Derived categories and their geometrical applications (part 2)

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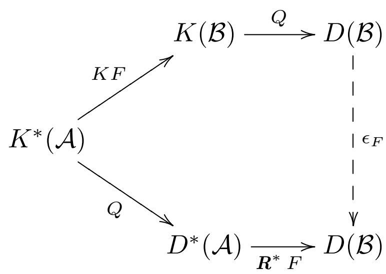
1 Derived functors

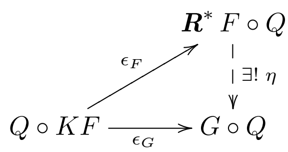
Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between two abelian categories. Then it induces a morphism of triangulated categories[1] $KF : K(\mathcal{A}) \to K(\mathcal{B})$. But, in general, KF *will not* take quasi-isomorphisms to quasi-isomorphisms, so it will not directly localise to give a functor $D(\mathcal{A}) \to D(\mathcal{B})$. In which case we want to have the next best thing:

Definition 1.1. Let $K^*(\mathcal{A})$ be a localising subcategory of $K(\mathcal{A})$. In other words, the natural functor $D^*(\mathcal{A}) = K^*(\mathcal{A})_{Qis} \to K(\mathcal{A})_{Qis} = D(\mathcal{A})$ is fully faithful. The *right derived functor* $\boldsymbol{R}^* F$ of F is a morphism $\boldsymbol{R}^* F : D^*(\mathcal{A}) \to D(\mathcal{B})$ of triangulated categories equipped with a natural transformation $\epsilon_F : Q \circ KF \to \boldsymbol{R}^* F \circ Q$



(1.1)

which has the following universal property: if $G : D^*(\mathcal{A}) \to D(\mathcal{B})$ is another morphism of triangulated categories equipped with a natural transformation $\epsilon_G : Q \circ KF \to G \circ Q$ then there exists a unique natural transformation $\eta : \boldsymbol{R}^* F \to G$ which makes the triangle



(1.2)

commute.

Similarly, the *left derived functor* $\boldsymbol{L}^* F$ is a morphism $\boldsymbol{L}^* F : D^*(\mathcal{A}) \to D(\mathcal{B})$ of triangulated categories equipped with a natural transformation $\epsilon_F : \boldsymbol{L}^* F \circ Q \to Q \circ KF$ satisfying a similar universal property.

Remarks:

1. If a derived functor exists, it is unique up to a unique isomorphism.

[1] An additive functor which commutes with shifts and preserves exact triangles.

2. If F is exact, then KF extends trivially to $D(A) \to D(B)$ and is its own left and right derived functor.

3. It is traditional to compute $\boldsymbol{R} F$ of left exact functors and $\boldsymbol{L} F$ of right exact functors so as to retain the connection with the original functor F and its classical derived functors. For example, if F is left exact we have $H^0(\boldsymbol{R} F(A)) = F(A)$ for every $A \in \mathcal{A}$, while if F is right exact we have $H^0(\boldsymbol{L} F(A)) = F(A)$.

Theorem 1.1 (Existence of derived functors ([Har66], I, Theorem 5.1))**.** *Suppose there exists a triangulated subcategory $L \subseteq K^*(\mathcal{A})$ such that*

1. Every object of $K^*(\mathcal{A})$ admits a quasi-isomorphism into an object of L.

2. If $I \in L$ is an acyclic complex[2], then $F(I)$ is also acyclic.

Then the right derived functor $(\boldsymbol{R}^* F, \epsilon_F)$ exists. Moreover, for every $I \in L$ the natural transformation

$$\epsilon_F(I) : \quad Q \circ F(I) \to \boldsymbol{R}^* F \circ Q(I)$$

is an isomorphism in $D(\mathcal{B})$.

