Optimal Control of the Obstacle Problem in a Perforated Domain

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Abstract

We study the problem of optimally controlling the solution of the obstacle problem in a domain perforated by small periodically distributed holes. The solution is controlled by the choice of a perforated obstacle which is to be chosen in such a fashion that the solution is close to a given profile and the obstacle is not too irregular. We prove existence, uniqueness and stability of an optimal obstacle and derive necessary and sufficient conditions for optimality. When the number of holes increase indefinitely we determine the limit of the sequence of optimal obstacles and solutions. This limit depends strongly on the rate at which the size of the holes shrink.

1 Introduction

1.1 The obstacle problem

Given a bounded open set Ω in $\mathbb{R}^n$, $n \geq 2$, and a function $\psi \in H^1(\Omega)$ such that $\psi \leq 0$ on $\partial\Omega$, the obstacle problem is to find u in the set

$$\mathcal{K}_\psi = \{u \in H_0^1(\Omega) : u \geq \psi\},$$

satisfying the variational inequality

$$\int_\Omega \nabla u \cdot \nabla(v - u)dx \geq 0, \quad \text{for all } v \in \mathcal{K}_\psi. \tag{1}$$

This u is the unique minimizer of the Dirichlet integral

$$\int_\Omega |\nabla v|^2 dx$$

over $\mathcal{K}_\psi$, and it is a well known fact that u is the (pointwise) smallest superharmonic function in Ω that stays above the obstacle ψ. We define the solution operator F by $F(\psi) = u$ when u solves (1) with obstacle ψ. We shall also consider the above problem with a source term, namely

$$\int_\Omega \nabla u \cdot \nabla(v - u)dx \geq \langle f, v - u\rangle, \quad \text{for all } v \in \mathcal{K}_\psi, \tag{2}$$

where $f \in H^{-1}(\Omega)$ is given and $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Then u is the unique minimizer of

$$\int_\Omega \frac{1}{2}|\nabla v|^2 dx - \langle f, v \rangle$$

over $\mathcal{K}_\psi$, and the least f-supersolution staying above ψ. For an overview of variational inequalities and the obstacle problem, see [KS00], and for its regularity properties, [Caf98].

1.2 Perforated domains and homogenization

From the domain Ω, a perforated domain is obtained by removing certain subsets, denoted T_i^ε, $i = 1, \ldots, N_\varepsilon$ of Ω, thereby yielding a set Ω_ε with holes. We choose these subsets to be balls uniformly distributed in $\mathbb{R}^n$. A perforated domain is then constructed as follows: For each $\varepsilon > 0$ let $\{P_i^\varepsilon\}_i$ be mutually disjoint cubes with sides of length 2ε covering $\mathbb{R}^n$. At the center of each cube we put a ball T_i^ε of radius $a^\varepsilon < \varepsilon$, i.e. $T_i^\varepsilon = B_{a^\varepsilon}(x_i)$. The set

$$\Omega_\varepsilon = \Omega - \cup_i T_i^\varepsilon$$

is then our perforated domain. Other examples can be found in [CM97].

Having constructed a perforated domain one may study Dirichlet and obstacle problems related to Ω_ε, with solutions u^ε. As $\varepsilon \to 0$, one wants to determine the limit u^0 of u^ε in terms of an equation it solves, called limit problem. This procedure is a particular form of homogenization. Homogenization of the obstacle problem in perforated domains has been addressed in many papers, see e.g. Carmine and Colombini [CC80], Attouch and Picard [AP83], Cioranescu and Murat [CM97] and Caffarelli and Lee [CL08].

1.3 Optimal control of the obstacle

Optimal control of the obstacle was studied by Adams, Lenhart and Yong in [ALY98]. The objective was to minimize the functional

$$J(\psi) = \int_\Omega (z - u)^2 + |\nabla \psi|^2 dx$$

over obstacles $\psi \in H_0^1(\Omega)$ with corresponding solutions u of (1). The function z, denoted *profile*, is a given element of $L^2(\Omega)$. Employing terminology of control theory, the obstacle ψ is referred to as *control* and the solution u is called *state*. It was proved in [ALY98] that J has a unique minimizer (control) $\hat{\psi}$, that coincides with the state, i.e. $\hat{u} = \hat{\psi}$. Moreover, this $\hat{u}$ is uniquely determined by a triple $(\hat{u}, \hat{p}, \hat{\mu})$, where $\hat{u}, \hat{p} \in H_0^1(\Omega)$ and $\hat{\mu} \in \mathcal{E}^+$ (a measure of finite energy) satisfying

$$\begin{cases} -\Delta \hat{u} = \hat{\mu} \quad \text{in } \Omega, \\ -\Delta \hat{p} = \hat{\mu} + \hat{u} - z \quad \text{in } \Omega, \\ \displaystyle\int_\Omega \hat{p} \, d\hat{\mu} = 0, \quad \hat{p} \geq 0. \end{cases} \tag{3}$$

1.4 Formulation of the problem and applications

In this paper we study the optimal control of solutions to the following variational inequality:

$$\begin{cases} u^\varepsilon \in \mathcal{K}^\varepsilon_\psi = \{u \in H^1_0(\Omega) : u \geq \psi\chi_{\cup T^\varepsilon_i}\}, \\ \displaystyle\int_\Omega \nabla u^\varepsilon \cdot \nabla(v - u^\varepsilon)dx \geq 0, \quad \text{for all } v \in \mathcal{K}^\varepsilon_\psi. \end{cases} \tag{4}$$

For any $\psi \in H^1_0(\Omega)$ (4) has a unique solution u^ε. It is the unique minimizer of

$$\int_\Omega |\nabla v|^2 dx$$

over $\mathcal{K}^\varepsilon_\psi$ and also the least superharmonic function above $\psi\chi_{\cup T^\varepsilon_i}$. Thus a mapping

$$F^\varepsilon : \begin{cases} H^1_0(\Omega) & \to & \mathcal{K}^\varepsilon_\psi \\ \psi & \to & u^\varepsilon \quad (\text{obstacle} \mapsto \text{solution}), \end{cases}$$

may be defined. For a given target profile $z \in L^2(\Omega)$, we pose the following problem:

$$\begin{cases} \text{Find } \psi \in H^1_0(\Omega) \text{ such that} \\ J(\psi) := \displaystyle\int_\Omega (F^\varepsilon(\psi) - z)^2 + |\nabla\psi|^2 dx \leq J(\varphi) \text{ for all } \varphi \in H^1_0(\Omega). \end{cases} \tag{5}$$

In particular, as $\varepsilon \to 0$, we want to determine the asymptotic behaviour of the optimal control ψ^ε and corresponding solution $F^\varepsilon(\psi^\varepsilon)$ in terms of homogenized equations.

