CRITICAL PERCOLATION IN THE PLANE. I. CONFORMAL INVARIANCE AND CARDY'S FORMULA. II. CONTINUUM SCALING LIMIT.

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ABSTRACT. We study scaling limits and conformal invariance of critical site percolation on triangular lattice. We show that some percolation-related quantities are harmonic conformal invariants, and calculate their values in the scaling limit. As a particular case we obtain conformal invariance of the crossing probabilities and Cardy's formula. Then we prove existence, uniqueness, and conformal invariance of the continuum scaling limit.

1. INTRODUCTION

In this paper we study critical ($p = p_c = \frac{1}{2}$) site percolation on triangular lattice. First we introduce and evaluate harmonic conformal invariants, given by the limits of certain probabilities as mesh of the lattice tends to zero. As a corollary we obtain crossing probabilities (predicted by J. Cardy in [5]) and show their conformal invariance (conjecture attributed to M. Aizenman by R. Langlands, Ph. Pouliot, and Y. Saint-Aubin in [12]). Then we show the existence, uniqueness, and conformal invariance of the continuum scaling limit. Some other similar conformal invariants and a different approach to the scaling limit will be discussed in a subsequent paper. For the general background on percolation consult the book [8], for topics related to crossing probabilities, conformal invariance, and scaling limits see [12, 1, 7] and other references mentioned below.

The key property of the percolation-related conformal invariants considered is that they depend harmonically on a parameter z (a point inside some domain Ω), which allows to determine them uniquely from their boundary behavior, and forces them to be conformally invariant.

To motivate search for such harmonic conformal invariants, let us first note that one can construct some assuming existence and conformal invariance of the percolation scaling limit. For example, given a simply connected domain Ω with two boundary points a and b one can define function $h(y,z) = h_{a,b}(y,z)$ to be the expected number of different clusters spanning from the boundary arc ab to the boundary arc ba and separating y from z (to set up this problem rigorously one actually needs to average numbers of "rightmost" and "leftmost" cluster boundaries separating y from z and taken with appropriate signs). Then by conjectured conformal invariance we can assume Ω to be a horizontal strip of width 1. Symmetry arguments and translation invariance show that

$$\begin{aligned} h(y,z) &= h(y,w) + h(w,z) \ , \\ h(y,z) &= h(y+w, z+w) \ , \\ h(y,z) &= 0, \quad \text{if } y - z \in i\mathbb{R} \ , \end{aligned}$$

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it easily follows that

$$h(y,z) = c\,\mathrm{Re}(y-z)\ , \tag{1}$$

and hence function $h(y,z)$ is harmonic in both y and z. So the reasoning above suggests a good candidate for a harmonic conformal invariant (though it does not hint at the value of c).

One can expect that in the discrete case similar invariants should be (almost) discrete harmonic functions, and one should be able to obtain a "discrete" proof of that. Indeed, modification of the methods described below shows that the scaling limit (as mesh of the lattice tends to zero) of the discrete version of h is harmonic and satisfies (1) with $c = \sqrt{3}/4$. Note that when y and z are taken on the boundary of Ω, the function h gives the expected number of different percolation clusters, crossing a conformal rectangle.

There are other similar and more complicated invariants, which we will discuss in the next version of this paper. For now we will concentrate on a single harmonic conformal invariant, which is a complexification of crossing probabilities. Namely we will show that the probability of a point inside a conformal triangle to be separated by a percolation cluster spanning two sides from the third one is approximately a (discrete) harmonic function. Then boundary behavior considerations allow to reconstruct the function uniquely and force it to be conformally invariant. There seems to be no short motivation for this particular function to be harmonic, but the proof is simpler than for other analogous invariants. Also when the point z is taken on the boundary, it gives the crossing probabilities of conformal rectangles. Hence Cardy's formula [5] and conformal invariance of crossing probabilities (in the scaling limit) follow immediately. Moreover, one obtains enough information to prove existence, uniqueness, and conformal invariance of the scaling limit.

There is a classical approach (due to Shizuo Kakutani, [9]) to Brownian Motion along the similar lines: exit probabilities for Brownian Motion started at z are harmonic functions in z with easily determinable boundary values. One can prove this by showing the same statement for the Random Walk (discrete Laplacian of the exit probability is trivially zero), and then passing to a limit (Kakutani works directly with Brownian Motion). However, our proof of discrete harmonicity is more complicated than the Random Walk analogue: instead of checking that Laplacian vanishes, we find harmonic conjugates (which turn out to be similar conformal invariants). This requires working with the first order derivatives only, while Laplacian involves the second order ones. Furthemore, instead of checking Cauchy-Riemann equations (which can be done), we check that contour integrals vanish: it requires even less precision, and allows for an easier passage to a limit.

Interestingly, instead of a pair of harmonic conjugate functions, we get a "harmonic conjugate triple." It seems that $2\pi/3$ rotational symmetry enters in our paper not because of the specific lattice we consider, but rather manifests some symmetry laws characteristic to (continuum) percolation.

After introducing harmonic conformal invariants, we obtain enough information to construct the continuum scaling limit, and show its uniqueness and conformal invariance. There are different objects which can represent the continuum scaling limit of percolation, see the discussion in [2, 1, 3, 4], and the references therein. In the discrete setting one defines percolation clusters as maximal connected subgraphs of some fixed color, and the question is what sort of object will nicely represent the scaling limit.

If one wants to get the full information about the "percolation configuration" in the scaling limit, there seem to be two approaches (with equivalent results).

The first one is a variation of the straightforward way, when one represents a percolation configuration as a collection of compact connected subsets of the plane, representing different clusters.

This has to be modified, since it happens in the scaling limit that two parts of a percolation cluster touch without connecting (in this particular place), since they are separated by a curve of opposite color. However, the modification of this approach, suggested by Michael Aizenman in [1], works: one should represent percolation clusters by a collection of all curves contained inside clusters of some fixed color. Then percolation configuration is a collection of all curves inside all clusters of the fixed color, such collections will almost surely satisfy certain "compatibility" relations. One can also add for convenience the collection of all curves of the opposite color.

