Matematiska Institutionen, KTH

Essential questions for the exam 5B1491

1. Formulate and prove some version of the maximum principle for the Laplace equation.

2. State and prove Lax-Milgrams theorem related to the boundary value problem

$$\begin{aligned} -\Delta u + \gamma u &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

3. Why are iterative methods useful to solve the finite element equations for (1)? Describe the multigrid method to solve the finite element equations for (1).

4. Prove an a priori error estimate of a FEM for

$$\begin{aligned} -(au')' + \gamma u &= f, \quad \text{in } \ (0,1), \\ u(0) = u'(1) &= 0. \end{aligned} \tag{2}$$

5. Formulate a FEM for (2).

6. Formulate an adaptive FEM algorithm for (2).

7. Formulate Lax's equivalence theorem. State in particular what wellposedness, stability, consistence and convergence means.

8. Formulate a finite difference method for a heat equation and use Lax equivalence theorem to motivate stability, convergence and consistency of the approximation.

9. Show by an example that the standard FEM Galerkin method does not work for the differential equation

$$u' - \epsilon u'' = 0,$$

when ϵ is much smaller than the mesh size.

10. Formulate a cure of the problem in question 9.

11. Consider a weak solution of the scalar conservation law

$$\begin{aligned} u_t + f(u)_x &= 0, \qquad x \in \boldsymbol{R},\ t > 0 \\ u(\cdot, 0) &= u_0 \end{aligned}$$

and the following two entropy conditions

(E) u is the limit of u^ϵ, i.e. $u = \lim_{\epsilon \to 0+} u^\epsilon$, where

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon.$$

(E') For all convex $\eta : \boldsymbol{R} \to \boldsymbol{R}$

$$\eta(u)_t + q(u)_x \leq 0$$

holds in weak sense, where $q' = \eta' f'$.

Show that E implies E'.

12. Show by an example non uniqueness of weak solutions of

$$\begin{aligned} &u_t + (\frac{u^2}{2})_x = 0, \\ &u(\cdot, 0) = u_0. \end{aligned} \tag{3}$$

13. Describe a formulation that gives uniqueness for (3).

14. Relate question 13 and the Lax-Friedrich method.

15. Formulate Lax-Friedrichs' method for the Hamilton-Jacobi equation

$$v_t + f(v_x) = 0, \quad x \in \boldsymbol{R}, \quad t > 0,$$

and derive the method from Lax-Friedrichs' method

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}(f(u_{i+1}^n) - f(u_{i-1}^n)) + \frac{(\Delta x)^2}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)/(\Delta x)^2,$$

for the conservation law

$$u_t + f(u)_x = 0, \quad x \in \boldsymbol{R}.$$

16. Present shortly an application of Hamilton-Jacobi equations.

17. Formulate the basic properties a Wienerprocess.

18. Show by Itos formula that if u solves Kolmogorov's backward equation with data $u(T, x) = g(x)$, then

$$u(t, x) = E[g(X(T))|\ X(t) = x], \tag{4}$$

where X is the solution of a certain SDE (which?).

19. Discuss the cost to compute u in (4) by a Monte Carlo method and by a finite difference or finite element method.

20. Solve with characteristics the linear symmetric hyperbolic system

$$\begin{aligned} &u_t + Au_x = 0, \qquad x \in \boldsymbol{R},\ t > 0 \\ &u(\cdot, 0) = u_0, \end{aligned}$$

where A is a constant $n \times n$ matrix, u_0 are given initial data and $u = u(x, t) \in \boldsymbol{R}^n$.

21. Prove a stability estimate for linear symmetric hyperbolic systems in $\boldsymbol{R}^d$, $d > 1$, with constant coefficients.