Functional Analysis 2000 (5B1472)
Weeks 4, 5, 6

1. Basic notions.

Reading: Friedman 4.1, 4.2, 4.3, 4.4

Syllabus: Linear space, normed linear space, metric linear space, equivalent norms, Banach space, Banach algebra.

Subspace, factor space, norm on factor space, basis and dimension for finite-dimensional spaces, Hamel basis. *Theorem: Every Banach space has a Hamel basis.*

Linear transformation (operator), continuous linear transformation, bounded linear transformation, operator norm, space $\mathcal{B}(X,Y)$. *Theorem: Let T be a linear transformation from a normed linear space to a normed linear space. Then the following are equivalent: (1) T is continuous. (2) T is continuous at some point. (3) T is bounded.*

Notes: Linear space is finite-dimensional if and only if all norms are equivalent (problem 1). Moreover, we will see later that for infinite dimensional linear spaces a few non-equivalent norms (and topologies) are of interest.

A set A is called Hamel basis in a linear space X if every element of X can be uniquely represented as a finite sum of the multiples of the basis elements. Every Banach space has a Hamel basis (Theorem 4.2.3). Unfortunately, since we consider only finite sums, such bases in infinitely dimensional Banach spaces have uncountably many elements and are not of much use – see problem 3 below.

One can try to deal with this problem by replacing the word finite with the word countable (note that to sum countable sequences we need to have metric, and the order of summation is also important!). One can try to say that set A forms a basis in a normed linear space X if every element of X can be uniquely represented as a countable converging sum of the multiples of the basis elements, such that the sum does not depend on the order of summation.

For Hilbert spaces this works - we will discuss it later (the reason is that using scalar product one can define orthogonal projection and hence orthogonal basis, much like in $\mathbb{R}^n$). Unfortunately, there are Banach spaces, which do not have a basis (a difficult theorem due to Per Enflo), so one should build a general theory without assuming existence of a basis. Fortunately, all useful spaces (like L^p) admit a basis.

2. Baire category. (Reminder)

Reading: Friedman 3.4

Syllabus: First and second Baire category, meager and residual sets. The Baire category theorem.

Notes: Many questions in Banach space theory involve proving that sets have non-empty interiors – see problem 2. Very powerful machinery for that is provided by the Baire category theorem.

Let X be a topological space (you can think of a metric space instead). Set is called nowhere dense if its closure has no interior. Then $E \in X$ is of *the first Baire category* if it is a countable union of nowhere dense sets, otherwise it is of *the second Baire category*. More modern terminology is *meager set* = set of the first category, *residual* *set* = complement of a meager set (note: residual $\neq$ of the second category!). The property is said to be (topologically) *generic*, if it holds for a residual set.

Often existence of an element with some property is shown by proving that this property is generic.

Whether E is of the first or the second category, depends on the space X: a line is residual and of second category inside itself, but meager inside a plane. Simlarly, $\mathbb{Z}$ is of the second category inside itself (every point is open!), but of the first category inside $\mathbb{R}$.

On a space which has both metric and measure (e.g., $\mathbb{R}$) there are two possible meanings of the word generic, and often one property is metrically generic (i.e. holds for a full measure of points), while the opposite is topologically generic (i.e. holds for a residual set of points). Caution: there exist a full measure subset A of $[0, 1]$, which is meager, and a zero measure subset B of $[0, 1]$, which is residual!

The Baire category theorem. [3.4.2] Complete metric space is non-meager (i.e. of second category) inside itself.

Corollary. In a complete metric space countable intersection of open dense subsets is residual (hence non-empty).

3. Three basic principles.

Reading: Friedman 4.5, 4.6

Syllabus: Uniform boundedness principle. Open mapping theorem. Closed graph theorem. *Corollary: Strongly convergent sequence of operators has a limit. Corollary (Inverse mapping theorem): Inverse of a bounded linear 1-to-1 operator between Banach spaces is also bounded.*

Notes:

1. *The principle of uniform boundedness (Banach-Steinhaus theorem). [4.5.1]* Let $\{T_\alpha\}$ be a family of bounded linear operators from a Banach space X to a n.l.s. Y. If for every $x \in X$ (more generally for every x in a second category subset of X) $\|T_\alpha x\| \leq M_x\|x\|$, then there is an M such that $\|T_\alpha\| \leq M$.

2. *The open mapping theorem. [4.6.1]* Let X and Y be Banach spaces, T be a bounded linear operator from X onto Y. Then T is open (i.e. maps open sets to open sets).