Another approach is to represent a percolation configuration as a collection of "nested" closed curves – external perimeters of clusters (of both colors). In the discrete case such curves will be simple, in the scaling limit they cease to be simple but remain "non-self-traversing." These curves are the unique curves which correspond to crossings by both colors, and they have a canonical orientation (depending on the color of the "outer" side). Then the percolation culster is a space inside some curve γ which is outside of all curves of opposite orientation lying inside γ.

The philosophy of constructing the scaling limit is to deduce first existence of subsequential limits by compactness arguments, and to show that quantities we know determine the law of the scaling limit uniquely, so it does not depend on chosen subsequence. We know values of crossing probabilies in the scaling limit, and the crossing events generate any reasonable σ-algebra for percolation configurations. However it is not immediate that there is a unique law with given probabilities of crossing events (e.g. horizontal and vertical strips generate the Borel σ-algebra in the plane, but knowing measures of those does not determine a Borel measure uniquely).

Thus it is easier to build the scaling limit gradually, so we will start with some partial aspects of the "full percolation configuration," which are also of independent interest, and sometimes are even more convinient to work with. One can think about these either as about scaling limit of similar objects for discrete percolation, or as about restriction of a full percolation configuration to a coarser σ-algebra.

Among these objects are the outer boundary of a percolation cluster, the external perimeter of a percolation cluster, and a lamination of a domain (telling which points on the boundary are connected by clusters inside the domain).

Setup and notation. We study critical site percolation on the triangular lattice with mesh δ. For general background, consult G. Grimmett's and H. Kesten's books [8, 11]. Vertices are colored in two colors, say blue and yellow, independently with equal probability $p = p_c = \frac{1}{2}$. By H. Kesten's [10] this probability is critical for the site percolation on the triangular lattice, but it is interesting to note that we do not use this fact. By a *blue simple path* (in a particular percolation configuration) we mean a sequence $\{b_j\}$ of vertices colored blue, with adjacent b_j and b_{j+1}. We say that a path goes or connects to some set, if the dual lattice hexagon around the last (or the first, depending on the context) vertex intersects this set. We will often identify this sequence of vertices with the corresponding curve – i.e. the broken line $b_1 b_2 \dots$.

For brevity we denote by τ the cube root of unity: $\tau := \exp(2\pi i/3)$, and assume that one of the lattice directions is parallel to the real axis.

For points a, b (or prime ends) on the boundary of a simply connected domain Ω we will call *"arc ab,"* the part of $\partial\Omega$ between the points a and b (in counterclockwise direction and including the endpoints). It need not be an arc in the strict sense, though. Note that our considerations do not differ whether one considers points or prime ends. By $[ab]$ we denote the interval joining a and b.

We denote by $\mathcal{H}$ the space of all Hölder curves up to parameterization, endowed with the uniform metric (infimum of $\sup \|\gamma_1(t)-\gamma_2(t)\|_\infty$ over all possible Hölder parameterizations of two curves by the interval $[0,1]$), and by $\mathcal{B}$ the corresponding Borel σ-algebra.

Note that for any M the set $\mathcal{H}_M$ of curves admitting Hölder parameterization with norm at most M is compact in $\mathcal{H}$ (in the uniform topology).

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2. Harmonic conformal invariants

Consider a simply connected domain Ω and three accessible boundary points (or prime ends), labeled counterclockwise $a(1), a(\tau), a(\tau^2)$.

If the domain Ω has a smooth boundary, there are harmonic functions

$$h^\Omega(a(\alpha), a(\tau\alpha), a(\tau^2\alpha), z) = h_\alpha(z) = h_{a(\alpha)}(z) \ , \ \alpha \in \{1, \tau, \tau^2\} \ ,$$

which are the unique solutions of the following mixed Dirichlet-Neumann problem:

$$\begin{cases} h_\alpha = 1 \text{ at } a(\alpha) \ , \ h_\alpha = 0 \text{ on the arc } a(\tau\alpha)a(\tau^2\alpha) \\ \dfrac{\partial}{\partial(\tau\nu)} h_\alpha = 0 \text{ on the arc } a(\alpha)a(\tau\alpha) \\ \dfrac{\partial}{\partial(-\tau^2\nu)} h_\alpha = 0 \text{ on the arc } a(\tau^2\alpha)a(\alpha) \end{cases} , \tag{2}$$

where ν is the counterclockwise-pointing unit tangent to $\partial\Omega$. Moreover, functions h_1, h_τ, h_{τ^2} in some sense form a "harmonic conjugate triple," see the discussion below.

As Wendelin Werner pointed out to us, function h_α is also given by the probability that Brownian Motion started at z and reflected on the arcs $a(\alpha)a(\tau\alpha)$, $a(\tau^2\alpha)a(\alpha)$ at $frac\pi 3$-angle pointing towards $a(\alpha)$ hits $a(\alpha)$ before the arc $a(\tau\alpha)a(\tau^2\alpha)$.

Problem (2) is conformally invariant, so we can alternatively pose it (and this works even if Ω has non-smooth boundary) by conformally transferring it from Ω to some nice domain. The solution of the problem (2) takes on a particularly nice form for an equilateral triangle Ω' with vertices $a'(1), a'(\tau), a'(\tau^2)$. Then the corresponding functions become linear functions h'_α, which are equal to 1 at one of the vertices and zero at the opposite side, i.e. are properly rescaled distances to the sides of the triangle. For a general simply connected domain Ω there is a unique conformal map φ of Ω to the triangle Ω', sending $a(\alpha)$ to $a'(\alpha)$. Therefore we remark that

Remark 1. *h_α can be defined by $h_\alpha := h'_\alpha \circ \varphi$.*

It is also easy to see directly (or by mapping to an equilateral triangle) that for a halfplane the solution of (2) is a hypergeometric function.

For a triangular lattice with mesh δ, we define an event $Q_\alpha(z)$, $\alpha \in \{1, \tau, \tau^2\}$, $z \in \Omega$, as an occurrence of a blue *simple* path going from the arc $a(\alpha)a(\tau\alpha)$ to the arc $a(\tau^2\alpha)a(\alpha)$, and separating z from the arc $a(\tau\alpha)a(\tau^2\alpha)$. Then we define functions

$$H^{\Omega,\delta}(a(\alpha), a(\tau\alpha), a(\tau^2\alpha), z) = H^\delta_\alpha(z) = H_\alpha(z)\ ,\ \alpha \in \{1, \tau, \tau^2\}\ ,\ z \in \Omega\ ,$$

to be the probabilities of $Q_\alpha(z)$'s. Note that H_α's are constant in each of the lattice triangles, so we will mainly consider their values at the vertices of the dual hexagonal lattice (= centers of triangles). We emphasize again, that we consider only simple paths, neglecting therefore the points separated by the "dangling ends." Otherwise the functions would have the same boundary values as H_α's but would be strictly bigger inside the domain (and therefore not harmonic).

