LINEAR ALGEBRA 5B1307, JANUARY 2001

Homework 01/01/31

1. Read about the so-called *parallelogram equality* (p. 106 in Axler's book): for all $u, v \in V$

$$\|u+v\|^2 \ + \ \|u-v\|^2 \ = \ 2 \ (\|u\|^2 \ + \ \|v\|^2) \ .$$

Recall that a norm in a vector space is a function

$$\| \cdot \| : V \to [0, +\infty)$$

such that

(1) $\|v\| = 0$ if and only if $v = 0$;
(2) $\|\alpha u\| \ = \ |\alpha| \|u\|$ for all $u \in V$, $\alpha \in F$;
(3) $\|u+v\| \ \leq \|u\| \ + \ \|v\|$.

Prove that a norm which satisfies the parallelogram inequality necessarily comes from an inner product.

(This is exercise 8, p. 123 in Axler's book.)

2. Prove that if $P \in \mathcal{L}(V)$; $P^2 = P$; and any of two conditions below is satisfied, then P is an orthogonal projection.

(1) every vector in $nullP$ is orthogonal to every vector in $rangeP$;
(2) $\|Pv\| \ \leq \ \|v\|$ for all $v \in V$.

(This is exercises 17-18, p. 124.)

3. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove

(1) U is invariant under T if and only if $P_U T P_U \ = \ T P_U$.
(2) U and $U^\perp$ are both invarinat under T if and only if P_U and T commute.

(This is exercises 19-20, p.124.)

4. Find relations between eigenvalues of T and T^*.

5. Let $G = (V, E)$ be a finite graph with a vertex-set V and an edge-set E. Each edge has two possible orientations, so that an oriented edge is understood as an ordered pair of vertices $e = (e_+, e_-)$. The set $\mathbb{E}$ of oriented edges has thus cardinality $2|E|$. An oriented edge $e = (e_+, e_-)$ can be naturally coupled with $\bar{e} = (e_-, e_+)$.

Consider the space $C^0(G, \mathbb{R})$ of real-valued functions on the vertices of G, and the space $C^1(G, \mathbb{R})$ of real-valued functions on the edges of G which respect the orientation.

$$C^0(G, \mathbb{R}) \ = \ \{ \ f : V \to \mathbb{R} \ \} \ ,$$

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$$C^1(G,\mathbb{R}) \;=\; \{\; \phi : \mathbb{E} \to \mathbb{R} \;:\; \phi(\bar{e}) = -\phi(e) \;\} \;.$$

These are inner-product spaces with the respective inner-products:

$$\langle f, g \rangle_{C^0} \;=\; \sum_{v \in V(G)} f(v) g(v) \quad \text{for all} \quad f, g \in C^0(G) \;;$$

$$\langle \phi, \psi \rangle_{C^1} \;=\; \frac{1}{2} \sum_{e \in \mathbb{E}(G)} \phi(e) \psi(e) \quad \text{for all} \quad \phi, \psi \in C^1(G) \;.$$

Consider the operator

$$d : \; C^0(G,\mathbb{R}) \to C^1(G,\mathbb{R})$$

defined by

$$df(e) \;=\; f(e_+) - f(e_-) \;,$$

its adjoint operator

$$d^* : \; C^1(G,\mathbb{R}) \to C^0(G,\mathbb{R}) \;,$$

and the Laplacian

$$\Delta : \; C^0(G,\mathbb{R}) \to C^0(G,\mathbb{R})$$

defined by

$$\Delta \;=\; d^* d \;.$$

(1) Show that "delta-functions " of vertices (respectively edges) form an orthonormal basis in C^0 (respectively C^1). (Recall that a "delta-function" is a function which takes value 1 in one point and value 0 in all other points.)
(2) Find the matrices of operators d, d^*, and Δ in the orthonormal bases described in (1). What is the meaning of these matrices in terms of the graph G?
(3) Compute dimensions of $range d^*$ and $null d^*$.