutator subgroup of Γ, i.e. the subgroup generated by the commutators $[x,y] = xyx^{-1}y^{-1}$, for x, y in Γ. A group Γ is *uniformly perfect* if there exists N such that every $\gamma \in \Gamma$ can be written as a product of at most N commutators.

Proposition (Matsumoto-Morita [19]) If Γ is uniformly perfect, then the kernel of the comparison map $H_b^2(\Gamma,\mathbb{R}) \to H^2(\Gamma,\mathbb{R})$ is equal to 0.

Example $SL(n,\mathbb{Z})$, for $n \geq 3$, is uniformly perfect [21]. This is proven by induction on n, the case $n = 3$ relying on the fact that any matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

in $\mathrm{SL}(3,\mathbb{Z})$ can be written as a product of 41 (!) elementary matrices [7], which are commutators.

Note that in contrast, $\mathrm{SL}(2,\mathbb{Z})$ cannot be uniformly perfect since, being isomorphic to the amalgamated product $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$, it contains nontrivial quasimorphisms.

More generally, if Γ is a perfect group, the *commutator length* of an element $\gamma \in \Gamma$ is defined as

$$\ell_{[\Gamma,\Gamma]}(\gamma) = \min\{n : \gamma \text{ is a product of } n \text{ commutators}\}.$$

The *stable length* of γ is then defined as

$$\|\gamma\| = \lim_{n\to\infty} \frac{\ell_{[\Gamma,\Gamma]}(\gamma^n)}{n}.$$

Note that in particular, if Γ is uniformly perfect, then $\|\gamma\| = 0$ for every $\gamma \in \Gamma$.

Theorem (Bavard [1]) The comparison map $H_b^2(\Gamma,\mathbb{R}) \to H^2(\Gamma,\mathbb{R})$ is injective if and only if $\|\gamma\| = 0$ for every $\gamma \in [\Gamma,\Gamma]$.

Actions by homeomorphisms on the circle

This section is based on work by Etienne Ghys. (See his beautifully written papers [10, 12, 11] on the subject for more details.)

Let Γ be a discrete group. Denote by $\mathrm{Homeo}_+(S^1)$ the group of orientation preserving homeomorphisms of the circle. We want to study the dynamics of actions of Γ on the circle S^1 by orientation preserving homeomorphisms, in other words, homomorphisms

$$h : \Gamma \longrightarrow \mathrm{Homeo}_+(S^1).$$

Actions by $\Gamma = \mathbb{Z}$

A homomorphism $\mathbb{Z} \to \mathrm{Homeo}_+(S^1)$ is completely determined by the image of the generator $1 \in \mathbb{Z}$, so that the study of actions by $\mathbb{Z}$ on $\mathrm{Homeo}_+(S^1)$ is equivalent to the understanding of the iterations of a single orientation preserving homeomorphisms $f \in \mathrm{Homeo}_+(S^1)$.

Rotations

Consider S^1 as the quotient $\mathbb{R}/\mathbb{Z}$ and for $\alpha \in \mathbb{R}$, define the rotation by $2\pi\alpha$ as $R_\alpha : S^1 \to S^1$, by mapping $x \mathrm{mod}\mathbb{Z}$ to $R_\alpha(x\mathrm{mod}\mathbb{Z}) = (x + a)\,\mathrm{mod}\mathbb{Z}$. We have the following dichotomy:

$$\begin{aligned} &\alpha \in \mathbb{Q} \text{ if and only if all orbits are finite,} \\ &\alpha \notin \mathbb{Q} \text{ if and only if all orbits are dense.} \end{aligned}$$

Rotation number

To generalize this dichotomy to arbitrary orientation preserving homoemorphisms, we consider the rotation number, which is an invariant introduced by Poincaré and roughly speaking measures how much rotation there is in a homeomorphism.

Denote by $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ the orientation preserving homeomorphisms of the real line $\mathbb{R}$ which commute with integral translations, that is, if $\tilde{f} \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$, then $\tilde{f}(x)+1=\tilde{f}(x+1)$, for every $x \in \mathbb{R}$. There is a natural projection $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_+(S^1)$. The kernel of this projection consists of integral translations of $\mathbb{R}$, so that we have the following central extension of $\mathrm{Homeo}_+(S^1)$ by $\mathbb{Z}$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \longrightarrow \mathrm{Homeo}_+(S^1) \longrightarrow 1.$$

Let $f \in \mathrm{Homeo}_+(S^1)$, and choose a lift $\tilde{f} \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of f and a base point $x_0 \in \mathbb{R}$. Define the *rotation number* of f as

$$\rho(f) = \lim_{n\to\infty} \frac{\tilde{f}^n(x_0)}{n} \mathrm{mod}\mathbb{Z}.$$

To see that the rotation number of f is well defined, we need to check that the limit exists, and does not depend on either x_0 or the choice of the lift. The two latter facts are easy and are left as an exercise. For the first fact we use the following lemma to deduce that the limit

$$\lim_{n\to\infty} \frac{\tilde{f}^n(x_0)}{n} \in \mathbb{R}$$

exists. (This is called the *translation number* of $\tilde{f}$.)

Lemma Let $C>0$ be a positive number and $\{a_n\}_{n\in\mathbb{N}}$ a sequence of nonnegative numbers such that

$$a_{n+m} \le a_n + a_m + C.$$

Then the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to

$$\lim_{n\to\infty} \frac{a_n}{n} = \liminf \frac{a_n}{n} = \inf \frac{a_n}{n}.$$

Proof. Note that the sequence $\{a_n/n\}_{n\in\mathbb{N}}$ is a nonnegative, bounded sequence:

$$0 \le \frac{a_n}{n} \le \frac{a_{n-1}+a_1+C}{n} \le \dots \le \frac{na_1+(n-1)C}{n} \le a_1 + C,$$

and hence the lim inf exists. As we only need the limit to exist, we prove that it is equal to the lim inf (and leave the verification that it is further equal to the infimum to the reader). Let $\epsilon>0$. We will show that $a_n/n < \liminf\{a_n/n\} + \epsilon$ for n big enough. By definition of the lim inf, there exists $N > 3C/\epsilon$ such that

$$\frac{a_N}{N} < \liminf \frac{a_n}{n} + \frac{\epsilon}{3}.$$

Pick $n \ge (3/\epsilon)\max\{a_r \mid r=0,1,...,N-1\}$. There exists $q,r\in\mathbb{N}$ such that $n=qN+r$ with $0\le r\le N-1$. We compute

$$\begin{aligned}\frac{a_n}{n} &\le \frac{a_{qN}+a_r+C}{n} \le \frac{qa_N+a_r+qC}{qN+r} \le \frac{a_N}{N}+\frac{a_r}{r}+\frac{C}{N}\\ &\le (\liminf \frac{a_n}{n}+\frac{\epsilon}{3})+\frac{\epsilon}{3}+\frac{\epsilon}{3},\end{aligned}$$

which finishes the proof of the lemma.