Theorem 1. *As $\delta \to 0$, functions H^δ_α converge uniformly in Ω to functions h_α.*

The statement above is a bit unrigorous, since the functions H^δ_α are defined only on discrete lattice depending on δ.

Since the problem (2) is conformally invariant, we conclude that

Corollary 1. *The limit of H^δ_α is a conformal invariant of the points $a(1), a(\tau), a(\tau^2), z$ and the domain Ω.*

Consider a conformal rectangle, i.e. a simply connected domain Ω with four points (or prime ends) $a(1), a(\tau), a(\tau^2), x$ on the boundary, labeled counterclockwise. Then the *crossing probability*, i.e. the probability of having a blue cluster connecting the arc $xa(1)$ to the arc $a(\tau)a(\tau^2)$ is equal to $H^\delta_{\tau^2}(x)$ (or rather to the boundary value of $H^\delta_{\tau^2}$ at x). Therefore

Corollary 2. *As $\delta \to 0$, the crossing probability tends to $h_{\tau^2}(x)$, and hence is conformally invariant.*

For a particular "nice" domain, like a rectangle or a half plane, it is easy to find h_α's and then check that crossing probabilities satisfy the Cardy's formula [5]. This is particularly easy for an equilateral triangle, where h_β's are linear:

Corollary 3 (Cardy's formula in Carleson's form)**.** *Consider an equilateral triangle abc with side length one and a point $x \in [bc]$. As $\delta \to 0$, the probability of a blue crossing from $[xb]$ to $[ac]$, tends to $|xb|$.*

Sometimes it is more convenient to work with a different formulation of the results above. For a discrete percolation configuration in conformal triangle abc there is always a unique vertex w on the arc bc with neighbors accessible by a blue crossing from ac and a yellow crossing from ab (it is just the endpoint of a blue crossing from ac to bc, closest to ab, or equivalently of a yellow crossing from ab to bc, closest to ac). It follows immediately from the discussion above that

Corollary 4 (Cardy-Carleson law)**.** *The law of w converges as $\delta \to 0$ to a distribution, which is a conformal invariant of configuration (Ω, a, b, c). For an equilateral triangle abc this distribution is uniform on the side bc.*

Remark 2. *Note also, that the probability of a point $z \in \Omega$ to be contained between the two crossings tends to $1 - h_b(z) - h_c(z) \equiv h_a(z)$.*

One can also ask about the speed of convergence in the statements above.

Remark 3. *One can check that the speed of convergence in the Theorem 1 and its Corollaries above is $O(\delta^\varepsilon)$ for some $\varepsilon > 0$, and then obtain values of crossing probabilities for the discrete case up to such error. After establishing values of exponents for percolation one can get $\varepsilon = 2/3$.*

The remaining part of this Section is devoted to the proof of Theorem 1.

2.1. **Proof of Theorem 1.** Take $\beta \in \{1, \tau, \tau^2\}$. If z is a center of some lattice triangle, and $z + \eta$ is the center of one of the adjacent triangles, denote by $P_\beta(z, \eta)$ probability of the event $Q_\beta(z + \eta) \setminus Q_\beta(z)$. Then the discrete derivative of H_β can be written as the following difference of the probabilities:

$$\frac{\partial}{\partial \eta} H_\beta(z) \ := \ H_\beta(z + \eta) - H_\beta(z) \ = \ P_\beta(z, \eta) - P_\beta(z + \eta, -\eta) \ . \tag{3}$$

Remark 4. *Since the discrete derivative is expected to have order $\asymp \delta$, and is a difference of the two probabilities on the right hand side, one might be tempted to assume that the latter also have order $\asymp \delta$. Surprisingly, they have a much bigger order $\asymp \delta^{2/3}$, so there is a non-trivial cancelation.*

Lemma 2.1 ($2\pi/3$-Cauchy-Riemann equations)**.** *Let z be a center of some triangle, and η be a vector from z to the center of one of the adjacent triangles. Then for any $\beta \in \{1, \tau, \tau^2\}$*

$$P_\beta(z, \eta) \ = \ P_{\tau\beta}(z, \tau\eta) \ . \tag{4}$$

Remark 5. *Lemma 2.1 holds not only for triangular lattice, but for any graph which is a triangulation.*

Remark 6. *Evaluating one more discrete derivative (by shifting the boundary of the domain, rather than the point), one can check that for nearby points z' and z''*

$$P_\beta(z', \eta') \ = \ P_\beta(z'', \eta') \ + \ O(\delta^{1+\varepsilon}) \ . \tag{5}$$

Together with the Lemma 2.1 this gives

$$\frac{\partial}{\partial \eta} H_\beta(z) = P_\beta(z, \eta) - P_\beta(z + \eta, -\eta) \stackrel{(4)}{=} P_{\tau\beta}(z, \tau\eta) - P_{\tau\beta}(z + \eta, -\tau\eta)$$

$$\stackrel{(5)}{=} P_{\tau\beta}(z, \tau\eta) - P_{\tau\beta}(z + \tau\eta, -\tau\eta) + O(\delta^{1+\varepsilon}) = \frac{\partial}{\partial(\tau\eta)} H_{\tau\beta}(z) + O(\delta^{1+\varepsilon}) \ .$$

Therefore the functions H_1, H_τ, H_{τ^2} satisfy discrete Cauchy-Riemann equations (or rather their version for triples) up to δ^ε (actually even up to $\delta^{2/3}$). But in passing to a scaling limit it is better to use global manifestations of analyticity, so we work with vanishing contour integrals instead.