3. *The closed graph theorem. [4.6.4]* Let X and Y be Banach spaces, T be a linear operator from X to Y. If the graph of T is closed, then T is bounded.

Proof 1. Set $E_n := \{x \in X : \|T_\alpha x\| \leq n \,\forall\alpha\}$. Then $\cap_{n=1}^{\infty} E_n$ is non-meager, so for some N the set E_N has interior, i.e. contains a ball $B(y, r)$. Thus for $x \in B(0, r)$ one has $\|T_\alpha x\| = \|T_\alpha(y + x) - T_\alpha y\| \leq 2N$, and $\|T_\alpha\| \leq 2N/r$.

Remark 3. To appreciate the closed graph theorem consider the following theree statements:

(a) x_n converges to some element x.
(b) Tx_n converges to some element y.
(c) $Tx = y$. A priori, to proof that T is continuous (=bounded) we must show that (a) implies (b) and (c). The closed graph theorem says that it is sufficient to show that (a) and (b) imply (c)!

Problem set #1.

Problem 1. Show that all norms on a linear space X are equivalent if and only if it is finite dimensional.
Hint. To show that infinite dimensional space admits non-equivalent norms, take a Hamel basis $\{e_\alpha\}$ and consider norms $\|x\| = \|\sum x_\alpha e_\alpha\| := \sum |x_\alpha| k_\alpha$ for different collections of coefficients k_α. To show that all norms on a finite-dimensional space are equivalent, To show that all norms on finite dimensional space (i.e. on $\mathbb{R}^n$) are equivalent, use compactness of the unit ball (Friedman 4.3.3).
Problem 2. Let X and Y be normed linear spaces. Then a linear operator T is bounded if and only if preimage of a ball $T^{-1}(\{y \in Y : \|y\| \leq 1\})$ has a nonempty interior.
Problem 3. Show that a Hamel basis for an infinite dimensional Banach space has to be uncountable.
Hint. Use Baire category.
Problem 4. Show that polynomials in one variable with real coefficients make a linear space (usually denoted $\mathbb{R}[x]$, where x is the "variable"). Find a countable Hamel basis basis in $\mathbb{R}[x]$. Show that there exists no norm making $\mathbb{R}[x]$ a Banach space. Find some norm, that makes $\mathbb{R}[x]$ a normed linear space.
Problem 5. Let X, Y, Z be Banach spaces, and $B : X \times Y \to Z$ be a bilinear (i.e. $B(x, y)$ depends linearly both on $x \in X$ and on $y \in Y$) continuous map. Prove that there is $M < \infty$ such that $\|B(x, y)\| \leq M\|x\|\|y\|$.
Problem 6. Let A be a closed subspace and B a finite dimensional subspace of a normed linear space X. Show that $A + B$ (which is $:= \{a + b : a \in A, b \in B\}$) is a closed subspace.
Problem 7. [Friedman 4.3.2, 4.3.3] Let A be a finite dimensional subspace of a normed linear space X and $x \in X$. Show that there is a point $a \in A$ such that $\|a - x\| = \inf\{\|b - x\| : b \in A\}$. Prove that if the norm is strictly convex (see Friedman problem 4.3.3) then such a is unique.
Problem 8. Consider the space $X := C[0, 1]$ of continuous functions on $[0, 1]$ and let $A := \{f \in X : f(0) = 0\}$. Show that A is a closed subspace and that $X/A \cong \mathbb{R}$
Problem 9. Hilbert space, roughly speaking, is a Banach space with a scalar product (see 6.1 for precise definition). Prove the *Hellinger-Toeplitz theorem*: If A is a symmetric (meaning $(x, Ay) = (Ax, y)$) linear operator from a Hilbert space H to itself, then A is bounded. Give 2 proofs: with closed graph theorem and with uniform boundness principle.
Remark. This theorem has important consequences in quantum mechanics: there are operators (like the energy) which must obey $(x, Ay) = (Ax, y)$, but are unbounded. By the Hellinger-Toeplitz theorem such operators cannot be everywhere defined. It creates certain technical difficulties: e.g. the sum $A + B$ is defined only on the intersection of subspaces $D(A)$ and $D(B)$ where A and B are defined, and this intersection a priori may be empty.
Additional problems: Friedman 4.1.3, 4.1.4, 4.2.1, 4.2.2, 4.2.5, 4.2.6, 4.3.1, 4.3.2, 4.3.3, 4.3.4, 4.3.7, 4.4.1, 4.4.3, 4.4.4, 4.6.1, 4.6.2, 4.6.4, and all other problems in 4.1-6.

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