Exercise: Check that the definition of the rotation number $\rho(f)$ of f does not depend on either x_0 or the choice of the lift.

Example: As above, let $R_\alpha : S^1 \to S^1$be the rotation by $2\pi\alpha$, for some $\alpha \in \mathbb{R}$. Then the map

$$\begin{array}{rccc} \tilde{R}_\alpha : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & x+\alpha \end{array}$$

is a lift of R_α, and choosing as base point $x_0 = 0$, we see that $\tilde{R}_\alpha^n(0) = n\alpha$, and hence

$$\rho(R_\alpha) = \lim_{n\to\infty} \frac{\tilde{R_\alpha}^n(0)}{n} \mathrm{mod}\mathbb{Z} = \lim_{n\to\infty} \frac{n\alpha}{n} \mathrm{mod}\mathbb{Z} = \alpha \mathrm{mod}\mathbb{Z}.$$

Theorem (Poincaré) Let $f \in \mathrm{Homeo}_+(S^1)$. The following dichotomy holds:

$\rho(f) \in \mathbb{Q}/\mathbb{Z}$ if and only if	there exists a finite orbit and all the orbits which are not periodic are asymptotic to a periodic one.
$\rho(f) \notin \mathbb{Q}/\mathbb{Z}$ if and only if	the action is minimal (definition below) or there exists an exceptional minimal set which is a Cantor set.

Minimal sets

More generally, consider now a group action given by a group homomorphism $h : \Gamma \to \mathrm{Homeo}_+(S^1)$. A *minimal set* is a subset of S^1 which is a minimal nonempty, closed, $h(\Gamma)$-invariant subset. The action is said to be *minimal* if S^1 itself is a minimal set, or equivalently, all orbits are dense.

For a general group action $h : \Gamma \to \mathrm{Homeo}_+(S^1)$, one of the following holds:

1. There is a finite orbit.

2. The action is minimal

3. There is a unique exceptional minimal set, that is an invariant Cantor set K such that the orbit of each point in K is dense in K.

Note that the existence of a (in general non unique) minimal set follows from Zorn's Lemma. Given a minimal set K, denote by $dK = K \setminus \mathrm{interior}(K)$ its topological boundary, and by K' its set of limit points. Since both dK and K' are closed, $h(\Gamma)$-invariant subset, it follows from the minimality of K that there is the following trichotomy:

1. $K' = \emptyset$. This implies that K is finite, so there is a finite orbit.

2. $dK = \emptyset$. This implies that $K = S^1$ and hence that all orbits are dense. (Since the closure of an orbit is a closed, $h(\Gamma)$-invariant subset.)

3. $K = K' = dK$. This means that K is a compact perfect subset of the circle with empty interior and is one definition of a Cantor set. The uniqueness of K follows (modulo a small argument) from that K is contained in the closure of any orbit.

Lecture III

The (bounded) Euler class

Consider again the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \longrightarrow \mathrm{Homeo}_+(S^1) \longrightarrow 1.$$

It is classical that isomorphism classes of central extensions are in one-to-one correspondence with cohomology classes in $H^2(\mathrm{Homeo}_+(S^1), \mathbb{Z})$. In fact, it is easy to exhibit a cocycle representing the cohomology class corresponding to the given central extension: Choose a set theoretic section $s : \mathrm{Homeo}_+(S^1) \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of the projection. Note that s can impossibly be a homomorphism (since otherwise $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ would be the product of $\mathbb{Z}$ and $\mathrm{Homeo}_+(S^1)$. Instead, define

$$e : \mathrm{Homeo}_+(S^1)^2 \longrightarrow \mathbb{Z}$$

to measure how far s is from being a homomorphism by the relation

$$T_{e(f_1,f_2)} s(f_1 f_2) = s(f_1)s(f_2),$$

for every $f_1, f_2 \in \mathrm{Homeo}_+(S^1)$, where $T_n : \mathbb{R} \to \mathbb{R}$, for $n \in \mathbb{Z}$ denotes the integral translation by n.

Exercise: Check that $de = 0$. In particular, e is an (inhomogeneous) cocycle and hence determines a cohomology class $[e] \in H^2(\mathrm{Homeo}_+(S^1), \mathbb{Z})$, called the *Euler class*.

For example, let us choose the following section $s : \mathrm{Homeo}_+(S^1) \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$: For $f \in \mathrm{Homeo}(S^1)$, define $s(f) = \widetilde{f} \in \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ to be the unique homeomorphism of the real line such that

$$\widetilde{f}(0) \in [0, 1[\,.$$

Let now $e : \mathrm{Homeo}_+(S^1)^2 \to \mathbb{Z}$ denote the cocycle obtained from this particular section.

Proposition The cocycle $e : \mathrm{Homeo}_+(S^1)^2 \to \mathbb{Z}$ takes only the values 0 and 1.

Proof Take $f_1, f_2 \in \mathrm{Homeo}_+(S^1)$. Since $\widetilde{f_1 f_2}$ and $\widetilde{f_1}\widetilde{f_2}$ differ by an integral translation, we must have

$$\widetilde{f_1 f_2} T_n = \widetilde{f_1}\widetilde{f_2},$$

for some n, and the proposition amounts to showing that either $n = 0$ or $n = 1$. By definition,

$$\widetilde{f_1}(0) \in [0, 1[\,,$$

and since $\widetilde{f_2}(0) \in [0, 1[$ and $\widetilde{f_2}(1) \in [1, 2[$, we have $\widetilde{f_2}\,[0, 1[) \subset [0, 2[$, so that we get on the one hand that

$$\widetilde{f_1}\widetilde{f_2}(0) \in [0, 2[\,.$$

On the other hand, we have

$$\widetilde{f_1 f_2}(n) \in [n, n+1[\,.$$

Evaluating the equality $\widetilde{f_1 f_2} T_n = \widetilde{f_1}\widetilde{f_2}$ on 0 gives

$$[n, n+1[\, \ni \widetilde{f_1 f_2}(n) = \widetilde{f_1 f_2} T_n(0) = \widetilde{f_1}\widetilde{f_2}(0) \in [0, 2[\,,$$

so that n has to be equal to 0 or 1 as claimed.

In particular, the cocycle e also determines a bounded cohomology class $[e]_b \in H_b^2(\mathrm{Homeo}_+(S^1), \mathbb{Z})$. It turns out that the bounded class $[e]_b$ is a natural generalization of the rotation number. Before proving that in the Proposition below, let us first compute the bounded cohomology group $H_b^2(\mathbb{Z}, \mathbb{Z})$.

