LINEAR ALGEBRA 5B1307, JANUARY 2001

Homework 01/01/23

1. Let V be a vector space and let U be subspace of V. Consider the quotient V/U of V by U. It is defined as the quotient of V by the equivalence relation

$$v \sim v' \text{ iff } v - v' \in U \ .$$

(1) Prove that V/U is a vector space.
(2) Define the natural projection of V to V/U and describe its null-space and range.
(3) Let $T : V \to W$ be a linear map between two vector spaces. Show that $V/nullT \cong rangeT$ and T factorizes through $nullT$.
(4) Consider linear maps $T : V \to W$ and $T' : V \to W'$ between V and vector spaces W, W'. Show that there exists a linear map $S : W \to W'$ such that $T' = ST$.

2. Give a necessary and sufficient condition for the union of two subspaces to be a subspace.

3. Let $G = (V, E)$ be a finite graph with a vertex-set V and an edge-set E. Each edge has two possible orientations, so that an oriented edge is understood as an ordered pair of vertices $e = (e_+, e_-)$. The set $\mathbb{E}$ of oriented edges has thus cardinality $2|E|$. An oriented edge $e = (e_+, e_-)$ can be naturally coupled with $\bar{e} = (e_-, e_+)$.

Consider the space $C^0(G, \mathbb{R})$ of real-valued functions on the vertices of G, and the space $C^1(G, \mathbb{R})$ of real-valued functions on the edges of G which respect the orientation.

$$C^0(G, \mathbb{R}) \ = \ \{ \ f : V \to \mathbb{R} \ \} \ ,$$

$$C^1(G, \mathbb{R}) \ = \ \{ \ \phi : \mathbb{E} \to \mathbb{R} \ : \ \phi(\bar{e}) = -\phi(e) \ \} \ .$$

The spaces $C^0(G, \mathbb{R})$ and $C^1(G, \mathbb{R})$ are called respectively the space of real-valued 0-forms and the space of real-valued 1-forms on G. Consider a map

$$d : \ C^0(G, \mathbb{R}) \to C^1(G, \mathbb{R})$$

called "exterior differential", and defined by

$$df(e) \ = \ f(e_+) - f(e_-) \ .$$

(1) Show that $C^0(G, \mathbb{R})$ and $C^1(G, \mathbb{R})$ are vector spaces over $\mathbb{R}$ and compute their dimensions.
(2) Show that d is a linear map and describe its null-space.

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4. Let V, W be finite dimensional vector spaces. Prove that

$$dim\mathcal{L}(V,W) \;=\; dimV dimW \;.$$

5. What is the relation between the matrices of a linear operator with respect to two different bases?

6. Consider vector spaces V, $dimV = n$ and W, $dimW = m$ over F. What is the relation between vector spaces $\mathcal{L}(V,W)$ and $\mathcal{M}(m,n,F)$.