In practice, the subcategory L of $K^*(\mathcal{A})$ almost always comes from a class of objects in $\mathcal{A}$. Indeed, we make the following definition:

Definition 1.2. A subset P of the objects of $\mathcal{A}$, stable under finite direct sums, is called *right (resp. left) adapted* to $F : \mathcal{A} \to \mathcal{B}$ if:

1. Any $A \in \mathcal{A}$ injects $O \hookrightarrow A \to B$ into some $B \in P$ (resp. is a quotient $B \twoheadrightarrow A \to 0$ of some $B \in P$).

2. F takes short exact sequences in P into short exact sequences.

Proposition 1.3. *If a right (resp. left) adapted class P exists for F, then $\boldsymbol{R}^+ F$ (resp. $\boldsymbol{L}^- F$) exists and is computed by applying F to complexes in P.*

Certain objects are always adapted to any functor F provided there are enough of them:

Proposition 1.4. *If $\mathcal{A}$ has enough injectives (resp. projectives) then the class of all these objects is right (resp. left) adapted to any additive functor F.*

2 Derived category of a scheme and its derived functors

Let $(X, \mathcal{O}_X)$ be a scheme. By $D^*(X)$ we shall denote the derived category of $K^*(\mathcal{O}_X\text{-}\mathbf{Mod})$. By $D^*_{qc}(X)$ (resp. $D^*_{coh}(X)$) we mean the derived category of the full subcategory of $K^*(\mathcal{O}_X\text{-}\mathbf{Mod})$ consisting of complexes with quasi-coherent (resp. coherent) cohomologies.

NB: What is usually meant by "the" derived category of a scheme X is $D^b_{coh}(X)$ - the derived category of bounded complexes of $\mathcal{O}_X$-modules whose cohomologies are coherent.

Proposition 2.1 ([God58],II,7.1.1)**.** *Category $\mathcal{O}_X$-**Mod** has enough injectives.*

Proposition 2.2 ([Har66], II, Prop. 1.2)**.** *Every sheaf of $\mathcal{O}_X$-modules is a quotient of a flat sheaf of $\mathcal{O}_X$-modules.*

[2] A complex all of whose cohomologies are zero.

Proposition 2.3. *If X is a quasi-projective variety, then every coherent sheaf is a quotient of a locally free sheaf. If X is moreover smooth, then every object in $K^b_{coh}(X)$ is quasi-isomorphic to a bounded complex of locally free sheaves.*

Proof. If X' is projective it has an ample line bundle $\mathcal{L}$ and every coherent sheaf on X' can be written as a quotient of $\mathcal{O}^n_{X'} \otimes \mathcal{L}^m$ for some $n, m \in \mathbb{Z}$. If $X \to X'$ is an open immersion, then it is flat and therefore the pullback to X is exact. Therefore, since also every coherent sheaf on U is a restriction of a coherent sheaf on X, we can form a quotient over X' and then pull it back to U.

Thus every coherent sheaf $\mathcal{F}$ on X has a (possibly infinite) resolution by locally frees. If X is smooth of dimension n, then in every local ring $\mathcal{O}_{X,x}$ we have $\mathrm{Tor}^i(\mathcal{F}_x, \mathbf{k}(x))$ vanish for all $i > n$. This means that we can truncate the locally free resolution of $\mathcal{F}$ at n-th term and keep it locally free. The last claim follows. $\square$

We now list standard derived functors which are defined on X:

1. $\boldsymbol{R}\,\mathrm{Hom}(-,-)$: Group Hom in $\mathcal{O}_X$-**Mod** induces a bifunctor $\mathrm{Hom}(-,-):\ K(X)^o \times K(X) \to K(\mathbf{AbGp})$. By using injective object resolutions we can compute its derived functor in second variable to obtain functor $\boldsymbol{R}\,\mathrm{Hom}_2(-,-):\ K(X)^o \times D^+(X) \to D(\mathbf{AbGp})$. This is 'exact' in first variable, in the sense that it takes acyclic complexes in $K(X)^o$ to the zero object in $D(\mathbf{AbGp})$ (see [Har66], I, Lemma 6.2). Therefore it trivially passes to a quotient $\boldsymbol{R}\,\mathrm{Hom}_{2,1}(-,-):\ D(X)^o \times D^+(X) \to D(\mathbf{AbGp})$.