Lemma There is a canonical isomorphism $H_b^2(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$.

Proof: Consider the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$. Chasing in the induced short exact sequence of cocomplexes

$$0 \to C^*(\mathbb{Z}, \mathbb{Z}) \to C^*(\mathbb{Z}, \mathbb{R}) \to C^*(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \to 0,$$

it is classical and elementary to check that one gets a long exact sequence

$$0 \to H^0(\mathbb{Z}, \mathbb{Z}) \to H^0(\mathbb{Z}, \mathbb{R}) \to H^0(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \to H^1(\mathbb{Z}, \mathbb{Z}) \to H^1(\mathbb{Z}, \mathbb{R}) \to H^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \to ...$$

Furthermore, the same long exact sequence is valid for the bounded cohomology groups:

$$0 \to H_b^0(\mathbb{Z}, \mathbb{Z}) \to H_b^0(\mathbb{Z}, \mathbb{R}) \to H_b^0(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \to H_b^1(\mathbb{Z}, \mathbb{Z}) \to H_b^1(\mathbb{Z}, \mathbb{R}) \to H_b^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \to ...$$

Recall that since $\mathbb{Z}$ is amenable, its real valued bounded cohomology vanishes in degree above 0, so that we get from the above long exact sequence that

$$H_b^*(\mathbb{Z}, \mathbb{Z}) \cong H_b^{*-1}(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = H^{*-1}(\mathbb{Z}, \mathbb{R}/\mathbb{Z}).$$

In particular, for $* = 2$, we get

$$H_b^2(\mathbb{Z}, \mathbb{Z}) \cong H^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z}.$$

(And for $* > 2$, one has $H_b^*(\mathbb{Z}, \mathbb{Z}) = 0$.)

Proposition Let $h : \mathbb{Z} \to \mathrm{Homeo}_+(S^1)$ be a homomorphism. Then

$$h^*([e]_b) \in H_b^2(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$$

is the rotation number of $h(1)$.

Proof: By definition, the connecting homomorphism (or Bockstein homomorphism) $H^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \to H_b^2(\mathbb{Z}, \mathbb{Z})$ has the following description: For a real number $x \in \mathbb{R}$, denote by $x\mathrm{mod}1$ the only real number in the interval $[0, 1[$ such that $x - x\mathrm{mod}1$ is an integer. In fact, $x - x\mathrm{mod}1$ is the (lower) integral part of x, which we denote by $\lfloor x \rfloor$. The connecting homomorphism is now given as

$$\begin{array}{rcl} H^1(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z} & \longrightarrow & H_b^2(\mathbb{Z}, \mathbb{Z}) \\ x\mathrm{mod}\mathbb{Z} & \longmapsto & \{(n, m) \mapsto (nx)\mathrm{mod}1 + (mx)\mathrm{mod}1 - (n + m)x\mathrm{mod}1\}. \end{array}$$

But since $x - x\mathrm{mod}1 = \lfloor x \rfloor$, the latter cocycle can be rewritten as

$$(n, m) \mapsto \lfloor (n + m)x \rfloor - \lfloor nx \rfloor - \lfloor mx \rfloor. \tag{1}$$

Let us describe the converse map $H_b^2(\mathbb{Z}, \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$. Take a cohomology class $[c] \in H_b^2(\mathbb{Z}, \mathbb{Z})$ represented by a bounded cocycle $c : \mathbb{Z}^2 \to \mathbb{Z}$. Since $H^2(\mathbb{Z}, \mathbb{Z}) = 0$, the cocycle c is the coboundary $c = db$ of an (unbounded in general) cochain $b : \mathbb{Z} \to \mathbb{Z}$. Define

$$\begin{array}{rcl} \bar{b} : \mathbb{Z} & \longrightarrow & \mathbb{R} \\ k & \longmapsto & \lim_{n\to\infty} \frac{b(kn)}{n}, \end{array}$$

and note that the limit exists for the same reason as the rotation (or translation) number treated above. Now it is clear that $\overline{b}$ is a homomorphism, and that the difference $\overline{b} - b$ is a bounded function. However, $\overline{b}$ will not be integral in general, so we consider instead

$$\begin{array}{rccc} \lfloor \overline{b} \rfloor : & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ & k & \longmapsto & \lfloor \overline{b}(k) \rfloor . \end{array}$$

The difference $\lfloor \overline{b} \rfloor - b$ is still bounded, and hence db and $d\lfloor \overline{b} \rfloor$ represent the same cohomology class in $H^2_b(\mathbb{Z}, \mathbb{Z})$. Note that the cocycle $d\lfloor \overline{b} \rfloor$ has exactly the form described in (1), and the inverse of the connecting homomorphism is given by

$$\begin{array}{rcl} H^2_b(\mathbb{Z}, \mathbb{Z}) & \longrightarrow & \mathbb{R}/\mathbb{Z} \\ {[db]} & \longmapsto & -\lim_{n\to\infty} \frac{b(n)}{n} \mathrm{mod}\mathbb{Z}. \end{array}$$

Suppose now that we are given a representation $h : \mathbb{Z} \longrightarrow \mathrm{Homeo}_+(S^1)$, and denote by $f = h(1)$ the image of the generator 1. Since $\widetilde{f^k}$ and $\left(\widetilde{f}\right)^k$ differ by an integral translation, there exists $u : \mathbb{Z} \to \mathbb{Z}$ such that

$$T_{u(k)} \circ \widetilde{f^k} = \left(\widetilde{f}\right)^k . \tag{2}$$

By the definition of $h^*(e)(n, m)$, we have

$$T_{h^*(e)(n,m)} = \widetilde{f^n} \circ \widetilde{f^m} \circ \left(\widetilde{f^{n+m}}\right)^{-1} = \widetilde{f^n} \circ \widetilde{f^m} \circ \left(\widetilde{f^{n+m}}\right)^{-1} \circ \left(\widetilde{f}\right)^{-n} \circ \left(\widetilde{f}\right)^{-m} \circ \left(\widetilde{f}\right)^{n+m},$$

and since the different lifts commute with each other, this expression can be rewritten as

$$\widetilde{f^n} \circ \left(\widetilde{f}\right)^{-n} \circ \widetilde{f^m} \circ \left(\widetilde{f}\right)^{-m} \circ \left(\widetilde{f^{n+m}}\right)^{-1} \circ \left(\widetilde{f}\right)^{n+m} = T_{-u(n)} \circ T_{-u(m)} \circ T_{u(n+m)},$$

which shows that

$$-du = h^*(e).$$

Evaluating the homeomorphisms given in (2) on 0 gives

$$u(k) + \widetilde{f^k}(0) = \left(\widetilde{f}\right)^k (0),$$

so that since $\widetilde{f^k}(0) \in [0, 1[$, it follows that

$$u(k) = \left\lfloor \left(\widetilde{f}\right)^k (0) \right\rfloor .$$

We can now conclude that the image of $h^*([e]_b) \in H^2_b(\mathbb{Z}, \mathbb{Z})$ under the inverse of the connecting homomorphism is

$$\lim_{n\to\infty} \frac{u(n)}{n} \mathrm{mod}\mathbb{Z} = \lim_{n\to\infty} \frac{\left(\tilde{f}\right)^n (0)}{n} \mathrm{mod}\mathbb{Z} = \rho(f),$$

which finishes the proof of the lemma.

