COMBINATORIAL GROUP THEORY
(5B1459), JANUARY-MARCH 2002

Tatiana Smirnova-Nagnibeda

PROBLEM LIST

1. In a free group of rank r, check that there are $2^r - 1$ subgroups of index 2.

2. In case X is a bouquet of 2 circles, check that there are precisely three non-isomorphic classes of coverings of degree 2, corresponding to the three subgroups of index 2 in the fundamental group of X, namely in a free group of rank 2.

3. Show that two free groups of finite or countable infinite rank are isomorphic if and only if they have the same rank.

4. Let $G = *_{i \in I} G_i$ be a free product of $|I| > 2$ groups, none of which is reduced to one element. Show that

(1) G is not abelian;
(2) G contains an element of infinite order;
(3) the center of G is reduced to one element.

5. Show that free groups are torsion free, i.e., contain no non-trivial element of finite order.

6. Show that the following is a presentation of the trivial group:

$$\langle a, b, c \mid a^3, b^3, c^4, acac^{-1}, aba^{-1}bc^{-1}b^{-1} \rangle.$$

7. The braid group on three strings. Show that the following three presentations define isomorphic groups. This is the so-called *braid group on three strings* which is of great importance in topology (for example, being the fundamental group of the complement, in $\mathbb{R}^3$, of the so-called "trefoil knot".)

$$\begin{array}{l} \langle x, y, z \mid x = yzy^{-1},\ y = zxz^{-1},\ z = xyx^{-1} \rangle \\ \langle x, y \mid xyx = yxy \rangle \\ \langle a, b \mid a^3 = b^2 \rangle \end{array}.$$

8. Let $G = SL(2, \mathbb{Z})$.

(1) Show that the elements

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

generate G.

(2) Recall that the matrices

$$M_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a free subgroup of rank 2 in G. Find its index.

9. The modular group. Prove that

(1) $PSL(2,\mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$;

Hint: One can prove it directly, or as a particular case of exercise 10.

(2) $PSL(2,\mathbb{Z})$ contains a subgroup which is free of rank 2.

10. Hecke groups. Consider an integer $q \geq 3$, the number $\lambda = 2cos(\pi/q)$, two elements

$$a_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in $G = PSL(2,\mathbb{R})$ and the subgroup Γ_λ of G generated by these two elements. Show that

$$\Gamma_\lambda \cong (\mathbb{Z}/q\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) .$$

11*. Automorphisms of the real line. Let f be the piecewise linear homeomorphism of the interval $[0,1]$ defined by

$$f(t) = \begin{cases} 4t & \text{if } t \in [0, \frac{1}{5}], \\ \frac{4}{5} + \frac{1}{4}(t - \frac{1}{5}) & \text{if } t \in [\frac{1}{5}, 1]. \end{cases}$$

Let γ_1 be the piecewise linear homeomorphism of the real line defined by

$$\gamma_1(t) = [t] + f(\{t\}) \text{ , for all } t \in \mathbb{R}$$

where $[t]$ denotes the integral part of a real number t, and $\{t\} = t - [t] \in [0,1[$ its fractional part.

Set also

$$\gamma_2 = T\gamma_1 T^{-1}$$

where T denotes the translation $t \mapsto t - \frac{1}{2}$.

Show that γ_1, γ_2 generate a free group of rank 2 in the group $Homeo_+(\mathbb{R})$ of all orientation-preserving piecewise linear homeomorphisms of the real line.

12. Free groups of rotations. Prove that $SO(3)$, the group of rotations in $\mathbb{R}^3$, contains free subgroups of rank n for any $n \geq 2$.

This easy exercise has some far-going applications, e.g., in finding a paradoxical decomposition of a sphere (c.f. Banach-Tarski paradox).

Hint: Check that the matrices

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

generate in $SO(3)$ a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$.

13. Let G be a group and let H be a subgroup of finite index in G. Show that

(1) If G is finitely generated, then H is finitely generated.
(2) H is finitely presentable if and only if G is finitely presentable.

Hint: One possibility is to read about Reidemeister-Schreier method. See e.g. Magnus-Karrass-Solitar, Section 2.3, or Stillwell, Sections 4.3.6 and 4.3.8.

14. Let G be a finitely generated group and let $G_1 < G_2 < ... < G_n < ... < G$ be an increasing sequence of subgroups in G such that $G = \cup_{n\geq 1} G_n$. Show that for n large enough $G_n = G$.

15. Residually finite groups. A group G is called *residually finite* if for any $g \in G, g \neq e_G$, there exists a finite group F and a homomorphism $\phi : G \to F$ such that $\phi(g) \neq e_F$.

Prove that

(1) A subgroup of a residually finite group is again residually finite.
(2) $SL(n, \mathbb{Z})$ is residually finite for all $n \geq 1$.
    Hint: $\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$ induces $SL(n, \mathbb{Z}) \to SL(n, \mathbb{Z}/q\mathbb{Z})$ for each $q \geq 2$.
(3) Free groups are residually finite.
    Hint: deduce from (1) and (2).
    Remark. Once (3) is proved, Mal'cev's Theorem then implies that free groups are ***hopfian***, i.e., any homomorphism $\mathbb{F}_n \to \mathbb{F}_n$ which is onto is necessarily a bijection.

16. Check that

$$SL(2,\mathbb{Z}) \cong (\mathbb{Z}/6\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z})$$

with

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

generating the cyclic group of order 6 and

$$f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

generating the cyclic group of order 4.