The simplest application of problem (5) is to find a function $\psi \in H^1_0(\Omega)$ such that a membrane over Ω, attached at $\partial\Omega$, assumes a form close to z when pushed up by the obstacle $\psi\chi_{\cup T^\varepsilon_i}$ consisting of thin cylindrical columns. It turns out that problem (5) has a unique solution ψ^ε such that $u^\varepsilon = F^\varepsilon(\psi^\varepsilon) = \psi^\varepsilon$. Additionally, u^ε may represented as $u^\varepsilon = G\mu^\varepsilon = \int_\Omega G(x, y)d\mu^\varepsilon(z)$, where G is the Green's function of Ω and $\mu^\varepsilon \in \mathcal{E}^+_\varepsilon$ (see notations section). The measure μ^ε is determined by

$$\int_\Omega (G\mu^\varepsilon - z)^2 + |\nabla G\mu^\varepsilon|^2 dx \leq \int_\Omega (G\nu^\varepsilon - z)^2 + |\nabla G\nu^\varepsilon|^2 dx, \tag{6}$$

for all $\nu^\varepsilon \in \mathcal{E}^+_\varepsilon$. This leads to applications for phenomena governed by Poisson's equation. The problem is then to optimally control a quantity u^ε by measures μ^ε (sources) with support in $\cup\overline{T^\varepsilon_i}$ such that $-\Delta u^\varepsilon = \mu^\varepsilon$ in Ω and $u^\varepsilon = 0$ on $\partial\Omega$. For example we may think of heating a tall structure with cylindrical fibres. The temperature in a cross section with heat source μ^ε from the fibres is $G\mu^\varepsilon$, which we want to be close to z without using too much energy.

Optimal control of PDE in perforated domains has been studied by several authors, e.g. Kesavan and Saint Jean Paulin, [KSJP99] and Saint Jean Paulin and Zoubairi, [SJPZ02]. In these papers the control appears as a source term in an equation with prescribed boundary values on the holes, and one seeks to minimize a combination of the energy of the solution and the L^2 - norm of the control. The set of admissible controls is a convex set $U^{\text{ad}}_\varepsilon \subset L^2(\Omega)$. As was outlined above, the present problem may be formulated as controlling the Poisson equation with a source term, the set of admissible source terms being the convex $\mathcal{E}_\varepsilon^+ \subset H^{-1}(\Omega)$. However, the objective functional is a combination of the energy of the solution and the closeness of the solution in L^2 - norm to a given profile, not the control. Also there are no conditions on the holes when using this formulation, only on the source term. The techniques used in the papers [KSJP99] and [SJPZ02] are also rather different from those of the present.

1.5 Main results

First we prove that Problem (5) has a unique solution ψ^ε such that $u^\varepsilon := F^\varepsilon(\psi^\varepsilon) = \psi^\varepsilon$. That is, the unique control ψ^ε that solves Problem (5) agrees with its corresponding state u^ε. The next theorems provide us with a means of computing this control, and give a complete description of its possible limits, as $\varepsilon \to 0$.

Theorem (Characterization for a fixed ε)

Let u^ε solve Problem (5)*. Then u^ε is uniquely determined by the following elliptic system:*

$$\begin{cases} -\Delta u^\varepsilon = \mu^\varepsilon \quad \text{in } \Omega,\ \operatorname{supp}(\mu^\varepsilon) \subset \cup \overline{T_i^\varepsilon} \\ -\Delta p^\varepsilon = \mu^\varepsilon + u^\varepsilon - z \quad \text{in } \Omega, \\ \displaystyle\int_\Omega p^\varepsilon d\mu^\varepsilon = 0,\ p^\varepsilon \geq 0 \ \text{in } \cup T_i^\varepsilon. \end{cases} \tag{7}$$

Here $u^\varepsilon \in H_0^1(\Omega)$, $p^\varepsilon \in H_0^1(\Omega)$ and $\mu^\varepsilon \in (H^{-1}(\Omega))^+$.

Theorem (Homogenized equations)

As $\varepsilon \to 0$, the limit of u^ε depends on the rate at which $a^\varepsilon \to 0$ as follows: There is a critical rate of decay a_*^ε of the radii of the holes T_i^ε such that

i) *If $a_*^\varepsilon = o(a^\varepsilon)$ then $u^\varepsilon \to u$ strongly in $H_0^1(\Omega)$ where u is uniquely determined by the system* (3)*.*

ii) *If $a^\varepsilon = a_*^\varepsilon$, then $u^\varepsilon \rightharpoonup u$ where u is uniquely determined by the pair $u, p \in H_0^1(\Omega)$ solving*

$$\begin{cases} -\Delta u = \nu p^-, \quad \text{in } \Omega \\ -\Delta p = \nu p^- + u - z, \quad \text{in } \Omega. \end{cases} \tag{8}$$

iii) *If $a^\varepsilon = o(a_*^\varepsilon)$, then $u^\varepsilon \rightharpoonup 0$.*

For the definition of a_*^ε and ν, see (10) and (11) in Section 2. These results are proved in Sections 3-4, and are formulated as one in Theorem 4.3.

In the next theorem the functional J in (5) is replaced by

$$J_\alpha(\psi) := \int_\Omega (F^\varepsilon(\psi) - z)^2 + \alpha|\nabla\psi|^2 dx,$$

enabling a trade-off between closeness to the profile and regularity of the obstacle. The aforementioned results remain valid after this modification, but (7) and (8) become (26) and (27) respectively.

Theorem (Stability w.r.t. the profile z)

Let u_i be given by either (7) or (8), for $z = z_i$, $i = 1, 2$. Then

$$\|u_1 - u_2\|^2_{H^1_0(\Omega)} \leq \frac{1}{2\alpha}\|z_1 - z_2\|^2_{L^2(\Omega)}$$

and

$$\|u_1 - u_2\|^2_{L^2(\Omega)} \leq \|z_1 - z_2\|^2_{L^2(\Omega)}.$$

Thus, the optimal control is stable with respect to the profile, and this stability persists in the limit $\varepsilon \to 0$. Of course, the less effort we put on minimizing the norm of the control, the less stability we get.

1.6 Outline of the paper

It is natural to first ask what the limit of u^ε solving (4) is for some fixed obstacle ψ (not depending on ε). The answer is Theorem 2.5. An analoguous theorem for viscosity solutions was proved by Caffarelli and Lee in the paper [CL08], where they also extended the result to the fully nonlinear case. The papers [AP83] and [CM97] treat problems similar to (4) using variational formulations, which is the appropriate notion for the optimal control problem studied in this paper. Theorem 2.5 can be derived from these papers relatively easily, and we will show how it follows from a theorem by Cioranescu and Murat in [CM97]. The proof of this consists of transferring problem (4) to one studied in [CM97], and invoke Theorem 4.1. therein. The same theorem will also be the main tool when proving convergence of the optimal control later on. This is the the topic of Section 2.