Definition A (not necessarily bijective nor continous) map $\varphi : S^1 \to S^1$ is said to be a *degree* 1 *monotone map* if there exists a lift $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ which is (not necessarily strictly) monotone.

Definition Let Γ be a group, and $h_1, h_2 : \Gamma \to \mathrm{Homeo}_+(S^1)$ two homomorphisms. We say that h_1 is *semi-conjugated* to h_2 if there exists nonempty invariant subsets $K_1, K_2 \subset S^1$ for h_1, h_2 respectively, and a degree 1 monotone map $\varphi : S^1 \to S^1$ such that φ induces a bijection between K_2 and K_1 and $h_1(\gamma) \circ \varphi = \varphi \circ h_2(\gamma)$, for every $\gamma \in \Gamma$.

Exercise Check that semi-conjugation indeed is an equivalence relation. Reflexivity and transitivity are immediate. For the symmetry, use that if φ induces a bijection between invariant subsets K_2 and K_1, and $h_1(\gamma) \circ \varphi = \varphi \circ h_2(\gamma)$, for a degree 1 monotone map φ, then the map $\varphi^* : S^1 \to S^1$ realizing the semi-conjugacy between h_2 and h_1 has to be defined as the inverse of φ on K_1, and on its complement $S^1 \setminus K_1$ by its lift $\tilde{\varphi}^* : \mathbb{R} \to \mathbb{R}$ given as

$$\tilde{\varphi}^*(x) = \sup\left\{x' \in \widetilde{K}_2 \mid \tilde{\varphi}(x') \leq \tilde{\varphi}(x)\right\},$$

where $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ is a lift of φ, and $\widetilde{K}_2$ is the preimage of K_2 under the projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$.

Warning! The classical definition of semi-conjugacy, requires the existence of a *continuous* degree 1 monotone map intertwining the actions of h_1 and h_2. But this definition is not symmetric, and we need a symmetric definition, since we want to characterize actions for which the pullback of the bounded Euler class is equal (and equality is a symmetric condition). In the first papers of Ghys on the subject (for example [10]), and in most of the references thereafter, semi-conjugacy, in the context of bounded cohomology, is defined by relaxing the continuity condition. But this does not work! By doing so, one obtains that all actions are semi-conjugated to the constant action $1 : \Gamma \to \mathrm{Homeo}_+(S^1)$ mapping Γ constantly on the identity. Indeed, for any $x_0 \in S^1$, the constant map $\varphi_{x_0}(x) \equiv x_0$ is a degree one monotone map, and

$$\varphi_{x_0} = 1 \circ \varphi_{x_0} = \varphi_{x_0} \circ h(\gamma).$$

The definition above is essentially what one can find in Ghys' later paper [12].

Theorem (Ghys) Let Γ be a finitely generated group, and $h_1, h_2 : \Gamma \to \mathrm{Homeo}_+(S^1)$ two homomorphisms. Then

$$h_1^*([e]_b) = h_2^*([e]_b) \in H_b^2(\Gamma, \mathbb{Z})$$

if and only if h_1 and h_2 are semi-conjugated to each other.

Proof of Ghys' Theorem. We start by showing that if $h_1, h_2 : \Gamma \to \mathrm{Homeo}_+(S^1)$ are semi-conjugated to each other, then $h_1^*([e]_b) = h_2^*([e]_b)$. Let thus $\varphi : S^1 \to S^1$ be a degree one monotone map satisfying $h_1(\gamma) \circ \varphi = \varphi \circ h_2(\gamma)$, for every $\gamma \in \Gamma$, and inducing a bijection on respective invariant subsets of S^1. Let $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ be the unique lift of φ with $\widetilde{\varphi}(0) \in [0, 1[$. Because $\widetilde{\varphi}$ induces a bijection on the lifts of the invariant subsets (otherwise it is not true in general), the two products of the lifts differ by an integral translation, and there exists $u : \Gamma \to \mathbb{Z}$ such that

$$\widetilde{h_1(\gamma)} \circ \widetilde{\varphi} \circ T_{u(\gamma)} = \widetilde{\varphi} \circ \widetilde{h_2(\gamma)}.$$

Claim 1 $du = h_2^*(e) - h_1^*(e)$.

Proof: Let $\gamma_1, \gamma_2 \in \Gamma$. The above equality applied to γ_1, γ_2 and $\gamma_1\gamma_2$ gives

$$\begin{aligned}
\widetilde{\varphi} &= \widetilde{h_1(\gamma_1)} \circ \widetilde{\varphi} \circ \left(\widetilde{h_2(\gamma_1)}\right)^{-1} \circ T_{u(\gamma_1)},\\
\widetilde{\varphi} &= \widetilde{h_1(\gamma_2)} \circ \widetilde{\varphi} \circ \left(\widetilde{h_2(\gamma_2)}\right)^{-1} \circ T_{u(\gamma_2)},\\
\widetilde{\varphi} &= \left(\widetilde{h_1(\gamma_1\gamma_2)}\right)^{-1} \circ \widetilde{\varphi} \circ \widetilde{h_2(\gamma_1\gamma_2)} \circ T_{-u(\gamma_1\gamma_2)}.
\end{aligned}$$

Substituting the second equation in the first one, and the last one in the second one we get

$$\widetilde{\varphi} = \widetilde{h_1(\gamma_1)} \circ \widetilde{h_1(\gamma_2)} \circ \Big(\widetilde{h_1(\gamma_1\gamma_2)}\Big)^{-1} \circ \widetilde{\varphi} \circ \widetilde{h_2(\gamma_1\gamma_2)} \circ \Big(\widetilde{h_2(\gamma_2)}\Big)^{-1} \circ \Big(\widetilde{h_2(\gamma_1)}\Big)^{-1} \circ T_{u(\gamma_1)+u(\gamma_2)-u(\gamma_1\gamma_2)}.$$

By definition,

$$T_{e(h_1(\gamma_1),h_1(\gamma_2))} = \widetilde{h_1(\gamma_1)} \circ \widetilde{h_1(\gamma_2)} \circ \Big(\widetilde{h_1(\gamma_1\gamma_2)}\Big)^{-1}$$

and also

$$T_{e(h_2(\gamma_1)h_2(\gamma_2))} = \widetilde{h_2(\gamma_1)} \circ \widetilde{h_2(\gamma_2)} \circ \Big(\widetilde{h_2(\gamma_1\gamma_2)}\Big)^{-1},$$

so that we obtain

$$\widetilde{\varphi} = \widetilde{\varphi} \circ T_{h_!^*(e)(\gamma_1,\gamma_2)-h_2^*(e)(\gamma_!,\gamma_2)+u(\gamma_1)+u(\gamma_1)-u(\gamma_1\gamma_2)},$$

which proves the claim.