Proof: Name the vertices of the lattice triangle which contains z by the letters X, Y, Z starting with the one opposite to $z + \eta$ and going counterclockwise. For the event $Q' := Q_\beta(z + \eta) \setminus Q_\beta(z)$ to occur, the closest to the arc $a(\tau\beta)a(\tau^2\beta)$ blue simple path γ going from the arc $a(\beta)a(\tau\beta)$ to the arc $a(\tau^2\beta)a(\beta)$ should separate z from $z + \eta$. Firstly this means that there are two disjoint blue paths ("halves" of γ), which go from Y and Z to the arcs $a(\tau^2\beta)a(\beta)$ and $a(\beta)a(\tau\beta)$ respectively. Secondly, the vertex X is colored yellow (otherwise we can include it into the path γ), and is joined by a simple yellow path to $a(\tau\beta)a(\tau^2\beta)$.

So we conclude that the event Q' can be described as an occurrence of three disjoint simple paths, joining X, Y, and Z to the arcs $a(\tau\beta)a(\tau^2\beta)$, $a(\tau^2\beta)a(\beta)$, and $a(\beta)a(\tau\beta)$, and colored yellow, blue, and blue correspondingly.

But in the latter description we can easily change the colors of the paths while preserving the probability of Q'. In fact, for a given configuration we can chose the "counterclockwise-most" yellow path from X to the arc $a(\tau\beta)a(\tau^2\beta)$, and the "clockwise-most" blue path from Y to the arc $a(\tau^2\beta)a(\beta)$. Denote by Ω' the union of these two paths and the part of Ω between them, containing $a(\tau^2\beta)$. The existence of a blue path from Z to $a(\beta)a(\tau\beta)$ depends only on the coloring of $\Omega\setminus\Omega'$. Thus if we condition by the coloring of Ω', the probability of the existence of a blue path from Z to $a(\beta)a(\tau\beta)$ is the same as the probability of such a yellow path, since we can invert colors in $\Omega\setminus\Omega'$ and use that $p=\frac{1}{2}$.

Taking expectation over all possible configurations of Ω', we deduce, that the event Q' has the same probability as the occurrence of three disjoint simple paths, joining X, Y, and Z to the arcs $a(\tau\beta)a(\tau^2\beta)$, $a(\beta)a(\tau^2\beta)$, and $a(\beta)a(\tau\beta)$, and colored yellow, blue, and yellow correspondingly. But if one inverts colors in all of Ω (which preserves probabilities), this is the description of the event $Q_{\tau\beta}(z+\tau\eta)\setminus Q_{\tau\beta}(z)$, and we proved the Lemma. □

Lemma 2.2 (Hölder norm estimates). *There are constants ε and C depending on the domain Ω only, such that H_β has ε-Hölder norm at most C. The boundary values of H_β are zero on the arc $a(\tau\beta)a(\tau^2\beta)$ and tend to 1 at the point $a(\beta)$ as $\delta\to 0$.*

Remark 7. *It is also easy to show that on $\partial\Omega$*

$$H_1+H_\tau+H_{\tau^2}\to 1\ .$$

Proof: Clearly, H_β takes values in $[0,1]$. To prove that it is Hölder, we need to show that

$$\left|H_\beta(z)-H_\beta(z')\right|\ \le\ C'\left|z-z'\right|^\varepsilon\ , \tag{6}$$

and it is sufficient to do so for pairs z,z' well away from one of the boundary arcs, say $a(\alpha)a(\tau\alpha)$. The difference on the left hand side of (6) is equal to

$$H_\beta(z)-H_\beta(z')=\mathbb{P}\left(Q_\beta(z)\setminus Q_\beta(z')\right)-\mathbb{P}\left(Q_\beta(z')\setminus Q_\beta(z)\right)\ ,$$

so it is enough to estimate the terms on the right hand side of the above equation. But for either of the corresponding events to occur, the interval $[zz']$ should be joint to the arcs $a(\tau\beta)a(\tau^2\beta)$, $a(\beta)a(\tau^2\beta)$, and $a(\beta)a(\tau\beta)$, by the yellow, blue, and blue clusters correspondingly – the reasoning is similar to that of the Lemma 2.1. Particularly, there is a monochrome cluster connecting $[zz']$ to $a(\alpha)a(\tau\alpha)$.

But $[zz']$ can be separated from $a(\alpha)a(\tau\alpha)$ by $|\log|z-z'||-c$ disjoint discrete annuli of fixed moduli. By the Lemma 4.1 probability of the existence of a monochrome cluster traversing such an annulus is bounded from above by some $q<1$, regardless of size. So the occurrence of a monochrome cluster connecting $[zz']$ to $a(1)a(\tau)$, implies a simultaneous occurrence of $|\log|z-z'||-c$ independent events of probability at most $q<1$, and hence we infer

$$\left|H_\beta(z)-H_\beta(z')\right|\le 2q^{|\log|z-z'||-c}=2q^{-c}\left|z-z'\right|^{|\log q|}\ ,$$

and the desired estimate (6) follows.

Note that as a particular consequence of this estimate for the case $z'=z+\eta$ we obtain

$$P_\beta(z,\eta)\ \le\ C'\delta^\varepsilon\ . \tag{7}$$

To prove the Lemma it remains to check the boundary values of H_β. Points on the arc $a(\tau\beta)a(\tau^2\beta)$ cannot be separated from it by a blue path, so boundary values of H_β on this arc are identically zero. The boundary value of H_β at the point $a(\beta)$ tends to 1 as $\delta\to 0$ also because of the Lemma 4.1. Indeed, there are $|\log\delta| - c$ disjoint discrete annuli of fixed shape around $a(\beta)$, each of them contains a blue cluster circumventing it independently with probability bounded from below by some $p > 0$. Thus probability of $a(\beta)$ being separated from the arc $a(\tau\beta)a(\tau^2\beta)$ is at least $1 - (1-p)^{|\log\delta| - c}$, which tends to 1 as $\delta \to 0$. This concludes the Proof. □

Take some equilateral triangular contour Γ with vertices in the centers of the lattice triangles and with bottom side parallel to the real axis. Denote vertices of Γ by $x(1), x(\tau), x(\tau^2)$ counterclockwise, starting with the top one. For a function $H(z)$ we define the discrete contour integral by

$$\oint_{\Gamma}^{\delta} H(z)dz \;:=\; \delta \sum_{z\in x(\tau)x(\tau^2)} H(z) + \delta\tau \sum_{z\in x(\tau^2)x(1)} H(z) + \delta\tau^2 \sum_{z\in x(1)x(\tau)} H(z) \;,$$

where sums are taken over centers z of lattice triangles, lying in the corresponding intervals.

