LINEAR ALGEBRA 5B1307, JANUARY 2001

Lösningar till Kontrollskrivningen

1. Let V be a finite dimensional vector space over F, $T \in \mathcal{L}(V)$.

(1) A non-zero vector $v \in V$ is called *an eigenvector* of T if there exists a scalar $\lambda \in F$ such that $Tv = \lambda v$. (v is then an eigenvector of T corresponding to the eignevalue λ.)
(2) If $\lambda \in F$ is an eigenvalue of T, then the subspace $null(T - \lambda I)$ of V is called *the eigenspace* corresponding to λ.
(3) If $\lambda \in F$ is an eigenvalue of T, then a non-zero vector $v \in V$ is *a generalized eigenvector* of T corresponding to the eigenvalue λ if there exists $k \in \mathbb{N}$ such that $(T - \lambda I)^k v = 0$.
(4) If $\lambda \in F$ is an eigenvalue of T, then the subspace $null((T - \lambda I)^{dim V})$ of V is called *the space of generalized eigenvectors* corresponding to λ.
(5) If V is a complex vector space, $T \in \mathcal{L}(V)$, and $\lambda_1, ..., \lambda_m$ are the distinct eigenvalues of T, then

$$V = null((T - \lambda_1 I)^{dim V}) \oplus ... \oplus null((T - \lambda_m I)^{dim V}) \ .$$

2. Let $M(T)$ denote the matrix of the operator T in the standard basis of $\mathbb{C}^5$. Then, by applying T to the basis vectors, we find the columns of $M(T)$, and consequently:

$$M(T) = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 \end{pmatrix} .$$

By developing the determinant $det(M(T) - \lambda I)$ with respect to the first column twice, we find that the characteristic polynomial of $M(T)$ is

$$p(\lambda) = (2-\lambda)^2 \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = (2-\lambda)^2(1-\lambda)^3 \ .$$

Thus T has two disctinct eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with multiplicities respectively $d_1 = 3$ and $d_2 = 2$. Therefore the Jordan form $J(T)$ will be a 5×5 matrix consisting of two blocks: B_1, of size 3×3, which will have 1's on the diagonal, and B_2, of size 2×2 which will have 2's on the diagonal.

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The next step is to determine the decomposition of each block into Jordan cells. In order to determine the number ot Jordan cells in a block B_i we need to compute the dimension of the eigenspace $null(T - \lambda_i I)$. The system of equations

$$(M(T) - \lambda_1 I)v = 0$$

becomes

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which yields

$$z_1 = 0, \quad z_2 = 0, \quad z_3 + z_4 = 0, \quad z_5 = 0.$$

Consequently, the eigenspace corresponding to λ_1 is a one-dimensional subspace of $\mathbb{C}^5$ spanned by the eigenvector

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

Thus the block B_1 consists of the unique Jordan cell. It has therefore to be of maximal possible size 3×3 and to have form

$$B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Analogously one computes that the eigenspace $null(T - \lambda_2 I)$ corresponding to λ_2 is spanned by two linearly independent eigenvectors

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence the block B_2 consists of two Jordan cells which can therefore only be of size 1×1 each:

$$B_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

One can also conclude at this point that the minimal polynomial of T is

$$q(\lambda) = (2 - \lambda)(1 - \lambda)^3 .$$

The Jordan form of T is

$$J(T) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The operator T has the matrix $J(T)$ in a certain Jordan basis $(w_1, w_2, w_3, w_4, w_5)$ of $\mathbb{C}^5$ which it now remains to determine. We can take

$$w_1 := v_1 , \quad w_4 := v_2 , \quad w_5 := v_3 .$$

The vectors w_2 and w_3 are generalized eigenvectors corresponding to λ_1 and lie in the subspaces $null((T-\lambda_1 I)^2)$ and $null((T-\lambda_1 I)^3)$ respectively. They should be taken lineraly independent from w_1.

Theorem 5 tells us that we can look for w_2 in the form of a solution of the system of equations $(T-\lambda_1 I)w = w_1$, which in our case becomes:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} ,$$

which yields

$$w_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} .$$

The vector w_3 can be found as a solution of the system of equations $(T-\lambda_1 I)w = w_2$. The similar computation gives

$$w_3 = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} .$$

Verification: It can be checked easily that if we view the vectors of the Jordan basis obtained above as columns of a matrix S, then

$$M(T) = S\ J(T)\ S^{-1} .$$

3. Theorem about *polar decomposition*, proven in Chapter 3, tells that for every linear operator $T \in \mathcal{L}(V)$, there exists an isometry $S \in \mathcal{L}(V)$ such that $T = S\sqrt{T^*T}$. The singular values of T are by definition the eigenvalues of the operator $\sqrt{T^*T}$. To find a polar decomposition of an operator means to find the matrices, in the standard basis, of the positive operator $\sqrt{T^*T}$ and of an isometry S, which satisfy the formula above.

We begin by finding the matrix of $\sqrt{T^*T}$ in the standard basis. From the formula for $M(T)$ we find the matrix of the operator T^*T in the standnard basis:

$$M(T^*T) = \begin{pmatrix} 13 & 14 & 4 \\ 14 & 24 & 18 \\ 4 & 18 & 29 \end{pmatrix} .$$

As the eigenvalues of $\sqrt{T^*T}$ are known to us, we can easily find the eigenvalues of T^*T as the squares of the eigenvalues of $\sqrt{T^*T}$. The eigenvalues of T^*T are therefore $\{1,\ 16,\ 49\}$. After we know the eigenvalues of T^*T, it is easy to find the eigenvectors of T^*T corresponding to them:

$$\lambda_1 = 1\,, \quad v_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\,;$$

$$\lambda_2 = 16\,, \quad v_2 = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}\,;$$

$$\lambda_3 = 49\,, \quad v_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\,.$$

In the basis (v_1, v_2, v_3) of $\mathbb{C}^3$, the operator $\sqrt{T^*T}$ has a diagonal matrix

$$M(\sqrt{T^*T},\ (v_1, v_2, v_3)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}\,.$$

Therefore the matrix of the operator $\sqrt{T^*T}$ in the standard basis can be found as

$$M(\sqrt{T^*T}) = \begin{pmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

The operator T is invertible (e.g., because the determinant of $M(T)$ is non-zero), therefore the isometry S is uniquely determined by T, and one has

$$M(T) = M(S)\ M(\sqrt{T^*T})\,.$$

Thus

$$M(S) = M(T)\ M(\sqrt{T^*T})^{-1} = \begin{pmatrix} \frac{1}{14} & \frac{25}{28} & -\frac{5}{14} \\ \frac{13}{7} & -\frac{11}{14} & \frac{5}{7} \\ \frac{13}{7} & -\frac{25}{14} & \frac{12}{7} \end{pmatrix}\,.$$