LINEAR ALGEBRA 5B1307, 2001-2002, PERIOD 3

TENTAMENSSKRIVNING, 02.03.05

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inga hjälpmedel är tillåtna

1.

(1) State the theorem and present the main ideas of the proof of existence of an eigenvalue for a linear operator in a finite dimensional complex vector space.
(2) State the theorem and present the main ideas of the proof of existence of an invariant subspace of dimension 1 or 2 for a linear operator in a finite dimensional real vector space.
(3) Demonstrate the difference between the real and the complex case with help of an example.

2. Let V be a finite dimensional inner-product space.

(1) Give definitions of a positive operator and of an isometry.
(2) Let T be a linear operator on V. Give two necessary and sufficient conditions of existence of a square root of T.
(3) What are the singular values of a linear operator?
(4) State the theorem about the polar decomposition of a linear operator. For which operators is this decomposition unique? Justify your answer.
(5) Compute the square root of the linear operator in $\mathbb{R}^4$, given by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

in the standard basis.
(6) Suppose the linear operator $T \in \mathcal{L}(\mathbb{C}^3)$ is given by the matrix

$$\begin{pmatrix} 4/5 & 0 & -3/5 \\ -9/25 & 4/5 & -12/25 \\ 12/25 & 3/5 & 16/25 \end{pmatrix}$$

in the standard basis. Is T an isometry? Justify your answer.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

3.

(1) Let T be a linear map between two finite dimensional inner-product spaces, $T \in \mathcal{L}(U, V)$. Define the adjoint operator T^*. Prove correctness of this definition.

(2) Compute the adjoint of the operator $T \in \mathcal{L}(F^3)$ given by

$$T(x, y, z) = (x + y, y, x + y + z) .$$

4. Let V be a finite dimensional vector space over a field F, $T \in \mathcal{L}(V)$.

(1) Define the characteristic polynomial $p_T(x)$ of T.

(2) Define the minimal polynomial $m_T(x)$ of T.

(3) Give an example of an operator T with $p_T(x) \neq m_T(x)$.

(4) Give definitions of the determinant of T and of the trace of T using the characteristic polynomial.

(5) Show that $p_T(x) = det(xI - T)$.

5. Consider the operator T in $\mathbb{C}^4$ given by the formula

$$T(x, y, z, w) = (x + z, x - 7z - 2w, y + 5z + w, 2w) .$$

(1) Find the matrix of T in the standard basis.

(2) Find the characteristic polynomial $p_T(x)$.

(3) Find the minimal polynomial $m_T(x)$.

(4) Find the Jordan form of T.

(5) Find a Jordan basis for T.

(6) The operator S in $\mathbb{C}^4$ given by the formula

$$S(x, y, z, w) = (2x + z + w, 2y + 3z + 3w, 3x - y + z - w, -3x + y + z + 3w)$$

has the same eigenvalues as T, with the same respective multiplicities. Find the minimal polynomial and the Jordan form of S.

6. Let V be a vector space over F with a basis $(v_1, ...v_n)$.

(1) Explain what is the dual space V' and which basis of V' is called dual to $(v_1, ...v_n)$.

(2) Suppose $n = 2$. Find, in terms of the basis (v_1', v_2') dual to (v_1, v_2), the basis of V' dual to the basis $(v_1 + 2v_2, 3v_1 + 4v_2)$ of V.

7. Let a rectangular $x'y'$-coordinate system be obtained by rotating a rectangular xy-coordinate system counterclockwise through the angle $\theta = 3\pi/4$. Find the xy-coordinates of the point whose $x'y'$-coordinates are $(5, 2)$.