Next, Section 3 is devoted to existence and uniqueness as well as some initial characterization of the optimal control. For any $\varepsilon > 0$, we prove the existence of a unique optimal control ψ^ε solving problem (5) that agrees with the corresponding state u^ε.

In Section 4 we show that the optimal control (and state) u^ε is uniquely determined by a system of elliptic equations. We also determine the limit u of u^ε, as $\varepsilon \to 0$. Depending on the rate at which $a^\varepsilon \to 0$ there are three possible scenarios;

i) if a^ε is too big, then $u = 0$,

ii) if a^ε is too small, then u is determined by (3),

iii) if $a^\varepsilon = a^\varepsilon_*$ there is an intermediate situation where u is determined by a new system of elliptic equations:

$$\begin{cases} -\Delta u = \nu p^- & \text{in } \Omega,\ u|_{\partial\Omega} = 0, \\ -\Delta p = \nu p^- + u - z & \text{in } \Omega,\ p|_{\partial\Omega} = 0. \end{cases}$$

Finally in section 5 we give some stability results for the optimal control problem. We show that when two profiles z_1 and z_2 are close in the norm of $L^2(\Omega)$, the solutions u_1 and u_2 of the corresponding optimal control problems are close in the norm of $H^1(\Omega)$. We also address the question of putting different weights on the terms in the functional J in (5).

1.7 Notations

Ω	A bounded open subset of $\mathbb{R}^n,\ n \geq 2$.
$\{T_i^\varepsilon\}_i$	The collection of balls of radius $a^\varepsilon < \varepsilon$ centered at lattice points $x_i \in 2\varepsilon\mathbb{Z}^n$.
Ω_ε	$\Omega - \cup_i T_i^\varepsilon$.
χ_E	The characteristic function of the set E.
ψ	Obstacle. Either $\psi \in H_0^1(\Omega)$ or $\psi \in H^1(\Omega)$ and $\psi \leq 0$ on $\partial\Omega$.
$\mathcal{K}_\psi$	$\{u \in H_0^1(\Omega) : u \geq \psi \text{ in } \Omega\}$.
$\mathcal{K}_\psi^\varepsilon$	$\{u \in H_0^1(\Omega) : u \geq \psi\chi_{\cup_i T_i^\varepsilon} \text{ in } \Omega\}$.
F	The solution operator $H_0^1(\Omega) \to \mathcal{K}_\psi$ to problem (1).
F^ε	The solution operator $H_0^1(\Omega) \to \mathcal{K}_\psi^\varepsilon$ to problem (4).
$\mathcal{H}_\varepsilon^+$	$\{u \in H_0^1(\Omega) : u \text{ is superharmonic in } \Omega \text{ and harmonic in } \Omega_\varepsilon\}$.
$\mathcal{E}^+$	Measures of finite energy, i.e. positive measures $\mu \in H^{-1}(\Omega)$
$\mathcal{E}_\varepsilon^+$	$\{\mu \in \mathcal{E}^+ : \operatorname{supp}\mu \subset \cup\overline{T_i^\varepsilon}\}$.

2 Homogenization of perforated obstacles

Let $\psi \in H^1(\Omega)$ be a given obstacle, $\psi \leq 0$ on $\partial\Omega$. We recall theorem 4.1. in [CM97]. This theorem concerns an obstacle problem with a source term $f \in H^{-1}(\Omega)$ and obstacle $\psi\chi_{\Omega_\varepsilon}$ instead of $\psi\chi_{\cup T_i^\varepsilon}$. Thus one studies the solution of

$$\begin{aligned} &y^\varepsilon \in H_0^1(\Omega),\ y^\varepsilon \geq \psi\chi_{\Omega_\varepsilon}, \\ &\int_\Omega \nabla y^\varepsilon \cdot \nabla(w - y^\varepsilon)dx \geq \langle f, w - y^\varepsilon\rangle, \quad \text{for all } w \in H_0^1(\Omega),\ w \geq \psi\chi_{\Omega_\varepsilon}, \end{aligned} \tag{9}$$

for a given $f \in H^{-1}(\Omega)$. The limit of y^ε depends on how fast the holes T_i^ε shrink, and the main tool of [CM97] is the construction of certain test functions w^ε that oscillate between 0 and 1. These functions dictate the limit of y^ε. In the example given in the introduction, where $T_i^\varepsilon = B_{a^\varepsilon}(x_i)$, the limit of y^ε depends on the rate at which $a^\varepsilon \to 0$. For a given positive constant C_0, we set

$$a_*^\varepsilon = \begin{cases} \exp(-\frac{C_0}{\varepsilon^2}) & \text{if } n = 2, \\ C_0\varepsilon^{\frac{n}{n-2}} & \text{if } n \geq 3. \end{cases} \tag{10}$$

We also define

$$\nu = \begin{cases} \frac{\pi}{2}\frac{1}{C_0} & \text{if } n = 2, \\ \frac{S_n(n-2)}{2^n}C_0^{n-2} & \text{if } n \geq 3, \end{cases} \tag{11}$$

where S_n is the area of the unit sphere in $\mathbb{R}^n$.

Theorem 2.1 *Let y^ε solve* (9). *Then if $a^\varepsilon = a_*^\varepsilon$, $y^\varepsilon \rightharpoonup y$ weakly in $H_0^1(\Omega)$, where $y \in \mathcal{K}_\psi$ is the unique solution of*

$$\int_\Omega \nabla y \cdot \nabla(w-y) - \nu y^-(w-y)dx \geq \langle f, w-y\rangle, \quad \textit{for all } w \in \mathcal{K}_\psi. \tag{12}$$

Remark 2.2 *Similar results hold for other perforated domains Ω_ε, but in general ν will be a measure belonging to $W^{-1,\infty}(\Omega)$, the dual space of $W^{1,1}(\Omega)$.*

Remark 2.3 *If $a^\varepsilon = o(a_*^\varepsilon)$, then* (12) *holds with $\nu = 0$. That is, y solves* (2).

Remark 2.4 *If f depends on ε, i.e $f = f^\varepsilon$, and if $f^\varepsilon \to f^0$ strongly in $H^{-1}(\Omega)$, then the above conclusions remain valid with f^ε in place of f in* (9) *and f^0 in place of f in* (12). *This requires only a trivial alteration of the proofs in* [CM97].

We now prove the variational inequality analogue of Theorem 1.2 in [CL08].

