# LINEAR ALGEBRA 5B1307, JANUARY 2002

## Homework 1; 02/01/24

**1.** Give a necessary and sufficient condition for the union of two subspaces to be a subspace.

**2.** Let $V, W$ be finite dimensional vector spaces. Prove that

$$dim\mathcal{L}(V,W) \;=\; dimV dimW \;.$$

**3.** Consider vector spaces $V$, $dimV = n$, and $W$, $dimW = m$, over $F$. What is the relation between vector spaces $\mathcal{L}(V,W)$ and $\mathcal{M}(m,n,F)$?

**4.** Let $V$ be a finite dimensional vector space, and let $U_1, ..., U_m$ be subspaces of $V$. Suppose that

(1) $V \;=\; U_1 \;+\; U_2 \;+\; \ldots \;+ U_m$;
(2) $dimU_1 \;+\; dimU_2 \;+ \ldots +\; dimU_m \;=\; dimV$ .

Show that $V$ is the direct sum of subspaces $U_1, ..., U_m$.

**5.** Recall that a *norm* on a vector space $V$ is a function

$$\|\cdot\| : V \to [0, +\infty)$$

such that

(1) $\|v\| = 0$ if and only if $v = 0$;
(2) $\|\alpha u\| \;=\; |\alpha|\|u\|$ for all $u \in V$, $\alpha \in F$;
(3) $\|u + v\| \;\leq \|u\| \;+\; \|v\|$.

We say that a norm satisfies the so-called *parallelogram equality* if for all $u, v \in V$

$$\|u + v\|^2 \;+\; \|u - v\|^2 \;=\; 2\;(\|u\|^2 \;+\; \|v\|^2) \;.$$

Prove that a norm which satisfies the parallelogram equality necessarily comes from an inner product, i.e., that there exists an inner product $\langle\cdot,\cdot\rangle$ on $V$, such that for every $v \in V$, $\|v\| \;=\; \sqrt{\langle v, v\rangle}$.

**6.** Prove the following inequality:

$$(\sum_{j=1}^{n} a_j b_j)^2 \;\leq\; \sum_{j=1}^{n} j a_j^2 \;\sum_{j=1}^{n} \frac{b_j^2}{j} \;.$$

**7.** Check whether the following matrices

(1) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;
(2) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

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define inner product structure on $\mathbb{R}^2$.

**8.** Let $A$ be a symmetric $n \times n$ matrix with real coefficients, such that the corresponding quadratic form $\sum_{i,j=1}^{n} a_{ij} x_i x_j$ is positive definite (i.e., $\sum_{i,j=1}^{n} a_{ij} x_i x_j \geq 0$ for any $(x_1, ..., x_n) \in \mathbb{R}^n$, and $\sum_{i,j=1}^{n} a_{ij} x_i x_j = 0$ if and only if $x_1 = ...x_n = 0$.)

Prove that $a_{ik}^2 \leq a_{ii} a_{kk}$ for all $i, k = 1, ..., n$.