# Convergence rates for adaptive finite element methods

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# GOAL

For PDE

$$R(w) = 0,$$

minimize the number of elements, with error

$$|g(w) - g(\overline{w})| \leq \text{TOL}.$$

The talk answers

- rigorous convergence rate for general adaptive algorithm,

- how to measure convergence rates?

- why convergence of the error density is essential and subtle.

**Plan:** Adaptive integration, optimal rates, Adaptive FEM algorithm (hanging nodes with linear tensor elements).

# ADAPTIVE INTEGRATION

$$\frac{dw}{dt} = f(t)$$

$$\begin{aligned}\text{TOL} \;&=\; \text{Euler error} \;\simeq \sum_n (\Delta t)^2 \rho_n,\\ \rho_n \;&\equiv\; \frac{f'(t_n)}{2}.\end{aligned}$$

**Uniform:** $$N_u \simeq \frac{1}{\text{TOL}} \|\rho\|_{L^1(0,1)}$$

**Adaptive:** $$N_a \simeq \frac{1}{\text{TOL}} \|\rho\|_{L^{\frac{1}{2}}(0,1)}$$

$$f(t) = \frac{1}{\sqrt{t}} \quad \Rightarrow$$

$$\mathcal{O}(\text{TOL}^{-1}) = N_a \ll N_u = \mathcal{O}(\text{TOL}^{-2})$$

# OPTIMAL CONVERGENCE RATES

$$\text{error} \simeq \sum_n \underbrace{\rho_n (\Delta x)^{p+d}}_{r_n} \leq \mathsf{TOL}$$

**Uniform:** $$N_u \simeq \frac{\int_\Omega dx}{\mathsf{TOL}^{\frac{d}{p}}} \|\rho\|_{L^1}^{\frac{d}{p}}$$

**Adaptive:** $$N_a \simeq \frac{1}{\mathsf{TOL}^{\frac{d}{p}}} \|\rho\|_{L^{\frac{d}{p+d}}}^{\frac{d}{p}}$$

## More precise?

# ADAPTIVE PDE

- error estimates for adaptive FEM: ...

- convergence of adaptive algorithm: B-V, D, M-N-S

- convergence rates: C-D-D (wavelet).

# Error representation

$$\begin{aligned} R(w) &= 0 \\ g(w) - g(\overline{w}) &= \int R(\overline{w})\lambda \ dx \\ \lambda &= ? \end{aligned}$$

# ERROR REPRESENTATION

$$\begin{aligned} R(w) &\equiv -\mathrm{div}(a\nabla w) - f = 0 \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

$$g(w) \equiv (w, F),$$

$$\begin{aligned} R^*(\lambda) &\equiv -\mathrm{div}(a\nabla \lambda) - F = 0 \quad \text{in } \Omega \\ \lambda &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

$$(R(\overline{w}), v) = 0 \quad \text{for all } v \in \overline{V}$$

imply

$$\begin{aligned} g(w) - g(\overline{w}) &= (w - \overline{w}, F) \\ &= (w - \overline{w}, -\mathrm{div}(a\nabla\lambda)) \\ &= (-f - \mathrm{div}(a\nabla\overline{w}), -\lambda) \\ &= (R(\overline{w}), \pi\lambda - \lambda) \end{aligned}$$

# ERROR DENSITY FOR BILINEAR ELEMENTS

$$
\begin{aligned}
(R(\overline{w}), \pi\lambda - \lambda) &= \sum_K \int_K (-f - \mathsf{div}(a\nabla\overline{w}))(\pi\lambda - \lambda)dx \\
&\quad + \sum_K \int_{\partial K} a\frac{\partial\overline{w}}{\partial n}(\pi\lambda - \lambda)ds \\
&\approx \sum_K \overline{\rho}_K h_K^4
\end{aligned}
$$

$\overline{\rho}$ depends on $n$ !

**Convergence of $\overline{\rho}$ ?**

# ADAPTIVE MESH WITH HANGING NODES AND SQUARES

**Lemma.**

$$h^{-4}\int_K (\pi\lambda - \lambda)dx \to \frac{\Delta\lambda}{12}.$$

## Approximation of $\Delta\lambda$?

- $D^2\overline{\lambda}$ converge on uniform meshes,
- for non uniform meshes?