   Theorem 2.1. *For any $A \in D(X)$ and $B \in D^+(X)$ we have*

   $$H^i(\boldsymbol{R}\,\mathrm{Hom}(A,B)) = \mathrm{Hom}^i_{D(X)}(A,B) \stackrel{def}{:=} \mathrm{Hom}_{D(X)}(A, B[i]) \qquad (2.1)$$

2. $\boldsymbol{R}\,\mathcal{H}om$: Sheaf $\mathcal{H}om$ in $\mathcal{O}_X$-**Mod** induces a bifunctor $\mathcal{H}om(-,-):\ K(X)^o \times K(X) \to K(X)$. Similar to above we can use injective resolutions in second variable to construct $\boldsymbol{R}\,\mathcal{H}om(-,-):\ D(X)^o \times D^+(X) \to D(X)$. Locally free sheaves of finite rank are adapted to $\mathcal{H}om$ in first variable, so if $\mathrm{Coh}(X)$ has enough locally frees we can pass to a derived functor in the first variable first to yield $\boldsymbol{R}\,\mathcal{H}om(-,-):\ D_{coh}(X)^- \times D(X) \to D(X)$.

   Definition 2.4. For any object $A \in D(X)$ we define its dual $A^\vee$ to be $\boldsymbol{R}\,\mathcal{H}om_X(A, \mathcal{O}_X)$.

3. $\boldsymbol{R}\,f_*$ and $\boldsymbol{R}\,\Gamma$: Let $f:\ X \to Y$ be a morphism of schemes. We can use injective resolutions to construct $\boldsymbol{R}\,f_*:\ D^+(X) \to D(Y)$. If f is separated and quasi-compact, then $\boldsymbol{R}\,f_*$ takes $D^+_{qc}(X)$ to $D^+_{qc}(Y)$. If f is proper and Y locally noetherian $\boldsymbol{R}\,f_*$ takes $D^+_{coh}(X)$ to $D^+_{coh}(Y)$. As the global section $\Gamma(X,-)$ functor is a particular case of $f_*(-)$ for f being the morphism $X \to \mathrm{Spec}\ \mathbb{Z}$ (or $\mathrm{Spec}\ k$ if X is defined over a field k) all of the above applies to it as well.

4. $\overset{\boldsymbol{L}}{\otimes}$: Tensor product induces a bifunctor $K(X) \times K(X) \to K(X)$. Flat sheaves are left adapted to $-\otimes-$, so we can pass to a derived functor in either variable. Then, once again, it turns out that for bounded from above complexes we can pass trivially to derived categories in second variable too. We thus obtain $-\overset{\boldsymbol{L}}{\otimes}-$ as a functor $D^-(X) \times D^-(X) \to D(X)$. Alternatively, we can obtain it as a functor $D(X) \times D^b_{fTD}(X) \to D(X)$, where $D^b_{fTD}(X)$ is the full subcategory of $D(X)$ consisting of complexes quasi-isomorphic to a bounded complex of flat objects.

5. $\boldsymbol{L}\,f^*$: Let $f:\ X \to Y$ be a morphism of schemes. Then flat sheaves are left adapted to f^* and so we can construct a left derived functor $\boldsymbol{L}\,f^*(-):\ D^-(Y) \to D^-(X)$. It furthermore takes $D^-_{qc}(Y)$ and $D^-_{coh}(Y)$ to $D^-_{qc}(Y)$ and $D^-_{coh}(Y)$, respectively. If Y is non-singular then $\boldsymbol{L}\,f^*$ takes $D^b_{coh}(Y)$ to $D^b_{coh}(X)$.

References

[God58] R. Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1958.

[Har66] R. Hartshorne, *Residues and duality*, Springer-Verlag, 1966.