Theorem 2.5 *Let $\psi \in H_0^1(\Omega)$ and let $u^\varepsilon = F^\varepsilon(\psi)$, i.e. u^ε solves* (4) *with obstacle ψ. Then there are three possibilities:*

i) If $a_*^\varepsilon = o(a^\varepsilon)$ then $u^\varepsilon \rightharpoonup u$ where u solves the usual obstacle problem (1).

ii) If $a^\varepsilon = a_*^\varepsilon$, then $u^\varepsilon \rightharpoonup u$ where u is the unique solution of

$$-\Delta u = \nu(u-\psi)^- \quad \textit{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

iii) If $a^\varepsilon = o(a_*^\varepsilon)$, then $u^\varepsilon \rightharpoonup 0$.

Proof. Set $y^\varepsilon = u^\varepsilon - \psi$ and $f = \Delta\psi$. Then y^ε satisfies

$$\int_\Omega \nabla y^\varepsilon \cdot \nabla(w - y^\varepsilon)dx \geq \langle f, w - y^\varepsilon\rangle, \quad \text{for all } w \in H_0^1(\Omega) : w \geq -\psi\chi_{\Omega_\varepsilon}. \tag{13}$$

Since $\|u^\varepsilon\|_{H_0^1(\Omega)} \leq \|\psi\|_{H_0^1((\Omega)}$ for all $\varepsilon > 0$ (see the proof of Lemma 3.2), there exists $u \in H_0^1(\Omega)$ such that $u^\varepsilon \rightharpoonup u$ as $\varepsilon \to 0$ for a subsequence. Thus $y^\varepsilon \rightharpoonup y$ for some $y \in H_0^1(\Omega)$ and the same subsequence. If $a^\varepsilon = a_*^\varepsilon$, Theorem 2.1 tells us that y solves

$$\int_\Omega \nabla y \cdot \nabla(w-y)dx \geq \int_\Omega \nu y^-(w-y)dx + \langle f, w-y\rangle, \quad \text{for all } w \in H_0^1(\Omega) : w \geq -\psi, \tag{14}$$

Additionally, if $a^\varepsilon = o(a_*^\varepsilon)$ then y solves (14) with $\nu = 0$ (equation (2)). Since $u = y + \psi$ this means

$$\int_\Omega \nabla u \cdot \nabla(v-u)dx \geq \int_\Omega \nu(u-\psi)^-(v-u)dx, \quad \text{for all } v \in H_0^1(\Omega) : v \geq 0. \tag{15}$$

Let $\tilde{u}$ solve

$$-\Delta\tilde{u} = \nu(\tilde{u} - \psi)^{-}. \tag{16}$$

This problem has a unique solution $\tilde{u}$, see [Eva98] or any standard book in PDE. It is nonnegative by the maximum principle. Thus $\tilde{u}$ solves (15) so $u = \tilde{u}$ by uniqueness. In case $a^{\varepsilon} = o(a_*^{\varepsilon})$, $\nu = 0$ and so $u = 0$.

Now suppose $a_*^{\varepsilon} = o(a^{\varepsilon})$. We use the fact that y^{ε} is a supersolution of

$$-\Delta\bar{y}^{\varepsilon} = f \quad \text{in } \Omega_{\varepsilon}, \quad \bar{y}^{\varepsilon}|_{\partial\Omega_{\varepsilon}} = 0.$$

At the rate $a_*^{\varepsilon} = o(a^{\varepsilon})$, $\bar{y}^{\varepsilon} \rightharpoonup 0$ (see [CL08] page 60 and the pages 44-45 of Rauch and Taylor [RT75]). Thus $y \geq 0$, implying $u \geq \psi$. Also $-\Delta u \geq 0$ since each u^{ε} is superharmonic. Thus $u \geq F(\psi)$ - the least superharmonic function above ψ. To see that it is also smaller, we note that $F(\psi) \geq \psi\chi_{\cup T_i^{\varepsilon}}$ since $F(\psi) \geq 0$. As $F(\psi)$ is superharmonic it follows from Lemma 3.1 that $u^{\varepsilon} \leq F(\psi)$, take $\varepsilon \to 0$ to see that $u \leq F(\psi)$.

In each of the cases *i)*, *ii)* and *iii)* the limit u is unique. Thus $u^{\varepsilon} \rightharpoonup u$ without passing to a subsequence. □

Considering this theorem, one could guess that for a sequence of optimal controls ψ^{ε} and states u^{ε} we would have $(\psi^{\varepsilon}, u^{\varepsilon}) \rightharpoonup (\psi, u)$ where $-\Delta u = \nu(u - \psi)^{-}$ and

$$\int_{\Omega} (u - z)^2 + |\nabla\psi|^2 dx = \inf_{\{\varphi, v \in H_0^1(\Omega): -\Delta v = \nu(v - \varphi)^{-}\}} \int_{\Omega} (v - z)^2 + |\nabla\varphi|^2 dx.$$

This is however not true unless $u = \psi = 0$. We show in the next section that one always has $u^{\varepsilon} = \psi^{\varepsilon}$.

3 Basic properties of the optimal control

The following lemma is standard but we give a proof for the sake of convenience.

Lemma 3.1 *Let $\psi \in H_0^1(\Omega)$ and let $u^{\varepsilon} = F^{\varepsilon}(\psi)$. Then*

i) u^{ε} is the smallest superharmonic function in Ω that stays above the obstacle $\psi\chi_{\cup T_i^{\varepsilon}}$. It is harmonic in Ω_{ε}.

ii) if ψ is superharmonic in Ω and harmonic in Ω_{ε}, then $u^{\varepsilon} = \psi$.

iii) there is exists a positive measure μ^{ε} with $\operatorname{supp}(\mu^{\varepsilon}) \subset \cup\overline{T_i^{\varepsilon}}$ such that

$$-\Delta u^{\varepsilon} = \mu^{\varepsilon}$$

in the sense of distributions.

