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# Markov Chain Monte Carlo for rare-event simulation in heavy-tailed settings

#### Thorbjörn Gudmundsson

Department of Mathematics KTH Stockholm

#### Licentiat seminarium, December 2013

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The Big Picture			

 Computational problem. For instance, the probability of default or the expected loss given ruin.

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 Computational problem. For instance, the probability of default or the expected loss given ruin.

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Complex system : no analytical solution available



- Computational problem. For instance, the probability of default or the expected loss given ruin.
- Complex system : no analytical solution available
- Simulation techniques
  - i Monte Carlo
  - ii Conditional Monte Carlo
  - iii Splitting methods
  - iv Importance sampling

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- Computational problem. For instance, the probability of default or the expected loss given ruin.
- Complex system : no analytical solution available
- Simulation techniques
  - i Monte Carlo
  - ii Conditional Monte Carlo
  - iii Splitting methods
  - iv Importance sampling
  - v Markov chain Monte Carlo (NEW)

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#### Problem

Consider a random variable X with known distribution F and the objective of computing

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where  $\{X \in A\}$  is thought as rare in the sense that *p* is small. Event of ruin for instance.

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#### Problem

Consider a random variable X with known distribution F and the objective of computing

$$o = \mathbb{P}(X \in A),$$

where  $\{X \in A\}$  is thought as rare in the sense that *p* is small. Event of ruin for instance.

**Example.** Random walk  $S_n = Y_1 + \cdots + Y_n$  with non-negative steps *Y*'s with known heavy-tailed distribution  $F_Y$  and objective of computing

$$p=\mathbb{P}\Big(rac{S_n}{n}>a\Big),$$

where *a* is much larger than  $\mathbb{E}[Y]$ .

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#### Stochastic Simulation

Want to compute  $p = \mathbb{P}(X \in A)$ .

In absence of an analytical solution, stochastic simulation offers an alternative.

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#### Stochastic Simulation

Want to compute  $p = \mathbb{P}(X \in A)$ .

In absence of an analytical solution, stochastic simulation offers an alternative.

Monte Carlo: sample identically distributed and independent copies  $X_1, \ldots, X_N$  and compute

$$\hat{p} = \frac{1}{N} \sum_{k=1}^{N} I\{X_k \in A\}.$$

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#### Shortcomings of Monte Carlo

The relative error of the Monte Carlo estimator is unbounded as  $p \rightarrow 0$ :

$$\frac{\mathbb{V}ar(\hat{p})}{p^2} = \frac{1}{N} \Big( \frac{1}{p} - 1 \Big) \to \infty, \quad \text{as } p \to 0.$$

**Example.** Standard normal variable *X*, compute  $p = \mathbb{P}(X > a)$  using  $N = 10^6$  number of simulations

$$a = 1 : \hat{p} = 0.158, \quad \frac{\text{Stdev}(\hat{p})}{\hat{p}} = 0.002$$
  

$$a = 3 : \hat{p} = 0.0014, \quad \frac{\text{Stdev}(\hat{p})}{\hat{p}} = 0.027$$
  

$$a = 5 : \hat{p} = 0, \quad \frac{\text{Stdev}(\hat{p})}{\hat{p}} = \infty$$

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- Conditional Monte Carlo (Asmussen)
- Splitting methods (Creou et al)
- Importance sampling (Sigmund, Dupuis, Blanchet)

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#### Importance sampling

Goal: construct an efficient estimator  $\hat{p}$  of  $p = \mathbb{P}(X \in A)$ , in the sense that its relative error is bounded.

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#### Importance sampling

Goal: construct an efficient estimator  $\hat{p}$  of  $p = \mathbb{P}(X \in A)$ , in the sense that its relative error is bounded.

The importance sampling approach (Dupuis et al 2007)

■ Generate independent copies *X*<sub>1</sub>,..., *X*<sub>N</sub> from a sampling distribution *G*.

Compute empirical estimate

$$\hat{p} = rac{1}{N}\sum_{k=1}^{N}rac{dF}{dG}(X_k)\mathbb{I}\{X_k\in A\}.$$

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#### Importance sampling

Goal: construct an efficient estimator  $\hat{p}$  of  $p = \mathbb{P}(X \in A)$ , in the sense that its relative error is bounded. The importance sampling approach (Dupuis et al 2007)

Generate independent copies  $X_1, \ldots, X_N$  from a sampling

distribution G.

Compute empirical estimate

$$\hat{\rho} = rac{1}{N}\sum_{k=1}^{N}rac{dF}{dG}(X_k)\mathbb{I}\{X_k\in A\}.$$

$$\mathbb{E}_G[\hat{p}] = \int_A \frac{dF}{dG}(X) dG(X) = F(A) = p.$$

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#### Importance sampling continued

Reduces to finding a suitable sampling distribution G.