Lemma 2.3 (Contour integrals vanish). *For any equilateral triangular contour $\Gamma \subset \Omega$ of length ℓ with vertices in the centers of the lattice triangles, with bottom side parallel to the real axis, and any $\beta \in \{1, \tau, \tau^2\}$ one has*

$$\oint_{\Gamma}^{\delta} H_{\beta}^{\delta}(z)\,dz \;=\; \oint_{\Gamma}^{\delta} \frac{1}{\tau} H_{\tau\beta}^{\delta}(z)\,dz \;+\; O(\ell\,\delta^{\varepsilon}) \;.$$

Proof: Color all triangles in chess-board fashion, so that triangles with centers on Γ are colored black. Denote by $\mathcal{B}$ the set of all centers of black triangles, lying on or inside Γ, and by $\mathcal{W}$ the set of all centers of white triangles, lying inside Γ.

Fix some $\alpha \in \{1, \tau, \tau^2\}$. Take η to be of length $\delta/\sqrt{3}$ collinear with $e^{\pi i/6}\,\big(x(\tau^2\alpha) - x(\tau\alpha)\big)$ and denote $\eta' := e^{\pi i/3}\eta$. We can write using (3) and (4):

$$\begin{aligned}
\sum_{z\in\mathcal{B}\setminus x(\alpha)x(\tau^2\alpha)} \left(H_\beta(z+\eta) - H_\beta(z)\right) &\overset{(3)}{=} \sum_{z\in\mathcal{B}\setminus x(\alpha)x(\tau^2\alpha)} \left(P_\beta(z,\eta) - P_\beta(z+\eta,-\eta)\right) \\
&\overset{(4)}{=} \sum_{z\in\mathcal{B}\setminus x(\alpha)x(\tau^2\alpha)} \left(P_{\tau\beta}(z,\tau\eta) - P_{\tau\beta}(z+\eta,-\tau\eta)\right) \\
&\overset{(*)}{=} \sum_{z\in\mathcal{B}\setminus x(\alpha)x(\tau\alpha)} \left(P_{\tau\beta}(z,\tau\eta) - P_{\tau\beta}(z+\tau\eta,-\tau\eta)\right) \;+\; O(\ell\delta^{\varepsilon-1}) \\
&\overset{(3)}{=} \sum_{z\in\mathcal{B}\setminus x(\alpha)x(\tau\alpha)} \left(H_{\tau\beta}(z+\tau\eta) - H_{\tau\beta}(z)\right) \;+\; O(\ell\delta^{\varepsilon-1}) \;.
\end{aligned} \tag{8}$$

In the identity (*) we also used that two sides differ by at most $O(\ell\delta^{-1})$ terms (number of vertices on Γ), which are of the order $O(\delta^{\varepsilon})$ by(7). Similarly one shows that

$$\sum_{z\in\mathcal{W}} \big(H_\beta(z+\eta') - H_\beta(z)\big) \;=\; \sum_{z\in\mathcal{W}} \big(H_{\tau\beta}(z+\tau\eta') - H_{\tau\beta}(z)\big) \;+\; O(\ell\delta^{\varepsilon-1}) \;. \tag{9}$$

Now, combining (8) and (9) we can write (using "telescoping sums")

$$\sum_{z\in x(\alpha)x(\tau^2\alpha)} H_\beta(z) - \sum_{z\in x(\tau\alpha)x(\tau^2\alpha)} H_\beta(z) =$$

$$= \sum_{z\in\mathcal{B}\setminus x(\alpha)x(\tau^2\alpha)} (H_\beta(z+\eta)-H_\beta(z)) + \sum_{z\in\mathcal{W}} \big(H_\beta(z+\tau\eta')-H_\beta(z)\big)$$

$$= \sum_{z\in\mathcal{B}\setminus x(\alpha)x(\tau\alpha)} (H_{\tau\beta}(z+\tau\eta)-H_{\tau\beta}(z)) + \sum_{z\in\mathcal{W}} \big(H_{\tau\beta}(z+\tau\eta')-H_{\tau\beta}(z)\big) + O(\ell\delta^{\varepsilon-1})$$

$$= \sum_{z\in x(\alpha)x(\tau\alpha)} H_{\tau\beta}(z) - \sum_{z\in x(\alpha)x(\tau^2\alpha)} H_{\tau\beta}(z) \ + \ O(\ell\delta^{\varepsilon-1}) \ .$$

So for any $\alpha\in\{1,\tau,\tau^2\}$ one has

$$\sum_{z\in x(\alpha)x(\tau^2\alpha)} H_\beta(z) - \sum_{z\in x(\tau\alpha)x(\tau^2\alpha)} H_\beta(z) = \sum_{z\in x(\alpha)x(\tau\alpha)} H_{\tau\beta}(z) - \sum_{z\in x(\alpha)x(\tau^2\alpha)} H_{\tau\beta}(z) + O(\ell\delta^{\varepsilon-1}) \ . \tag{10}$$

Summing three copies of equation (10) above with coefficients and different values of α plugged in:

$$-\frac{\delta}{2}(\text{equation 10})|_{\alpha=1} - i\frac{\delta\sqrt{3}}{2}(\text{equation 10})|_{\alpha=\tau} + \frac{\delta}{2}(\text{equation 10})|_{\alpha=\tau^2} \ ,$$

we arrive at the desired identity and prove the Lemma. □

By Lemma 2.2 the Hölder norms of functions H^δ_α, $\alpha\in\{1,\tau,\tau^2\}$ are uniformly bounded, hence from any sequence of such functions with $\delta\to 0$ one can chose a uniformly converging subsequence. Therefore to show that the functions H^δ_α converge uniformly, as $\delta\to 0$, to the functions h_α and prove Theorem 1, it is sufficient prove the following

Lemma 2.4. *Assume that for some subsequence $\delta_j\to 0$ the functions $H^{\delta_j}_\alpha$ converge uniformly in Ω to some functions f_α. Then $f_\alpha\equiv h_\alpha$.*