# CURE BY LOCALIZED AVERAGE

$$\Psi \geq 0, \quad \int_{\mathbb{R}} \Psi(x)dx = 1$$

$$|\Psi'| \sim \frac{\Psi}{\alpha}, \quad \frac{h_{max}}{\alpha} \to 0$$

**Lemma.**

$$(\overline{D^2\overline{\lambda}})_j \equiv \bar{h}D^2\overline{\lambda} \cdot \Psi^j \to \lambda''(x_j)$$

**Proof.**

Step 1: $\bar{h}D^2$ is symmetric

Step 2:

$$\begin{aligned} \bar{h}D^2(\overline{\lambda} - \lambda) \cdot \Psi &= (\overline{\lambda} - \lambda) \cdot \bar{h}D^2\Psi \\ &= \mathcal{O}(\frac{h_{max}^2 \log h_{max}}{\alpha^2}) \to 0 \end{aligned}$$

Step 3:

$$\Delta\lambda \simeq \overline{D_1^2\overline{\lambda}} + \overline{D_2^2\overline{\lambda}}$$

# THE EDGE RESIDUAL

$$\sum_K \int_{\partial K} a \frac{\partial \overline{w}}{\partial n} (\pi\lambda - \lambda) ds$$

$$= \sum_{edges} \int_{edge} \frac{\partial \overline{w}}{\partial x_j} h D_j \left( a(\pi\lambda - \lambda) \right) ds$$

$$\simeq \sum_{edges} \int_{edge} \frac{\partial w}{\partial x_j} h D_j \left( a(\pi\lambda - \lambda) \right) ds$$

$$\simeq \sum_{edges} \int_{edge} a h D_j^2 w \; (\pi\lambda - \lambda) ds$$

$$\simeq \sum_K \frac{h^4}{24} a \overline{D_1^2 \overline{w}} \; \overline{D_2^2 \overline{\lambda}} + \dots$$

# PDE ERROR DENSITY

$$
\begin{aligned}
r_K &= h_K^4 \bar{\rho}_K \\
&\equiv h_K^4 \overline{(-f - \nabla a \cdot \nabla \overline{w})} (\overline{D_1^2 \overline{\lambda}} + \overline{D_2^2 \overline{\lambda}})/12 \\
&+ h_K^4 a \overline{D_1^2 \overline{w}}\ \overline{D_2^2 \overline{\lambda}}/24 + \ldots
\end{aligned}
$$

**Theorem.** Assume bilinear elements for structured mesh with at most one hanging node per edge, $w \in \mathcal{C}^2$, $\lambda \in \mathcal{C}^3$ and

$$
\begin{aligned}
\|w - \overline{w}\|_{W^{1,\infty}} &= o(1) \\
\|w - \overline{w}\|_{L^\infty} + \|\lambda - \overline{\lambda}\|_{L^\infty} &= \mathcal{O}(h_{max}^\gamma),
\end{aligned}
$$

then $\bar{\rho}$ converges for localization $\alpha \gg h_{max}^{\gamma/2}$, as $h_{max} \to 0$.

# PDE ADAPTIVE ALGORITHM WITH CONVERGENCE RATE

$$\textbf{if } r_K \geq \frac{\mathsf{TOL}}{N}, \text{then } \textbf{divide},$$
$$\textbf{Stop if } \max_K r_K \leq S_1 \frac{\mathsf{TOL}}{N}.$$

**Theorem.** Assume also $S_1 \geq 2^d C$ and

$$C^{-1} \leq \left|\frac{\rho_{\text{parent}}}{\rho_{\text{child}}}\right| \leq C,$$

then each refinement level decreases the maximal error indicator with the factor $C/2^{d+2}$ or stops with

$$\limsup_{\mathsf{TOL}\to 0+} \frac{N}{N_{opt}} \leq (2^{d+2}C)^{\frac{d}{2}},$$
$$\limsup_{\mathsf{TOL}\to 0+} \frac{\text{error}}{\mathsf{TOL}} \leq S_1.$$

# ALTERNATIVE ERROR DENSITIES

$s \in [0, 1]$:

$$\begin{aligned} \text{error} &= s\left(R(\overline{w}), \pi\lambda - \lambda\right) \\ &\quad + (1 - s)\left(\pi w - w, R^*(\lambda)\right) \end{aligned}$$

gives the same density asymptotically

$$a\Delta w \Delta\lambda + a\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 \lambda}{\partial x_2^2} + a\frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 \lambda}{\partial x_1^2}$$

and the approximation error $\overline{D_i^2 \overline{w}} \to \frac{\partial^2 w}{\partial x_i^2}$ dominates

# NON LINEAR PROBLEMS!