Proof. Taking $v = u^\varepsilon + \eta$ in (4) for some nonnegative $\eta \in H_0^1(\Omega)$, we see that

$$\int_\Omega \nabla u^\varepsilon \cdot \nabla \eta dx \geq 0,$$

implying $-\Delta u^\varepsilon \geq 0$ in the weak sense, i.e. u^ε is superharmonic. From the maximum principle we conclude that $u^\varepsilon \geq 0$ in all of Ω. Suppose v is another superharmonic function such that $v \geq \psi \chi_{\cup T_i^\varepsilon}$. Then since $\min(u^\varepsilon, v) \geq \psi \chi_{\cup T_i^\varepsilon}$, (4) implies

$$0 \leq \int_\Omega \nabla u^\varepsilon \cdot \nabla(\min(u^\varepsilon, v) - u^\varepsilon) dx = \int_{\{u^\varepsilon > v\}} \nabla u^\varepsilon \cdot \nabla(v - u^\varepsilon) dx.$$

But since v is superharmonic

$$0 \geq \int_\Omega \nabla v \cdot \nabla(\min(u^\varepsilon, v) - u^\varepsilon) dx = \int_{\{u^\varepsilon > v\}} \nabla v \cdot \nabla(v - u^\varepsilon) dx.$$

This implies that the set $\{u^\varepsilon > v\}$ is empty. Next we show that u^ε is harmonic in any ball $B \subset \Omega_\varepsilon$. To this end, let h^ε be the solution of

$$\begin{cases} \Delta h^\varepsilon = 0 & \text{in } B \\ h^\varepsilon = u^\varepsilon & \text{on } \partial B \end{cases}$$

and set

$$\bar{u}^\varepsilon(x) = \begin{cases} h^\varepsilon(x), & \text{if } x \in B \\ u^\varepsilon(x), & \text{otherwise.} \end{cases}$$

Then $\bar{u}^\varepsilon$ is a *Poisson modification* of u^ε w.r.t. B in Ω. It is well known, see [Hel69], that a Poisson modification of a superharmonic function is superharmonic. Since $u^\varepsilon \geq 0$, we have $h^\varepsilon \geq 0$ in B by the maximum principle, so $\bar{u}^\varepsilon \in \mathcal{K}_\psi^\varepsilon$. If u^ε was not harmonic in B, $\bar{u}^\varepsilon$ would be a smaller superharmonic function in $\mathcal{K}_\psi^\varepsilon$. This contradicts the minimality of u^ε. As for *ii*), it is an immediate consequence of *i*).

By the Riesz decomposition theorem there exists a positive measure μ^ε such that $-\Delta u^\varepsilon = \mu^\varepsilon$. It has the same support as the distribution $-\Delta u^\varepsilon$. □

Lemma 3.2 *For each fixed $\varepsilon > 0$, the functional J in* (5) *has a unique minimizer ψ^ε that coincides with the solution of* (4)*, i.e. $u^\varepsilon = F^\varepsilon(\psi^\varepsilon) = \psi^\varepsilon$.*

Proof. Taking $v = \psi^+$ in (4) we easily derive the estimate

$$\|u^\varepsilon\|^2_{H_0^1(\Omega)} \leq \int_\Omega \nabla \psi^+ \cdot \nabla u^\varepsilon dx \leq \|\psi^+\|_{H_0^1(\Omega)} \|u^\varepsilon\|_{H_0^1(\Omega)},$$

implying $\|u^\varepsilon\|_{H_0^1(\Omega)} \leq \|\psi\|_{H_0^1(\Omega)}$. By Lemma 3.1 $F^\varepsilon(u^\varepsilon) = u^\varepsilon$, so $J(u^\varepsilon) \leq J(\psi)$. This means that we may restrict our attention to obstacles in the set

$$\mathcal{H}_\varepsilon^+ = \{\psi \in H_0^1(\Omega) : \psi \text{ is superharmonic in } \Omega \text{ and harmonic in } \Omega_\varepsilon\}.$$

Consequently our problem has been reduced to minimizing

$$\bar{J}(\psi) := \int_\Omega (\psi - z)^2 + |\nabla \psi|^2 dx$$

over $\mathcal{H}_\varepsilon^+$. As $\mathcal{H}_\varepsilon^+$ is convex, closed and nonempty (it contains the zero function), and $\bar{J}$ is strictly convex, $\bar{J}$ has a unique minimizer u^ε in $\mathcal{H}_\varepsilon^+$. Thus J has a unique minimizer u^ε in $H_0^1(\Omega)$, such that $F^\varepsilon(u^\varepsilon) = u^\varepsilon$. □

4 Characterization and limit behaviour of the optimal control

Having established the existence of an optimal control u^ε, we want to determine u^ε in terms of an equation it solves. In particular, we want to determine the limit as $\varepsilon \to 0$ of u^ε.

First we look at the situation when $F^\varepsilon(\psi) \rightharpoonup F(\psi)$ for any $\psi \in H_0^1(\Omega)$, i.e. when $a_*^\varepsilon = o(a^\varepsilon)$. Then there is a particular way of determining the limit of the sequence $\{u^\varepsilon\}_\varepsilon$ of optimal controls.

We recall that u^ε satisfies

$$J(u^\varepsilon) = \inf_{v \in H_0^1(\Omega)} \int_\Omega (F^\varepsilon(v) - z)^2 + |\nabla v|^2 dx.$$

By Lemma 3.2 we know that $F^\varepsilon(u^\varepsilon) = u^\varepsilon$ and that u^ε is the unique solution of the following:

$$\begin{cases} \text{Find } u^\varepsilon \in \mathcal{H}_\varepsilon^+ \text{ such that} \\ \bar{J}(u^\varepsilon) := \displaystyle\int_\Omega (u^\varepsilon - z)^2 + |\nabla u^\varepsilon|^2 dx \le \bar{J}(v) \text{ for all } v \in \mathcal{H}_\varepsilon^+. \end{cases} \tag{17}$$

It is easy to see that this is equivalent to (6) in the introduction.

Lemma 4.1 *Let u^ε solve problem* (17) *and suppose $a_*^\varepsilon = o(a^\varepsilon)$. Then $u^\varepsilon \to u$ strongly in $H_0^1(\Omega)$ where u is the unique minimizer of*

$$J(\psi) = \int_\Omega (F(\psi) - z)^2 + |\nabla \psi|^2 dx$$

over H_0^1.

Proof. Since $0 \in \mathcal{H}_\varepsilon^+$ for any $\varepsilon > 0$, we have

$$\|u^\varepsilon\|_{H_0^1(\Omega)}^2 \le \bar{J}(u^\varepsilon) \le \bar{J}(0) = \|z\|_{L^2(\Omega)}^2.$$

Thus $\{u^\varepsilon\}$ has a subsequence that converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, to some $u \in H_0^1(\Omega)$. We shall write $\{u^\varepsilon\}$ also for this subsequence.

By the weak lower semicontinuity of the norm, and by the optimality of u^ε, we have:

$$\int_\Omega (u - z)^2 + |\nabla u|^2 dx \le \liminf_{\varepsilon \to 0} \bar{J}(u^\varepsilon) = \liminf_{\varepsilon \to 0} \int_\Omega (F^\varepsilon(u^\varepsilon) - z)^2 + |\nabla u^\varepsilon|^2 dx$$
$$\le \lim_{\varepsilon \to 0} \int_\Omega (F^\varepsilon(u) - z)^2 + |\nabla u|^2 dx = \int_\Omega (u - z)^2 + |\nabla u|^2 dx.$$

The last equality follows since $F^\varepsilon(u) \rightharpoonup F(u)$ and $F(u) = u$ (u is superharmonic). This proves that the convergence is strong.