Proof: Clearly the discrete contour integrals $\oint^\delta H^{\delta_j}_\beta$ converge to the usual contour integrals $\oint f_\beta$. Then using Lemma 2.3 we conclude (one may need to shift contour by $\le\delta$ so that its vertices are in the centers of lattice triangles, but this perturbation does not affect the limit), that for any equilateral triangular contour $\Gamma\subset\Omega$ of length ℓ, with bottom side parallel to the real axis, and any $\beta\in\{1,\tau,\tau^2\}$ one has

$$\oint_\Gamma f_\beta(z)\,dz \ = \ \oint_\Gamma \frac{1}{\tau} f_{\tau\beta}(z)\,dz \ . \tag{11}$$

Take any $\alpha\in\{1,\tau,\tau^2\}$. Adding two copies of (11) with coefficients and different values of β plugged in:

$$(\text{equation 11})|_{\beta=\alpha} - \left(\frac{1}{2}+\frac{i}{2\sqrt{3}}\right)\cdot(\text{equation 11})|_{\beta=\tau\alpha} \ ,$$

we obtain

$$\oint_\Gamma \left(f_\alpha(z) \ + \ \frac{i}{\sqrt{3}}\left(f_{\tau\alpha}(z)-f_{\tau^2\alpha}(z)\right)\right) dz \ = \ 0 \ . \tag{12}$$

By Morera's theorem we deduce that the function in (12) is analytic, and hence for any $\alpha \in \{1, \tau, \tau^2\}$ the function f_α is harmonic with harmonic conjugate $\frac{1}{\sqrt{3}}\left(f_{\tau\alpha}(z) - f_{\tau^2\alpha}(z)\right)$. It follows that for any $\alpha \in \{1, \tau, \tau^2\}$ and any unit vector η

$$\frac{\partial}{\partial\eta} f_\alpha \; = \; \frac{\partial}{\partial(\tau\eta)} f_{\tau\alpha} \; . \tag{13}$$

Hence

$$\frac{\partial}{\partial(\tau\nu)} f_\alpha \equiv \frac{\partial}{\partial\nu} f_{\tau^2\alpha} \equiv \frac{\partial}{\partial\nu} 0 \equiv 0 \text{ for points on the arc } a(\alpha)a(\tau\alpha) \; ,$$

$$\frac{\partial}{\partial(-\tau^2\nu)} f_\alpha \equiv \frac{\partial}{\partial(-\nu)} f_{\tau\alpha} \equiv \frac{\partial}{\partial(-\nu)} 0 \equiv 0 \text{ for points on the arc } a(\tau^2\alpha)a(\alpha) \; .$$

The identities above should be understood as holding in the limit as $z \in \Omega$ tends to the corresponding boundary arc. Also by the Lemma 2.2 the boundary values of f_α are equal to zero on the arc $a\,(\tau\alpha)\,a\,(\tau^2\alpha)$ and to 1 at the point $a\,(\alpha)$.

Summing it up, we conclude that f_α's satisfy the mixed Dirichlet-Neumann problem (2), which has a unique solution. Thus $f_\alpha = h_\alpha$, and we have proven the Lemma and the Theorem. □

3. Continuum scaling limits

Consider some domain Ω with three points (or prime ends) a, b, and c on the boundary, named counterclockwise. For a discrete percolation configuration we can define "the lowest blue crossing," $\gamma^{l.c.} = \gamma_\delta^{l.c.} = \gamma_{\mathcal{B}}^{l.c.}(a, b, c)$, as the closest to the arc ab, simple blue curve going from the arc ca to the arc bc. For simplicity, if no such crossing exists, we assume it to consist of one vertex, closest to c. This does not influence our reasoning since by the Lemma 4.1 the probability of the existence of a genuine crossing tends to 1 as $\delta \to 0$. Similarly, we define "the highest yellow crossing," $\gamma_{\mathcal{Y}}^{l.c.}(c, a, b)$, as the closest to the arc ca, simple yellow curve going from the arc bc to the arc ab.

These curves end at neighboring vertices on the arc bc, so we can join them to obtain a simple curve $\gamma^{o.b.} = \gamma_\delta^{o.b.} = \gamma_{\mathcal{Y},\mathcal{B}}^{o.b.}(a, b, c)$, which starts on the arc ab and ends on the arc ca, touching the arc bc in the middle. The law of $\gamma_\delta^{o.b.}$ is a probability measure $\mu_\delta^{o.b.}$, supported on the space $\mathcal{H}$ of Hölder curves (we identify a discrete curve on the δ-lattice with the corresponding broken line).

Theorem 2 (Outer boundary)**.** *As $\delta \to 0$, the law of $\gamma^{o.b.}$ converges to a law $\mu^{o.b.}$ on Hölder self-avoiding paths from ab to ac, touching bc at one point, which separates them into two simple "halves." The law $\mu^{o.b.}$ is a conformal invariant of the configuration (Ω, a, b, c).*

Note that in the scaling limit two halves of the $\gamma^{o.b.}$ are the outer boundaries (inside Ω) of all yellow percolation clusters intersecting the arc ab and viewed from c, and all blue percolation clusters intersecting the arc ca and viewed from b.

In the proof we automatically obtain, that the law of $\gamma^{o.b.}$ is CCI (Completely Conformally Invariant in the language of [17, 19]), and points where $\gamma^{o.b.}$ touches the three arcs satisfy the Cardy-Carleson law (Remark 2 also gives that $h_a(c)$ is the probability of a point to be separated by $\gamma^{o.b.}$ from b and c). These two properties determine the law uniquely (cf. [19]). Hence using [13, 19] we conclude that the law of $\gamma^{o.b.}$ coincides with two other laws, enjoying the same properties: with the laws of the hull of Oded Schramm's chordal SLE_6 (started at a, aimed at some point in bc and stopped upon hitting bc) or hull of the reflected Brownian motion (started at a, reflected on ab and ac at $\frac{\pi}{3}$-angle pointing towards bc, and stopped upon hitting bc), which was considered by Wendelin Werner in [19].