Claim 2 The cochain u is bounded.

Proof: Let $\gamma \in \Gamma$. By definition, $\widetilde{h_1(\gamma)}(0) \in [0,1[$ and $\widetilde{h_2(\gamma)}(0) \in [0,1[$. Also, recall that we have chosen $\widetilde{\varphi}$ such that $\widetilde{\varphi}(0) \in [0,1[$. We have on the one hand

$$\widetilde{\varphi} \circ \widetilde{h_2(\gamma)}(0) \in \widetilde{\varphi}([0,1[) \subset [0,2[\,,$$

and on the other hand

$$\widetilde{h_1(\gamma)} \circ \widetilde{\varphi} \circ T_{u(\gamma)}(0) = T_{u(\gamma)} \circ \widetilde{h_1(\gamma)} \circ \widetilde{\varphi}(0) \in T_{u(\gamma)} \circ \widetilde{h_1(\gamma)}([0,1[)) \subset T_{u(\gamma)}([0,2[)) = [u(\gamma), u(\gamma)+2[),$$

so the only chance for those two expressions to be equal is if the subsets $[0,2[$ and $[u(\gamma), u(\gamma)+2[$ have a nonempty intersection, which forces $u(\gamma) \in \{-1,0,1\}$.

Since the cocycles $h_1^*(e)$ and $h_2^*(e)$ differ by a bounded $\mathbb{Z}$-valued coboundary, they indeed define the same bounded cohomology class in $H_b^2(\Gamma,\mathbb{Z})$.

For the converse, suppose that $h_1^*([e]_b) = h_2^*([e]_b) \in H_b^2(\Gamma,\mathbb{Z})$. By defininition, there exists a bounded cochain $u : \Gamma \to \mathbb{Z}$ such that $h_2^*(e) - h_1^*(e) = du$. Let

$$0 \longrightarrow \mathbb{Z} \longrightarrow \overline{\Gamma} \longrightarrow \Gamma \longrightarrow 1$$

be the central extension corresponding to the cohomology class $h_1^*([e]) = h_2^*([e]) \in H^2(\Gamma,\mathbb{Z})$. Let $s_1, s_2 : \Gamma \to \overline{\Gamma}$ be the two set theoretic sections giving rise to the cocycles $h_1^*(e)$ and $h_2^*(e)$ respectively and note that $s_2(\gamma) = s_1(\gamma)i(u(\gamma))$, where i denotes the injection $i : \mathbb{Z} \to \overline{\Gamma}$.

Define, for $j = 1,2$, homomorphisms $\overline{h}_j : \overline{\Gamma} \to \mathrm{Homeo}_{\mathbb{Z}}\mathbb{R}$ by

$$\overline{h}_j(i(m)s_j(\gamma)) = T_m \circ \widetilde{h_j(\gamma)},$$

for every $\gamma \in \Gamma$.

Exercise Check that $\overline{h}_j$ indeed is a group homomorphism.

Define a map $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{\varphi}(t) = \sup\left\{(\overline{h}_1(\alpha^{-1}) \circ \overline{h}_2(\alpha))(t) \mid \alpha \in \overline{\Gamma}\right\},$$

for $t \in \mathbb{R}$.

Claim 3 $\widetilde{\varphi}$ is well defined, i.e the supremum is not $+\infty$.

Proof: The claim follows from the fact that for any $t \in \mathbb{R}$, the set $\left\{(\overline{h}_1(\alpha^{-1}) \circ \overline{h}_2(\alpha))(t) \mid \alpha \in \overline{\Gamma}\right\}$ is a bounded subset of $\mathbb{R}$. Since both $\overline{h}_1(\alpha^{-1})$ and $\overline{h}_2(\alpha)$ commute with integral translation, we have

$$(\overline{h}_1(\alpha^{-1}) \circ \overline{h}_2(\alpha))(t+m) = (\overline{h}_1(\alpha^{-1}) \circ \overline{h}_2(\alpha))(t) + m,$$

so that we can restrict to $t \in [-1, 0[$. Furthermore, both $\overline{h}_1(\alpha^{-1})$ and $\overline{h}_2(\alpha)$ are monotone and hence their composition also, so for $t \leq 0$ we have

$$(\overline{h}_1(\alpha^{-1}) \circ \overline{h}_2(\alpha))(t) \leq (\overline{h}_1(\alpha^{-1}) \circ \overline{h}_2(\alpha))(0),$$

which shows that we can without loss of generality suppose that $t = 0$. Since

$$\overline{h}_1(\alpha^{-1}) \circ \overline{h}_2(\alpha) = (\overline{h}_1((T_m\alpha)^{-1}) \circ \overline{h}_2(T_m\alpha),$$

for every $\alpha \in \overline{\Gamma}$, and furthermore every element in $\overline{\Gamma}$ can be written (uniquely) as $\alpha = i(m)s_1(\gamma)$, it is enough to bound the set $\left\{(\overline{h}_1(s_1(\gamma)^{-1}) \circ \overline{h}_2(s_1(\gamma))(0) \mid \gamma \in \Gamma\right\}$. On the one hand, we have

$$\overline{h}_1(s_1(\gamma))^{-1}(0) = \left(\widetilde{h_1(\gamma)}\right)^{-1}(0) \in \left]-1, 0\right],$$

and on the other hand

$$\overline{h}_2(s_1(\gamma)) = \overline{h}_2(s_2(\gamma)i(-u(\gamma))) = T_{-u(\gamma)} \circ \widetilde{h_2(\gamma)}.$$

Together, this implies that

$$\overline{h}_1(s_1(\gamma))^{-1}\overline{h}_2(s_1(\gamma))(0) \in \left]-u(\gamma) - 1, -u(\gamma) + 1\right[,$$

which proves the claim.

Claim 4 $\widetilde{\varphi}$ is monotone and commutes with integral translations.