Let $\varphi \in H_0^1(\Omega)$ and set $v = F(\varphi)$, $v^\varepsilon = F^\varepsilon(\varphi)$. Then $v^\varepsilon \to v$ in $L^2(\Omega)$. Moreover, by the optimality of u^ε,

$$\int_\Omega (u - z)^2 + |\nabla u|^2 dx = \lim_{\varepsilon \to 0} \int_\Omega (u^\varepsilon - z)^2 + |\nabla u^\varepsilon|^2 dx$$
$$\le \lim_{\varepsilon \to 0} \int_\Omega (v^\varepsilon - z)^2 + |\nabla \varphi|^2 dx = \int_\Omega (v - z)^2 + |\nabla \varphi|^2 dx.$$

We have proved that

$$\int_\Omega (F(u)-z)^2 + |\nabla u|^2 dx = \inf_{\varphi\in H_0^1(\Omega)} \int_\Omega (F(\varphi)-z)^2 + |\nabla\varphi|^2 dx.$$

Thus ψ coincides with the optimal control for the classical obstacle problem and may be characterized as in (3). Since u is unique, the whole sequence $\{u^\varepsilon\}$ converges to u, strongly in $H_0^1(\Omega)$. $\square$

Recall *iii*) of Lemma 3.1: since u^ε is superharmonic in Ω and harmonic off the holes $\cup\overline{T_i^\varepsilon}$ there is a measure of finite energy μ^ε with support in $\cup\overline{T_i^\varepsilon}$ such that $-\Delta u^\varepsilon = \mu^\varepsilon$. As in the paper [ALY98] u^ε is determined by a triple $(u^\varepsilon, p^\varepsilon, \mu^\varepsilon) \in H_0^1(\Omega)\times H_0^1(\Omega)\times \mathcal{E}_\varepsilon^+$. The proof relies on approximating the solution operator F^ε by a more regular one. We consider the operator F_δ^ε on $H_0^1(\Omega)$ defined by

$$F_\delta^\varepsilon : \begin{cases} H_0^1(\Omega) & \to & H_0^1(\Omega) \\ \psi & \to & u_\delta^\varepsilon \end{cases}$$

when

$$-\Delta u_\delta^\varepsilon = \frac{1}{\delta}\beta(u_\delta^\varepsilon - \psi\chi_{\cup T_i^\varepsilon}).$$

Here

$$\beta(r) = \begin{cases} 0, & r \geq 0 \\ -r^2, & -\frac{1}{2} \leq r < 0 \\ r+\frac{1}{4}, & r < -\frac{1}{2}. \end{cases}$$

The main advantages of this operator is that it continuous with respect to weak convergence of obstacles, and that it is fairly straight-forward to determine its differential. The steps to be taken now are:

- Show that $F_\delta^\varepsilon(\psi) \to F^\varepsilon(\psi)$ strongly as $\delta \to 0$, for any $\psi \in H_0^1(\Omega)$
- Replace F^ε by F_δ^ε in the optimal control problem and show that the optimal pair $(\psi_\delta^\varepsilon, u_\delta^\varepsilon)$ so obtained converges strongly to $(u^\varepsilon, u^\varepsilon)$, the optimal pair for solution operator F^ε.
- Derive optimality conditions for $(\psi_\delta^\varepsilon, u_\delta^\varepsilon)$ by introducing a Lagrange multiplier.
- Let $\delta \to 0$ to find optimality conditions for u^ε.

The proof is identical to that of [ALY98] until one finds that the system

$$\begin{cases} -\Delta u^\varepsilon = \mu^\varepsilon \quad \text{in } \Omega,\ \operatorname{supp}(\mu^\varepsilon) \subset \cup\overline{T_i^\varepsilon} \\ -\Delta p^\varepsilon = \mu^\varepsilon + u^\varepsilon - z \quad \text{in } \Omega. \end{cases} \tag{18}$$

is satisfied by the optimal control u^ε and an adjoint function p^ε.

Since u^ε minimizes $\bar{J}$ over $\mathcal{H}_\varepsilon^+$ we have

$$\liminf_{t\to 0} \frac{\bar{J}(u^\varepsilon + t(v^\varepsilon - u^\varepsilon)) - \bar{J}(u^\varepsilon)}{2t} \geq 0,$$

for all $v^\varepsilon \in \mathcal{H}_\varepsilon^+$. This implies

$$\int_\Omega \nabla p^\varepsilon \cdot \nabla(v^\varepsilon - u^\varepsilon)dx \geq 0.$$

Taking $v^\varepsilon = u^\varepsilon + \varphi^\varepsilon$ for any $\varphi^\varepsilon \in \mathcal{H}_\varepsilon^+$ we find $p^\varepsilon \geq 0$ in $\cup T_i^\varepsilon$. By choosing $v^\varepsilon = 0$ we obtain

$$\int_\Omega p^\varepsilon d\mu^\varepsilon = 0.$$

The uniqueness of the triple $(p^\varepsilon, \mu^\varepsilon, u^\varepsilon)$ is proved exactly as in [ALY98]. We have the following result:

Proposition 4.2 *The optimal control u^ε is uniquely determined by a triple $p^\varepsilon, \mu^\varepsilon, u^\varepsilon$ where $u^\varepsilon, p^\varepsilon \in H_0^1(\Omega)$ and μ^ε is a measure of finite energy satisfying*

$$\begin{cases} -\Delta u^\varepsilon = \mu^\varepsilon \quad \text{in } \Omega, \ \operatorname{supp}(\mu^\varepsilon) \subset \cup \overline{T_i^\varepsilon} \\ -\Delta p^\varepsilon = \mu^\varepsilon + u^\varepsilon - z \quad \text{in } \Omega, \\ \displaystyle\int_\Omega p^\varepsilon d\mu^\varepsilon = 0, \ p^\varepsilon \geq 0 \ \text{in } \cup T_i^\varepsilon. \end{cases} \tag{19}$$

Next we show that p^ε satisfies the variational inequality

$$\begin{cases} p^\varepsilon \geq -Gz\chi_{\Omega_\varepsilon} \\ \displaystyle\int_\Omega \nabla p^\varepsilon \cdot \nabla(q^\varepsilon - p^\varepsilon)dx \geq \int_\Omega (u^\varepsilon - z)(q^\varepsilon - p^\varepsilon)dx, \\ \text{for all } q^\varepsilon \in H_0^1(\Omega): \ q^\varepsilon \geq -Gz\chi_{\Omega_\varepsilon}. \end{cases} \tag{20}$$