Corollary 5. *The law $\mu^{o.b.}$ coincides with the law of the boundary of the hull of chordal SLE_6 (or reflected Brownian motion) started at a and stopped upon hitting bc.*

By the work [16] of Greg Lawler, Oded Schramm, and Wendelin Werner we know the dimension of the outer boundary of Brownian motion, so we deduce

Corollary 6. *The curves $\gamma^{o.b.}$ and $\gamma^{l.c.}$ (and hence the outer boundary of a percolation cluster) have Hausdorff dimension $\frac{4}{3}$ almost surely.*

Remark 8. *Lowest crossing depends only on the area "below" it, and all our considerations are stable under perturbations of domains, since Cardy's formula is. So one can continue working with the area "above" the lowest crossing, finding crossings for some configurations there. Using an inductive procedure it is not difficult to construct in this way scaling limits for laminations and backbones (and then in a similar inductive way pass to the full percolation configuration).*

Consider now a domain Ω with two boundary points a and b. For any percolation configuration there is a unique curve $\gamma^{e.p.}$ along the edges of the dual hexagonal lattice, which goes from a to b separating the blue clusters intersecting the arc ab from the yellow clusters intersecting the arc ba. This curve is called the "exploration process" (one can actually introduce time) or "external perimeter" (of all blue clusters intersecting ab, as viewed from ba).

We say that a path is *self-avoiding* if it does not have "transversal self intersections" (maybe a more appropriate term would be "non-self-traversing").

Theorem 3 (External perimeter)**.** *As $\delta \to 0$, the law of the exploration process $\gamma^{e.p.}$ converges to a law $\mu^{e.p.}$ on Hölder self-avoiding paths from a to b. The law is a conformal invariant of the configuration (Ω, a, b).*

It is possible to prove (using known crossing exponent for "5 arms") that the curve $\gamma^{e.p.}$ is almost surely not simple. In proving that the subsequential limit of the laws $\mu_\delta^{e.p.}$ is uniquely determined, we only use its properties (locality and Cardy's formula) valid also for Oded Schramm's SLE_6 process by [13, 18], and so we arrive at

Corollary 7. *The law $\mu^{e.p.}$ coincides with that of the Schramm's chordal SLE_6 started at a and aiming at b.*

As discussed above, the discrete percolation configuration in the whole plane can be represented by a collection of all external perimeters: (as a collection of nested pairwise disjoint oriented simple loops). It is not difficult (though a bit technical) to obtain from the results above the following

Theorem 4 (Collection of external perimeters)**.** *As $\delta \to 0$, the law of the collection of all external perimeters converges to a law μ on collections of mutually- and self-avoiding oriented nested Hölder loops. This law (and its restriction to any domain) is conformally invariant.*

We sketch the proof below. In a subsequent paper we intend to give a different, perhaps more conceptual, proof of this Theorem.

3.1. **Proof for the outer boundary.** By the Lemma 4.2 (which comes from the Theorem A.1 in [3] by M. Aizenman and A. Burchard) the family of measures $\{\mu_\delta^{o.b.}\}_\delta$ is weakly precompact in $\mathcal{H}$. Indeed, $\lim_{M\to\infty} \mu_\delta^{o.b.}(\mathcal{H} \setminus \mathcal{H}_M) = 0$ uniformly in δ, whereas the sets $\mathcal{H}_M$ are compact.

Thus one can chose a sequence $\delta_j \to 0$ so that the subsequence $\left\{\mu^{o.b.}_{\delta_j}\right\}$ converges weakly to a measure $\mu^{o.b.}$. It is sufficient to show that the latter measure is independent of the chosen subsequence. For the rest of the section we fix the sequence $\{\delta_j\}$.

Since the property of being self-avoiding is preserved in the limit, the measure $\mu^{o.b.}$ is supported on self-avoiding Hölder curves, starting on the arc ab, touching the arc bc and ending on the arc ac. The two "halves" of the curve $\gamma^{o.b.}$ will almost surely be simple (However, the whole curve need not be). Indeed, the probability of a simple curve becoming non-simple in the scaling limit is zero: it would imply six "arms," which happens with probability zero by Lemma 4.3. Similarly one shows, that almost surely the curve $\gamma^{o.b.}$ touches the arc bc at a unique point (otherwise there would be three "arms" going to a boundary point, which happens with probability zero). Denote by $\mathcal{H}^{o.b.}$ the set of the Hölder curves satisfying the properties above.

For two disjoint simple curves η (going from $p \in ab$ to $q \in bc$ and denote pq below) and η' (going from $r \in bc$ to $s \in ac$ and denoted rs below), let $A_{\eta,\eta'}$ be the event of $\gamma^{o.b.}$ lying completely inside the conformal pentagon $apqrs$. It is easy to see that the event $A_{\eta,\eta'}$ is equivalent to the a yellow crossing from ap to $pqrs$ within $apqrs$, which is closest to as, ending inside qr. Since this probability is given by the Cardy-Carleson law, it is uniquely determined and so is the probability of $A_{\eta,\eta'}$.

It remains to notice that the events $A_{\eta,\eta'}$ generate (by disjoint unions and complements) the restriction of the σ-algebra $\mathcal{B}$ to the set $\mathcal{H}^{o.b.}$, which supports the measure $\mu^{o.b.}$. Hence the measure $\mu^{o.b.}$ is independent of the subsequence chosen.

Note that we automatically obtain that $\mu^{o.b.}$ is conformally invariant, since so is the Cardy-Carleson law. The Corollaries follow, since the properties we used are satisfied by the hull of reflected Brownian motion (see [19]) and by the hull of the Schramm's SLE_6 (see [13]).

3.2. **Proof for the external perimeter.** By the Lemma 4.2 the family of measures $\left\{\mu^{e.p.}_{\delta}\right\}_{\delta}$ is weakly precompact in $\mathcal{H}$. Thus one can choose a sequence $\delta_j \to 0$ so that the subsequence $\left\{\mu^{e.p.}_{\delta_j}\right\}$ converges weakly to a measure $\mu^{e.p.}$. Again, it is sufficient to show that the latter measure is independent of the chosen subsequence. For the rest of the section we fix the sequence $\{\delta_j\}$.

Since such properties are preserved in the limit, the measure $\mu^{e.p.}$ is supported on self-avoiding Hölder curves, starting at a, and terminating at b. When $\gamma^{e.p.}(t)$ is some parametrization of such a curve by the interval $[0,1]$, denote by $H(t)$ its hull at time t, i.e. the closure of $\Omega \setminus \Omega(t)$. Here $\Omega(t)$ denotes the component of connectivity of $\Omega \setminus \gamma^{e.p.}[0,t]$, containing the point b. Then the hull grows, i.e. it is strictly larger for larger values of t, meaning that the endpoint of $\gamma^{e.p.}[0,t]$ is always "visible" from b. Indeed, otherwise the external perimeter would enter a zero width fjord, implying eight "arms," which happens with probability zero by Lemma 4.3.