Proof: Since, for every $\alpha \in \overline{\Gamma}$, both $\overline{h}_1(\alpha^{-1})$ and $\overline{h}_2(\alpha)$ commute with integral translation, the same is true for $\widetilde{\varphi}$. Moreover, Since both $\overline{h}_1(\alpha^{-1})$ and $\overline{h}_2(\alpha)$ are monotone, if $t \leq t'$, then

$$(\overline{h}_1(\alpha^{-1}) \circ \overline{h}_2(\alpha))(t) \leq (\overline{h}_1(\alpha^{-1}) \circ \overline{h}_2(\alpha))(t'),$$

for every $\alpha \in \overline{\Gamma}$, and hence also

$$\widetilde{\varphi}(t) = \sup_{\alpha \in \overline{\Gamma}} \left\{(\overline{h}_1(\alpha^{-1}) \circ \overline{h}_2(\alpha))(t)\right\} \leq \sup_{\alpha \in \overline{\Gamma}} \left\{(\overline{h}_1(\alpha^{-1}) \circ \overline{h}_2(\alpha))(t')\right\} = \widetilde{\varphi}(t'),$$

which proves the claim.

Claim 5 $\widetilde{\varphi} \circ \overline{h}_2(\overline{\gamma}) = \overline{h}_1(\overline{\gamma}) \circ \widetilde{\varphi}$, for every $\overline{\gamma} \in \overline{\Gamma}$.

Proof: Let $\overline{\gamma} \in \overline{\Gamma}$ and $t \in \mathbb{R}$. We have

$$\begin{aligned}
\widetilde{\varphi} \circ \overline{h}_2(\overline{\gamma})(t) &= \widetilde{\varphi}(\overline{h}_2(\overline{\gamma})(t)) \\
&= \sup\left\{(\overline{h}_1(\alpha^{-1}) \circ \overline{h}_2(\alpha))(\overline{h}_2(\overline{\gamma})(t)) \mid \alpha \in \overline{\Gamma}\right\} \\
&= \sup\left\{\overline{h}_1(\overline{\gamma}) \circ \overline{h}_1(\overline{\gamma}^{-1}\alpha^{-1}) \circ \overline{h}_2(\alpha\overline{\gamma}))(t)) \mid \alpha \in \overline{\Gamma}\right\} \\
&= \overline{h}_1(\overline{\gamma}) \sup\left\{(\overline{h}_1((\alpha')^{-1}) \circ \overline{h}_2(\alpha'))(t)) \mid \alpha \in \overline{\Gamma}\right\} \\
&= \overline{h}_1(\overline{\gamma}) \circ \widetilde{\varphi}(t),
\end{aligned}$$

where to obtain the fourth equality, we have done the change of variable $\alpha\overline{\gamma} = \alpha'$ and used the fact that $\overline{h}_1(\overline{\gamma})$ is monotone.

It is immediate from Claim 4 that $\widetilde{\varphi}$ descends to a unique degree one monotone map $\varphi : S^1 \to S^1$, and it follows from Claim 5 that

$$\varphi \circ h_2(\gamma) = h_1(\gamma) \circ \varphi,$$

for every $\gamma \in \Gamma$. Let now $\widetilde{K}_1$ be the image of $\widetilde{\varphi}$, and define $\widetilde{K}_2 \subset \mathbb{R}$ as

$$\widetilde{K}_2 = \left\{ \sup \{x \in \mathbb{R} \mid \widetilde{\varphi}(x) = y\} \middle| \, y \in \widetilde{K}_1 \right\}.$$

Note that $\widetilde{K}_1$ and $\widetilde{K}_2$ are invariant by $\overline{h}_1$ and $\overline{h}_2$ respectively, and $\widetilde{\varphi}$ induces a bijection between $\widetilde{K}_2$ and $\widetilde{K}_1$. Taking now K_1 and K_2 to be the images of $\widetilde{K}_1$ and $\widetilde{K}_2$ under the natural projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$, we have just shown that h_1 and h_2 are semi-conjugated, which finishes the proof of the theorem.

Lecture IV

Consequences of Ghys' Theorem

Corollary 1 Let $h : \Gamma \to \mathrm{Homeo}_+(S^1)$ be a homomorphism. Then $h^*([e]_b) = 0 \in H^2_b(\Gamma, \mathbb{Z})$ if and only if the action has a fixed point.

Proof Let $1 : \Gamma \to \mathrm{Homeo}_+(S^1)$ denote the trivial representation (mapping Γ constantly on the identity), and note that $h^*(1) = 0 \in H^2_b(\Gamma, \mathbb{Z})$. Thus, by Ghys' Theorem, $h^*([e]_b) = 0 \in H^2_b(\Gamma, \mathbb{Z})$ if and only if h is semi-conjugated to 1. Suppose that h is semi-conjugated to 1. Then there exists a degree 1 monotone map $\varphi : S^1 \to S^1$ such that

$$h(\gamma) \circ \varphi = \varphi \circ 1 = \varphi.$$

In particular, any point in the image of φ is a fixed point for h. Conversely, let x_0 be a fixed point fo h, and let $\varphi_{x_0} : S^1 \to S^1$ be the constant map onto x_0. This is a degree 1 monotone map. Furthermore, we have

$$h(\gamma) \circ \varphi_{x_0} = \varphi_{x_0} \circ 1.$$

Take now $K_1 = K_2 = \{x_0\}$. Note that this set is invariant under both h and 1, and clearly, φ_{x_0} induces a bijection between K_2 and K_1, so that h is semi-conjugated to 1.

Consequence (Ghys, Burger-Monod) For $n \geq 3$, any action by $\mathrm{SL}(n, \mathbb{Z})$ by orientation preserving homeomorphisms of the circle has to have a fixed point! Indeed, $H^2_b(\mathrm{SL}(n, \mathbb{Z}), \mathbb{Z}) = 0$.

If only some weaker vanishing of the bounded cohomology is known, then one can prove, similarly as above:

Corollary 2 Suppose that $H^2_b(\Gamma, \mathbb{R}) = 0$ and that the commutator subgroup $[\Gamma, \Gamma]$ has finite index in Γ. Then any action by orientation preserving homeomorphisms of Γ on the circle has a finite orbit.

The fact that the comparison map is injective in degree 2 for $\mathrm{SL}(n, \mathbb{Z})$ generalizes to higher rank lattice, so the following consequence is immediate:

Consequence (Ghys [11], Burger-Monod [6]) Let Γ be a lattice in a higher rank simple Lie group. Then every action by a Γ on the circle by orientation preserving homeomorphisms has a finite orbit.

In fact, the authors further show that if the action is further by C^1-diffeomorphisms, then the action factors through a finite group (this is much harder than proving the mere existence of a finite orbit). For homeomorphisms, it is still an open question.

Corollary 3 Suppose that $H_b^2(\Gamma, \mathbb{R}) = 0$. Then any action by orientation preserving homeomorphisms of Γ on the circle is semi-conjugated to an action by rotation.