17. Presentations with positive relations (von Dyck, 1882). Show that any finitely presented group

$$G = \langle s_1, ..., s_n \mid r_1, ..., r_m \rangle$$

has a finite presentation

$$\langle s_0, s_1, ..., s_n \mid r'_0, r'_1, ..., r'_m \rangle$$

such that each r'_j is a word in letters $s_0, ..., s_n$ (and without $s_0^{-1}, ..., s_n^{-1}$).

Hint: set $r'_0 = s_0 s_1 ... s_n$.

18. Let F be a free group of rank 2, and let F' denote the commutator subgroup of F. Prove that $F *_{F'} F$ is finitely generated but not finitely presented.

Hint: Use the theorem about finitely presented amalgamated products and the characterization of finitely generated normal subgroups in a free group.

19. Finitely presented HNN-extensions. Let G be a finitely presented group and let H be an HNN-extension of G over a subgroup $A < G$. Then H is finitely presented if and only if A is finitely generated.

Hint: See analogous result for amalgamated products.

20. B. H. Neumann's construction of an uncountable infinity of groups with two generators. Read Neumann's proof and apply it to prove the following:

(1) the group given by presentation

$$\langle s, t \mid s^3 = (s^{-1}t^{-1}st)^3 = 1\rangle$$

has uncountably many pairwise non-isomorphic quotients;

(2) $PSL(2, \mathbb{Z})$ has uncountably many pairwise non-isomorphic quotients.

Hint: for (2), one could apply Neumann's method to show that

$$\langle s, t \mid s^2 = t^3 = (st)^{30} = 1\rangle$$

has uncountably many pairwise non-isomorphic quotients, and then deduce the result for $PSL(2, \mathbb{Z})$ via Exercise 9(1).

21. The normal closure. Given a group G and a subset $X \subset G$, prove that the following three definitions of the *normal subgroup generated by* X (or the *normal closure of* X) are equivalent:

(1) $gp_G(X)$ is the intersection of all normal subgroups of G which contain X;
(2) $gp_G(X)$ is the smallest normal subgroup of G which contains X;
(3) $gp_G(X)$ is the subgroup of G generated by the set $\{g^{-1}xg : g \in G, x \in X\}$.

22. Let $X = X_1 \cup X_2$ be a union of two open simply connected sets, and suppose that $X_1 \cap X_2$ is arcwise connected. Prove that, for $x_0 \in X_1 \cap X_2$, the group $\pi_1(X, x_0)$ is trivial.

Deduce as a corollary, that a sphere of dimension ≥ 2 is simply connected (though not contractible). Use this to show that $\mathbb{R}^2$ is not homeomorphic to any $\mathbb{R}^n$ with $n \neq 2$.

23. Petersen graph is not a Cayley graph. Let P denote the Petersen graph. Its vertex set is the set or unordered pairs from $\{1, 2, 3, 4, 5\}$, and two vertices, $\{i, j\}$ and $\{k, l\}$, are connected by an edge if and only if $\{i, j\} \cap \{k, l\} = \emptyset$.

Show that P is not a Cayley graph.

24. Universal Coxeter group. Describe the Coxeter system corresponding to the Coxeter graph which is obtained from the complete graph K_n on n vertices by labelling all its edges by ∞.

Formulate the universal property of U_n in the class of Coxeter groups of rank n.

25. Show that there exist Coxeter systems (W, S) and (W', S') with $|S| \neq |S'|$ such that W and W' are isomorphic as abstract groups.

Hint: Think of the dihedral group D_{12}.

26. Growth functions of groups. Let G be a finitely generated group, and let S be a finite set of generators of G.

(1) Prove that the function $d_S : G \times G \to \mathbb{N}$ given by

$$d_S(g_1, g_2) \ = \ l_S(g_1^{-1} g_2) \ ,$$

where

$$l_S(g) \ = \ min\{n : g = s_{i_1} ... s_{i_n} \ \text{ with } \ s_{i_j} \in S \cup S^{-1}\} \ ,$$

is a metric on G (it is called *the word metric on G associated to S.*) Check that G acts by isometries on the metric space (G, d_S).

(2) Prove that *the growth function* $\beta_{(G,S)} : \mathbb{N} \to \mathbb{N}$ of the pair (G, S), defined by

$$\beta_{(G,S)}(k) \ = \ \sharp\{g \in G : l_S(g) \leq k\}$$

is submultiplicative, i.e.,

$$\beta_{(G,S)}(k_1 + k_2) \ \leq \ \beta_{(G,S)}(k_1) \ \beta_{(G,S)}(k_2) \ ,$$

for all $k_1, k_2 \in \mathbb{N}$.

(3) Recall that *the spherical growth function* of the pair (G, S) is defined as

$$\theta_{(G,S)}(k) \ = \ \sharp\{g \in G : l_S(g) = k\} \ .$$

Consider the generating function of the sequence $(\theta_{(G,S)}(k))_{k \in \mathbb{N}}$, namely the formal power series $\Theta_{(G,S)}(z) = \sum_{k=0}^{\infty} \theta_{(G,S)}(k) z^k$. Find a recursive expression for $\theta_{(\mathbb{Z}^2, \{(1,0),(0,1)\})}(k)$ and thus give a direct proof of the formula

$$\Theta_{(\mathbb{Z}^2, \{(1,0),(0,1)\})}(z) \ = \ \left(\frac{1+z}{1-z}\right)^2 \ .$$

(4) Prove that the growth function of a pair (G, S) is majorized by the growth function of the pair $(F(S), S)$. Deduce that the growth of a finitely gerenated group is at most exponential.