Denote by $\mathcal{H}^{e.p.}$ the set of the Hölder curves satisfying the properties above. It is easier to work with perimeters by parameterizing them (e.g. by diameter). One can do it in the following way. Namely, to each $\gamma \in \mathcal{H}^{e.p.}$ and $\epsilon > 0$ we associate a broken line $\gamma_\epsilon = \phi_\epsilon(\gamma)$ by the following inductive procedure. The first point is $\gamma^0_\epsilon := \gamma(t_0) = a$. Then γ^{j+1} is the first exit of $\gamma[t_j, 1]$ from the ball $B(\gamma^j, \epsilon)$. If it never exits this ball, we set $\gamma^{j+1} := b$ and end the procedure.

The law of curves γ_ϵ is uniquely determined. In fact it is easy to see by induction that the laws of t_j, $H(t_j)$, and hence γ^j are uniquely determined: in fact, the law of $H(t_{j+1}) \setminus H(t_j)$ coincides with the law $\mu^{o.b.}$ inside $B(\gamma_j, \epsilon) \setminus H(t_j)$. We use that percolation is local, and the hull depends only on percolation inside it (stability of Cardy's formula under perturbations of the boundary is also used here).

Thus the measure $\phi_\epsilon^{-1}(\mu)$ is uniquely determined. Recall that $\lim_{M\to\infty}\mu^{e.p.}(\mathcal{H}\setminus\mathcal{H}_M)=0$ and also note that γ_ϵ converges to γ uniformly in $\mathcal{H}_M$. Therefore, $\phi_\epsilon^{-1}(\mu)$ converge weakly to μ, and μ is uniquely determined.

As before, we automatically obtain that $\mu^{e.p.}$ is conformally invariant, since so are the Cardy-Carleson law and $\mu^{o.b.}$. The Corollary follows, since the properties we used are satisfied by the Schramm's SLE_6 (see [13]).

3.3. **Sketch of the proof for the full limit.** Again, [3] gives enough information to obtain a subsequential scaling limit: a measure μ on the mentioned space, and it is sufficient to prove that this measure is uniquely determined. One of a few possible ways is to proceed as follows.

For any domain Ω with two boundary points a and b to the law μ corresponds the law $\mu^{e.p.}=\mu_\Omega^{e.p.}$ of external perimeter in Ω, which is independent from the choice of a subsequence determining μ. For any particular external perimeter curve $\gamma^{e.p.}\in\mathcal{H}^{e.p.}(\Omega)$, consider the connected components $\{\Omega'\}$ of $\Omega\setminus\gamma^{e.p.}$. Usual "number of arms" considerations show that almost surely no other external perimeters can go from one such component to another within Ω (one has to also use even more classical estimate for 3 arms in a half-plane). Also the law $\mu^{e.p.}$ depends only on $\gamma^{e.p.}$ (or rather on its "infinitesimal neighborhood"). So for any component Ω' bordering the boundary on the arc $a'b'$ we can do the similar considerations, retrieving uniquely the law of the external perimeter within Ω' from a' to b' (or in the opposite direction, depending on the location of a' and b': whether it is ba or ab). There is a subtle point here: one has to check that in some (related to conformal geometry) sense law of external perimeter is stable under perturbations of a domain. Continuing by induction, one retrieves (in the limit) uniquely the law of all arcs of curves (from the full collection of external perimeters) with endpoints on the boundary of Ω. Now performing a new inductive procedure with this law, one can obtain the full scaling limit.

4. Appendix: Technical estimates

We gathered in this Section some known (see [1, 3, 8]) estimates we need.

Lemma 4.1 (Russo-Seymour-Welsh estimates). *For a conformal rectangle (or an annulus) of a fixed shape probability of the existence of a monochrome crossing is contained in an interval $[p,q]$. The constants $p>0$, $q<1$ depend on the shape, but not on the size of the rectangle or mesh of the lattice.*

This Lemma is needed to establish Hölder continuity of harmonic conformal invariants. For bond percolation such a statement can be found in 11.70 of G. Grimmett's book [8]. The proof is based on self-duality property (forcing the crossing probabilities of symmetric shapes to be $\frac{1}{2}$), so it equally applies to critical site percolation on triangular lattice.

Lemma 4.2. *For discrete percolation all the non-repeating paths supported on the connected clusters (or external perimeters of clusters) within some compact region can be simultaneously parameterized by Hölder continuous functions on $[0,1]$, whose Hölder norms are (uniformly in the mesh δ) stochastically bounded.*

This Lemma is used to establish precompactness of laws describing discrete percolations. It is stated as Theorem A.1 at the end of [3] by M. Aizenman and A. Burchard for non-repeating paths, and the main theorems of the paper clearly apply to external perimeters as well.

Lemma 4.3. *For two concentric balls of radii r and R probability of existence for discrete percolation of 5 crossings (not all of the same color) between their boundaries is $\leq \operatorname{const}(r/R)^2$ uniformly in the mesh δ of the lattice. Hence (e.g. by the Russo-Seymour-Welsh theory) the similar probability for 6 crossings is $\leq \operatorname{const}(r/R)^{2+\epsilon}$, and it immediately follows that in the scaling limit almost surely there is no point with 6 curves (not all of the same color) incoming.*

We use this Lemma (see [1] by M. Aizenman) to show that simple curves cannot "collapse" when we pass to the subsequential limit. For general number of arms, the exact exponents were predicted by M. Aizenman, B. Duplantier, and H. Aharony in [4] (see also related papers of B. Duplantier, e.g. [6] and references therein). Up to now there was no direct proof, but since exponents are known for SLE_6, it is not difficult to write a proof on the basis of the current paper and work of G. Lawler, O. Schramm, and W. Werner [13, 14, 15].

We only need a very special case of 5 arms which can be calculated directly (along with 2 or 3 arms in the half-plane, since they have exponents 1 or 2 and some "derivative representation.")

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