Milnor-Wood inequalities

Let M be a smooth manifold, and let $h : \pi_1(M) \to \mathrm{Homeo}_+(S^1)$ be a homomorphism. On the product $\widetilde{M} \times S^1$, there is a natural diagonal action of the fundamental group $\pi_1(M)$, given by Deck transformations on the universal cover $\widetilde{M}$ of M and by the action induced by h on the circle. The quotient $\pi_1(M) \setminus \widetilde{M} \times S^1$ is a natural circle bundle over M:

$$\begin{array}{c} \pi_1(M) \setminus \widetilde{M} \times S^1 \\ \downarrow \\ M. \end{array}$$

A circle bundle ξ over a smooth manifold M is said to be *flat* if it is induced by a homomorphism $h : \pi_1(M) \to \mathrm{Homeo}_+(S^1)$.

Given a circle bundle ξ over a smooth manifold M, the obstruction to the existence of a section of ξ defines a cohomology class $\varepsilon_2(\xi) \in H^2(M, \mathbb{Z})$ called the *Euler class*. Of course, the reader will not be surprised by the fact that if ξ is flat, induced by the homomorphism $h : \pi_1(M) \to \mathrm{Homeo}_+(S^1)$, then the Euler class $\varepsilon_2(\xi) \in H^2(M, \mathbb{Z})$ is precisely the image, under the natural map $H^2(\pi_1(M), \mathbb{Z}) \to H^2(M, \mathbb{Z})$, of the pullback by h^* of the cohomology class $[e] \in \mathrm{Homeo}_+(S^1)$ considered previously. Thus the map $H^2(\pi_1(M), \mathbb{Z}) \to H^2(M, \mathbb{Z})$ sends $h^*([e])$ to $\varepsilon_2(\xi)$.

If M is an oriented 2-dimensional manifold, then its second singular cohomology group $H^2(M, \mathbb{Z})$ is, by Poincaré duality, isomorphic to $\mathbb{Z}$. The isomorphism is given by taking the Kronecker product of a cohomology class with the fundamental class $[M] \in H_2(M, \mathbb{Z})$. If, again, ξ is a circle bundle over M, then we define its *Euler number* $\chi(\xi) \in \mathbb{Z}$ to be the image of the Euler class $\varepsilon_2(\xi)$ under this isomorphism.

Theorem (Wood [23]) Let ξ be a circle bundle over a closed oriented surface Σ_g of genus $g \geq 1$. Then ξ is flat if and only if

$$|\chi(\xi)| \leq 2(g-1).$$

Proof We will prove only the "only if" part. Suppose that ξ is induced by the homomorphism $h : \pi_1(\Sigma_g) \to \mathrm{Homeo}_+(S^1)$. Recall that a surface group has the following presentation

$$\pi_1(\Sigma_g) = \left\langle a_1, b_1, ..., a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle.$$

The theorem follows from the observation that

$$\widetilde{h(a_1)}\widetilde{h(b_1)}\widetilde{h(a_1^{-1})}\widetilde{h(b_1)}^{-1} \cdot ... \cdot \widetilde{h(a_g)}\widetilde{h(b_g)}\widetilde{h(a_g^{-1})}\widetilde{h(b_g^{-1})}$$

is an integral translation by precisely $\chi(\xi)$.

Circle bundles of special interest are circle bundles of oriented $\mathbb{R}^2$-vector bundles. Those are defined as follows: If E is an oriented $\mathbb{R}^2$-vector bundles over M, then the quotient

$$S(E) = E_0/\lambda x \sim x, \text{ for } x \in E_0 \text{ and } \lambda \in \mathbb{R}_+,$$

where E_0 denotes the nonzero vectors in E, is naturally an oriented circle bundle over M. Now, the obstruction to the existence of a section in $S(E) \to M$ is nothing else than the obstruction to the existence of a nonvanishing section in $E \to M$, which is the classical *Euler class* of the vector bundle, $\varepsilon_2(E) = \varepsilon_2(S(E)) \in H^2(M, \mathbb{Z})$. Again, the *Euler number* $\chi(E) \in \mathbb{Z}$ is the image of the Euler class $\varepsilon_2(E)$ under the canonical isomorphism $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$.

We could now directly apply Wood's inequality to vector bundles to obtain a bound on the Euler number of flat vector bundles, but this would not quite be good enough (by a factor 2). Instead, note that there is another natural circle bundle associated to the oriented vector bundle E, namely

$$PS(E) = E_0/\lambda x \sim x, \text{ for } x \in E_0 \text{ and } \lambda \in \mathbb{R}^*.$$

(The difference with $S(E)$ is that λ is now allowed to be a negative number. Thus, the vectors x and $-x$ will be identified in $PS(E)$, but not in $S(E)$.)

We can now prove Milnor's original inequality for oriented vector bundles, of which Wood's inequality above is a generalization to circle bundles. Recall that an oriented $\mathbb{R}^n$-vector bundle over a smooth manifold M is said to be flat if it is induced by a representation $\pi_1(M) \to \mathrm{GL}^+(n, \mathbb{R})$.

Theorem (Milnor [20]) Let E be an oriented $\mathbb{R}^2$-vector bundle over a surface Σ_g of genus $g \geq 1$. Then E is flat if and only if

$$|\chi(E)| \leq g - 1.$$

Proof Here also, we only prove the "only if" part. The point is that because $S(E) \to PS(E)$ is a double cover, $2 \cdot \chi(S(E)) = \chi(PS(E))$. If E is flat then it is induced from a representation $h : \pi_1(M) \to \mathrm{GL}^+(2, \mathbb{R})$ and the circle bundle $PS(E)$ is induced by a representation $h_0 : \pi_1(M) \to \mathrm{PGL}^+(2, \mathbb{R})$ given by composing h with the natural projection $\mathrm{GL}^+(2, \mathbb{R}) \to \mathrm{PGL}^+(2, \mathbb{R})$, where $\mathrm{PGL}^+(2, \mathbb{R})$ denotes the quotient of $\mathrm{GL}^+(2, \mathbb{R})$ by its center (the matrices of the form $\lambda \cdot \mathrm{Id}$, for $\lambda \in \mathbb{R}^*$). The natural action of $\mathrm{PGL}^+(2, \mathbb{R})$ on the projective line $P^1\mathbb{R} \cong S^1$ induces a homomorphism $\varphi : \mathrm{PGL}^+(2, \mathbb{R}) \to \mathrm{Homeo}_+(S^1)$ and the circle bundle $PS(E)$ is of course induced by the homomorphism $\varphi \circ h_0$, and is hence flat. Thus, we can apply Wood's inequality to $PS(E)$ and obtain

$$\chi(PS(E)) \leq 2(g - 1).$$

This finishes the proof of the theorem, since

$$|\chi(E)| = |\chi(S(E))| = \frac{1}{2} |\chi(PS(E))| \leq g - 1.$$

In higher dimensions, if E is an oriented $\mathbb{R}^n$-vector bundle over a smooth manifold M, then again, the obstruction to the existence of a nonvanishing section determines the *Euler class* $\varepsilon_n(E) \in H^n(M, \mathbb{Z})$. If moreover M is an oriented n-dimensional manifold, then the Euler number $\chi(E) \in \mathbb{Z}$ is the image of the Euler class under the canonical isomorphism $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$.