Here Gz is the Green potential of z. That is, $-\Delta Gz = z$ and $Gz \in H_0^1(\Omega)$. Note that $p^\varepsilon \geq -Gz$ follows from the maximum principle since $u^\varepsilon, \mu^\varepsilon \geq 0$. Also $-Gz\chi_{\Omega_\varepsilon} = 0$ in $\cup T_i^\varepsilon$ but here $p^\varepsilon \geq 0$. Suppose then $q^\varepsilon \geq -Gz\chi_{\Omega_\varepsilon}$ and $q^\varepsilon \in H_0^1(\Omega)$. Then since $q^\varepsilon \geq 0$ in $\cup T_i^\varepsilon$ we have $\int_\Omega q^\varepsilon d\mu^\varepsilon \geq 0$. Thus

$$0 \leq \int_\Omega (q^\varepsilon - p^\varepsilon)d\mu^\varepsilon = \int_\Omega (q^\varepsilon - p^\varepsilon)d(-\Delta p^\varepsilon + z - u^\varepsilon) = \int_\Omega \nabla p^\varepsilon \cdot \nabla(p^\varepsilon - q^\varepsilon) - (q^\varepsilon - p^\varepsilon)(u^\varepsilon - z)dx.$$

Arguing as in the proof of Lemma 4.1 we may assume $u^\varepsilon \rightharpoonup u$ for a subsequence. For the same reason we have $\mu^\varepsilon \rightharpoonup^* \mu$ for some $\mu \in \mathcal{E}^+$. By Theorem 2.1 and Remark 2.4, $p^\varepsilon \rightharpoonup p$ where p solves

$$\begin{cases} p \geq -Gz \\ \displaystyle\int_\Omega \nabla p \cdot \nabla(q - p)dx \geq \int_\Omega \nu p^-(q - p)dx + \int_\Omega (u - z)(q - p)dx, \\ \text{for all } q \geq -Gz, \end{cases} \tag{21}$$

when $a^\varepsilon = a_*^\varepsilon$. Letting $\tilde{p}$ solve

$$-\Delta\tilde{p} = \nu\tilde{p}^- + u - z$$

we see that $\tilde{p}$ solves (21) and so $p = \tilde{p}$ by uniqueness. Thus $\mu = \nu p^-$ and we obtain the system

$$\begin{cases} -\Delta u = \nu p^-, \quad \text{in } \Omega \\ -\Delta p = \nu p^- + u - z, \quad \text{in } \Omega. \end{cases} \tag{22}$$

If $a^\varepsilon = o(a^\varepsilon_*)$ then $\nu = 0$ and $u = 0$.

We can now state our main theorem.

Theorem 4.3 *The functional*

$$J(\psi) = \int_\Omega (F^\varepsilon(\psi) - z)^2 + |\nabla\psi|^2 dx$$

has a unique minimizer u^ε over $H^1_0(\Omega)$ such that $F^\varepsilon(u^\varepsilon) = u^\varepsilon$. The minimizer is uniquely determined by a triple $(u^\varepsilon, p^\varepsilon, \mu^\varepsilon)$ where $u^\varepsilon, p^\varepsilon \in H^1_0(\Omega)$ and μ^ε is a measure of finite energy satisfying

$$\begin{cases} -\Delta u^\varepsilon = \mu^\varepsilon \quad \text{in } \Omega,\ \operatorname{supp}(\mu^\varepsilon) \subset \cup\overline{T^\varepsilon_i} \\ -\Delta p^\varepsilon = \mu^\varepsilon + u^\varepsilon - z \quad \text{in } \Omega, \\ \displaystyle\int_\Omega p^\varepsilon d\mu^\varepsilon = 0,\ p^\varepsilon \geq 0 \ \text{in } \cup T^\varepsilon_i. \end{cases} \tag{23}$$

As $\varepsilon \to 0$, the limit of u^ε depends on the rate at which $a^\varepsilon \to 0$ as follows:

i) If $a^\varepsilon_* = o(a^\varepsilon)$ then $u^\varepsilon \to u$ strongly in $H^1_0(\Omega)$ where u is uniquely determined by the system (3).

ii) If $a^\varepsilon = a^\varepsilon_*$, then $u^\varepsilon \rightharpoonup u$ where u is uniquely determined by the pair $u, p \in H^1_0(\Omega)$ solving

$$\begin{cases} -\Delta u = \nu p^-, \quad \text{in } \Omega \\ -\Delta p = \nu p^- + u - z, \quad \text{in } \Omega. \end{cases} \tag{24}$$

iii) If $a^\varepsilon = o(a^\varepsilon_*)$, then $u^\varepsilon \rightharpoonup 0$.

Proof. All that remains is to prove the uniqueness of u, p satisfying (24). Suppose then (u, p) and (v, q) both solve (24). By defining $\gamma(\lambda) = \lambda^-$ for $\lambda \in \mathbb{R}$ we may write

$$p^-(x) - q^-(x) = \int_0^1 \frac{d}{dt}\gamma(tp(x) + (1-t)q(x))dt = \int_0^1 \gamma'(tp(x) + (1-t)q(x))dt(p(x) - q(x)).$$

(It is easy to see that this holds by approximation with smooth functions whenever γ is Lipschitz continuous). Since $\gamma'(\lambda) = -\chi_{\{\lambda<0\}} \leq 0$ we have

$$\int_\Omega (p^- - q^-)(p - q)dx = \int_\Omega \int_0^1 \gamma'(tp + (1-t)q)dt(p - q)^2 dx \leq 0.$$

Then

$$\begin{aligned} 0 \geq \nu \int_\Omega (p^- - q^-)(p - q)dx &= \langle -\Delta(u - v), p - q\rangle = \langle u - v, -\Delta(p - q)\rangle \\ &= \int_\Omega (u - v)(\nu(p^- - q^-) + u - v)dx = \langle -\Delta(u - v), u - v\rangle + \int_\Omega (u - v)^2 dx \\ &= \|u - v\|^2_{H^1(\Omega)}, \end{aligned}$$

so $u = v$ and thus $p^- = q^-$. Therefore $-\Delta p = -\Delta q$ and consequently $p = q$. $\square$

We remark that Theorem 4.3, possibly except parts i) and iii), holds for all perforated domains considered in [CM97]. Indeed, the proofs require nothing of the geometry of T_i^ε as long as Theorem 2.1 holds.

