LINJÄR ALGEBRA 5B1307

TENTAMENSSKRIVNING, 01.03.06

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inga hjälpmedel är tillåtna

1.

(1) Give two definitions of a basis in a vector space.
(2) Show that two definitions are equivalent.
(3) What is the dimension of a vector space?
(4) Show that the dimension of a vector space is well-defined.

2.

(1) Give a definition of the characteristic polynomial of a linear operator without using the notion of determinant (if necessary, treat separately the real and the complex cases).
(2) Formulate Cayley-Hamilton Theorem.
(3) Define the determinant of a linear operator T and show that the characteristic polynomial of T is given by the formula

$$p_T(\lambda) \;=\; det(\lambda I - T)\ .$$

3.

(1) Define a nilpotent operator and give an example.
(2) How does the Jordan matrix of a nilpotent operator look like?
(3) In which basis does the operator have this form?

4. Let V be an inner-product vector space.

(1) Give a necessary and sufficient condition for an operator $T \in \mathcal{L}(V)$ to be diagonalizable. (Consider separately the cases when V is a vector space over $\mathbb{R}$ and $\mathbb{C}$.)
(2) Demonstrate the difference between the complex and the real cases with help of an example.
(3) Explain the main ideas of the proof of the theorem in the real case.
(4) In the case when V is defined over $\mathbb{R}$, give a necessary and sufficient condition for T to be normal.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

5. Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ has the matrix

$$\begin{pmatrix} 5 & 0 & 1 \\ 1 & 1 & 0 \\ -7 & 1 & 0 \end{pmatrix}$$

in the standard basis. Find the Jordan matrix of the operator and the basis in which $M(T)$ has the Jordan form.

6.

(1) What is an isometry?
(2) Show that the set of all isometries of a vector space is a group.
(3) Determine the group of isometries $Isom(\mathbb{R}^2)$.
(4) Suppose that $T \in \mathcal{L}(\mathbb{R}^4)$ is given by the matrix

$$\begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -5/6 & 1/6 & 1/6 \\ 1/2 & 1/6 & 1/6 & -5/6 \\ 1/2 & 1/6 & -5/6 & 1/6 \end{pmatrix}$$

in the standard basis. Is T an isometry? Justify your answer.

7. Consider the point $x \in \mathbb{R}^2$ which has coordinates $(-2, 6)$ in the standard basis. Find the coordinates of x in the basis obtained from the standard basis by the counterclockwise rotation through the angle $\theta = 3\pi/4$.

8. Let V be a vector space of dimension n; let $\{v_i\}_{i=1}^n$ be a basis in V.

(1) Find a basis in the dual space V^*. Justify your answer.
(2) Suppose that $\{v_i\}_{i=1}^n$, $\{u_i\}_{i=1}^n$ are two bases in V; that A is the transition matrix from $\{v_i\}_{i=1}^n$ to $\{u_i\}_{i=1}^n$; and that $\{v_i^*\}_{i=1}^n$, $\{u_i^*\}_{i=1}^n$ are the corresponding bases in V^*. What is the transition matrix from $\{v_i^*\}_{i=1}^n$ to $\{u_i^*\}_{i=1}^n$?
(3) What is $(\mathcal{M}(n, n, \mathbb{C}))^*$? Justify your answer.