If E admits a flat structure, then the Euler class is in the image of

$$H^n(\pi_1(M), \mathbb{Z}) \to H^n(M, \mathbb{Z}).$$

Furthermore, Ivanov-Turaev and Gromov independently showed that the Euler class of flat bundles can be represented by a bounded cocycle, or in other words, it is in the image of

$$H_b^n(\pi_1(M),\mathbb{Z}) \to H^n(\pi_1(M),\mathbb{Z}) \to H^n(M,\mathbb{Z}).$$

(The proof of Ivanov-Turaev [17] is very direct and gives an explicit bounded cocycle representing the Euler class of flat bundles, while Gromov's proof [14] is more abstract and conceptual, and eventually generalizes to all characteristic classes of flat G-bundles when G is a real algebraic subgroup of $\mathrm{GL}(n,\mathbb{R})$.)

Lemma Let E be a flat oriented $\mathbb{R}^n$-bundle over an n-dimensional manifold M. If $\pi_1(M)$ is amenable, then $\chi(E)=0$.

Proof Consider the commutative diagramm

$$\begin{array}{ccccc} H_b^n(\pi_1(M),\mathbb{Z}) & \longrightarrow & H^n(\pi_1(M),\mathbb{Z}) & \longrightarrow & H^n(M,\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ 0 = H_b^n(\pi_1(M),\mathbb{R}) & \longrightarrow & H^n(\pi_1(M),\mathbb{R}) & \longrightarrow & H^n(M,\mathbb{R}). \end{array}$$

Since the Euler class $\varepsilon_n(E) \in H^n(M,\mathbb{Z})$ comes from a bounded class, its image in $H^n(M,\mathbb{R})$ goes through $0 = H_b^n(\pi_1(M),\mathbb{R})$ and is hence 0. Now recall that in top dimension, the inclusion of coefficients $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces an injection

$$H^n(M,\mathbb{Z}) \hookrightarrow H^n(M,\mathbb{R}),$$

so that also $\varepsilon_n(E) = 0 \in H^n(M,\mathbb{Z})$.

Affine structure

Let M be a smooth n-dimensional manifold. Recall that M admits an *affine structure*, if there exists an atlas for M such that the corresponding transition functions are affine transformations of $\mathbb{R}^n$. We say that M is an *affine manifold* if it admits an affine structure. Examples of affine manifolds are tori, Euclidean manifolds (for which the transition functions are Euclidean transformations of $\mathbb{R}^n$, i.e. products of rotations and translations), etc.

Affine manifolds are far from being understood, and there are many deep open conjectures on them. Let us look at one of them:

Chern Conjecture (1955) If M is a closed affine manifold, then $\chi(M)=0$.

Recall that $\chi(M)$ denotes the Euler-Poincaré characteristic of M and is defined as the alternating sum of the Betti numbers of M:

$$\chi(M) = \sum_{i=0}^{\dim(M)} (-1)^i \dim(H^i(M,\mathbb{R})).$$

Moreover, a reformulation of a Theorem of Hopf tells us that

$$\chi(M) = \chi(TM).$$

As far as I know, the conjecture is only known to hold in the following cases: 1) $\pi_1(M)$ is amenable (we will see a simple proof below), 2) M is a complete affine manifold (there is a one page not so simple proof by Kostant and Sullivan [18]), 3) M is a surface admitting a hyperbolic structure [20] (we will see how this easily follows from Milnor's inequality), 4) M is a Riemannian manifold which is locally isometric to a product of hyperbolic planes [5] (of course, 4) generalizes 3) and in fact follows from generalized Milnor-Wood inequalities). Note that in cases 3) and 4) $\chi(M) \neq 0$, so that it makes more sense to speak about the converse of Chern's conjecture, and what one really proves is that a closed manifold which is locally isometric to a product of hyperbolic planes cannot admit an affine structure.

The proofs presented here rely on two basic facts: First, if M is affine, then its tangent bundle TM admits a flat structure. Second, the Euler class of flat bundles can be represented by a bounded class.

Lemma If M is a closed affine manifold with amenable fundamental group, then $\chi(M) = 0$.

Proof Since M is affine, TM is flat, so that, as above, $\chi(M) = \chi(TM) = 0$.

This simple argument is apparently due to Benedetti and Petronio [1]. The original proof of Hirsch and Thurston [15] is substantially more difficult.

Corollary to Milnor's inequality The only surface admitting an affine structure is the 2-torus.

Proof If M is an affine surface, then its tangent bundle TM admits a flat structure. If M is the 2-sphere, then the precedent lemma forces $\chi(M) = 0$, but $\chi(S^2) = 2$. Suppose now that Σ_g is a closed surface of genus $g \geq 1$. Since TM is flat, it follows from Milnor's inequality that

$$|\chi(\Sigma_g)| = |\chi(T\Sigma_g)| \leq \frac{1}{2} |\chi(\Sigma_g)| ,$$

and this inequality is only possible if $\chi(\Sigma_g) = 2(1 - g) = 0$, so that $g = 1$.

As a last illustration of the potential of bounded cohomology, let me finish with a recent generalization of Milnor-Wood's inequality. The proof combines bounded cohomology, representation theory and Margulis' Superrigidity Theorem.

Theorem (Bucher-Gelander [5]) Let M be a closed Riemannian manifold of dimensions $2n$ locally isometric to a product of n copies of the hyperbolic plane. Let E be an oriented $\mathbb{R}^{2n}$-vector bundle over M. If E is flat, then

$$|\chi(E)| \leq \frac{1}{2^n} |\chi(M)| .$$

Corollary Let M be a closed Riemannian manifold of dimensions $2n$ locally isometric to a product of n copies of the hyperbolic plane. Then M does not admit an affine structure.

Proof The corollary follows from the generalized Milnor-Wood inequality exactly as in Milnor's case: Namely if such a manifold M admits an affine structure, then its tangent bundle TM admits a flat structure, so that by the Theorem,

$$|\chi(M)| = |\chi(TM)| \leq \frac{1}{2^n} |\chi(M)| ,$$

which is impossible since $\chi(M) \neq 0$ (this can for example be seen from Hirzebruch's proportionality principle [16]).

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