5 Stability

If two profiles z_1 and z_2 are close in L^2, then the corresponding optimal controls u_1 and u_2 are close in H^1. Indeed, this follows from the uniqueness proof of Theorem 4.3. Suppose (u_1, p_1) and (u_2, p_2) solve (24) with $z = z_1$ and $z = z_2$ respectively. Then

$$\begin{aligned}0 \geq \int_\Omega \nu(p_1^- - p_2^-)(p_1 - p_2)dx &= \langle -\Delta(u_1 - u_2), p_1 - p_2\rangle = \langle u_1 - u_2, -\Delta(p_1 - p_2)\rangle \\ &= \int_\Omega (u_1 - u_2)(\nu(p_1^- - p_2^-) + u_1 - u_2 - z_1 + z_2) \\ &= \|u_1 - u_2\|_{L^2}^2 + \|u_1 - u_2\|_{H_0^1}^2 - \int_\Omega (u_1 - u_2)(z_1 - z_2)dx.\end{aligned}$$

By Hölder's inequality and then Young's inequality we get

$$\begin{aligned}\|u_1 - u_2\|_{L^2(\Omega)}^2 + \|u_1 - u_2\|_{H_0^1(\Omega)}^2 &\leq \|z_1 - z_2\|_{L^2(\Omega)}\|u_1 - u_2\|_{L^2(\Omega)} \\ &\leq \frac{1}{2}\|z_1 - z_2\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u_1 - u_2\|_{L^2(\Omega)}^2.\end{aligned}$$

Thus we obtain

$$\|u_1 - u_2\|_{L^2(\Omega)}^2 + 2\|u_1 - u_2\|_{H_0^1(\Omega)}^2 \leq \|z_1 - z_2\|_{L^2(\Omega)}^2 \tag{25}$$

We also have

$$\|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Omega)}^2 + 2\|u_1^\varepsilon - u_2^\varepsilon\|_{H_0^1(\Omega)}^2 \leq \|z_1 - z_2\|_{L^2(\Omega)}^2,$$

whenever $(u_1^\varepsilon, p_1^\varepsilon, \mu_1^\varepsilon)$ and $(u_2^\varepsilon, p_2^\varepsilon, \mu_2^\varepsilon)$ solve (23) with $z = z_1$ and $z = z_2$. This is proved in the same way since it holds that

$$\int_\Omega (p_1^\varepsilon - p_2^\varepsilon)d(\mu_1^\varepsilon - \mu_2^\varepsilon)dx \leq 0.$$

One may also put different weigths on the terms in the functional J or $\bar{J}$. For a real number $\alpha > 0$ we consider the problem of minimizing

$$\bar{J}_\alpha(u) := \int_\Omega (u - z)^2 + \alpha|\nabla u|^2 dx, \quad \text{or equivalently} \quad \int_\Omega \frac{1}{\alpha}(u - z)^2 + |\nabla u|^2 dx,$$

over $\mathcal{H}_\varepsilon^+$. Then the minimizer u^ε is determined by the triple $(u^\varepsilon, p^\varepsilon, \mu^\varepsilon)$ satisfying

$$\begin{cases} -\Delta u^\varepsilon = \mu^\varepsilon \quad \text{in } \Omega, \ \operatorname{supp}(\mu^\varepsilon) \subset \cup\overline{T_i^\varepsilon} \\ -\Delta p^\varepsilon = \mu^\varepsilon + \dfrac{1}{\alpha}(u^\varepsilon - z) \quad \text{in } \Omega, \\ \displaystyle\int_\Omega p^\varepsilon d\mu^\varepsilon = 0, \ p^\varepsilon \geq 0 \text{ in } \cup T_i^\varepsilon. \end{cases} \tag{26}$$

When we pass to the limit $\varepsilon \to 0$, $u = \lim_{\varepsilon\to 0} u^\varepsilon$ is determined by (u,p) solving

$$\begin{cases} -\Delta u = \nu p^-, & \text{in } \Omega \\ -\Delta p = \nu p^- + \dfrac{1}{\alpha}(u-z), & \text{in } \Omega, \end{cases} \tag{27}$$

when $a^\varepsilon = a^\varepsilon_*$. The stability estimate (25) becomes

$$\frac{1}{\alpha}\|u_1 - u_2\|^2_{L^2(\Omega)} + 2\|u_1 - u_2\|^2_{H^1_0(\Omega)} \le \frac{1}{\alpha}\|z_1 - z_2\|^2_{L^2(\Omega)},$$

and the same for u_1^ε and u_2^ε. We deduce that

$$\|u_1 - u_2\|^2_{H^1_0(\Omega)} \le \frac{1}{2\alpha}\|z_1 - z_2\|^2_{L^2(\Omega)}$$

and

$$\|u_1 - u_2\|^2_{L^2(\Omega)} \le \|z_1 - z_2\|^2_{L^2(\Omega)},$$

so the optimal control is always stable with respect to $L^2(\Omega)$ norm but the $H^1_0(\Omega)$ norm may blow up as $\alpha \to 0$.

References

[ALY98] D. R. Adams, S. M. Lenhart, and J. Yong, *Optimal control of the obstacle for an elliptic variational inequality*, Appl. Math. Optim. **38** (1998), no. 2, 121–140.

[AP83] Hédy Attouch and Colette Picard, *Variational inequalities with varying obstacles: the general form of the limit problem*, J. Funct. Anal. **50** (1983), no. 3, 329–386.

[Caf98] L. A. Caffarelli, *The obstacle problem revisited*, J. Fourier Anal. Appl. **4** (1998), no. 4-5, 383–402.

[CC80] Luciano Carbone and Ferruccio Colombini, *On convergence of functionals with unilateral constraints*, J. Math. Pures Appl. (9) **59** (1980), no. 4, 465–500.

[CL08] Luis Caffarelli and Ki-ahm Lee, *Viscosity method for homogenization of highly oscillating obstacles*, Indiana Univ. Math. J. **57** (2008), no. 4, 1715–1741.

[CM97] Doina Cioranescu and François Murat, *A strange term coming from nowhere*, Topics in the mathematical modelling of composite materials, 1997, pp. 45–93.

[Eva98] Lawrence C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[Hel69] L. L. Helms, *Introduction to potential theory*, Pure and Applied Mathematics, Vol. XXII, Wiley-Interscience A Division of John Wiley & Sons, New York-London-Sydney, 1969.

[KS00] David Kinderlehrer and Guido Stampacchia, *An introduction to variational inequalities and their applications*, Classics in Applied Mathematics, vol. 31, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. Reprint of the 1980 original.

[KSJP99] S. Kesavan and J. Saint Jean Paulin, *Optimal control on perforated domains*, J. Math. Anal. Appl. **229** (1999), no. 2, 563–586.

[RT75] Jeffrey Rauch and Michael Taylor, *Potential and scattering theory on wildly perturbed domains*, J. Funct. Anal. **18** (1975), 27–59.

[SJPZ02] J. Saint Jean Paulin and H. Zoubairi, *Optimal control and "strange term" for a Stokes problem in perforated domains*, Port. Math. (N.S.) **59** (2002), no. 2, 161–178.