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#### Importance sampling continued

Reduces to finding a suitable sampling distribution *G*. The zero-variance distribution

$$F_A(x) = \mathbb{P}(X \leq x | X \in A).$$

If we can choose  $G = F_A$ , then  $\frac{dF}{dF_A}(X)\mathbb{I}\{X \in A\} = p$ , so

$$\hat{\rho} = rac{1}{N}\sum_{k=1}^{N}rac{dF}{dF_A}(X_k)\mathbb{I}\{X_k\in A\} = 
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with zero variance!

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#### Importance sampling continued

Reduces to finding a suitable sampling distribution *G*. The zero-variance distribution

$$\mathcal{F}_{\mathcal{A}}(x) = \mathbb{P}(X \leq x | X \in \mathcal{A}).$$

If we can choose  $G = F_A$ , then  $\frac{dF}{dF_A}(X)\mathbb{I}\{X \in A\} = p$ , so

$$\hat{\rho} = rac{1}{N}\sum_{k=1}^{N}rac{dF}{dF_A}(X_k)\mathbb{I}\{X_k\in A\} = 
ho,$$

with zero variance! Requires knowledge of  $\mathbb{P}(X \in A) \dots$ 

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The idea			

Want: sample from  $F_A(x) = \mathbb{P}(X \le x | X \in A)$ . Assuming the existence of a density, it takes the form

$$f_{\mathcal{A}}(x) = rac{f(x)\mathbb{I}\{x\in \mathcal{A}\}}{\mathbb{P}(X\in \mathcal{A})}.$$

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The idea			
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Want: sample from  $F_A(x) = \mathbb{P}(X \le x | X \in A)$ . Assuming the existence of a density, it takes the form

$$f_{\mathcal{A}}(x) = rac{f(x)\mathbb{I}\{x\in \mathcal{A}\}}{\mathbb{P}(X\in \mathcal{A})}.$$

The main idea is to construct a Markov chain  $(X_k)_{k\geq 1}$  for which  $f_A$  is the invariant density via MCMC. Then *extract* information about the normalising constant from the sample.

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Estimator			
Estimator			

■ Construct a Markov chain (X<sub>k</sub>)<sub>k≥1</sub> via MCMC sampler, with the zero-variance distribution F<sub>A</sub> as its invariant distribution.

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#### Estimator

- Construct a Markov chain (X<sub>k</sub>)<sub>k≥1</sub> via MCMC sampler, with the zero-variance distribution F<sub>A</sub> as its invariant distribution.
- For any  $v \ge 0$  such that  $\int_A v(x) dx = 1$ , consider

$$u((X_k)_{k\geq 1})=rac{1}{N}\sum_{k=1}^Nrac{v(X_k)\mathbb{I}\{X_k\in A\}}{f(X_k)}.$$

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#### Estimator continued

For  $\int_A v(x) dx = 1$  it holds

$$\mathbb{E}_{F_A}\Big[\frac{1}{N}\sum_{k=1}^N \frac{v(X_k)\mathbb{I}\{X_k \in A\}}{f(X_k)}\Big] = \int_A \frac{v(x)}{f(x)} \frac{f(x)}{p} dx$$
$$= \frac{1}{p} \int_A v(x) dx$$
$$= \frac{1}{p}.$$

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#### Estimator continued

• For  $\int_A v(x) dx = 1$  it holds  $\mathbb{E}_{F_A} \left[ \frac{1}{N} \sum_{k=1}^N \frac{v(X_k) \mathbb{I}\{X_k \in A\}}{f(X_k)} \right] = \int_A \frac{v(x)}{f(x)} \frac{f(x)}{p} dx$   $= \frac{1}{p} \int_A v(x) dx$   $= \frac{1}{p}.$ 

Define 
$$\hat{q} = \frac{1}{N} \sum_{k=1}^{N} \frac{v(X_k)\mathbb{I}\{X_k \in A\}}{f(X_k)}$$
 estimator of  $1/p$ .

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#### **Design issues**

Estimator 
$$\hat{q} = \frac{1}{N} \sum_{k=1}^{N} \frac{v(X_k)\mathbb{I}\{X_k \in A\}}{f(X_k)}$$
 of  $1/p$ .

 Choice of the MCMC sampler: crucial to control the dependence of the Markov chain, to ensure the large sample efficiency

$$\mathbb{V}ar(\hat{q}) o 0$$
, as  $N o \infty$ .

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#### **Design issues**

Estimator 
$$\hat{q} = \frac{1}{N} \sum_{k=1}^{N} \frac{v(X_k)\mathbb{I}\{X_k \in A\}}{f(X_k)}$$
 of  $1/p$ .

 Choice of the MCMC sampler: crucial to control the dependence of the Markov chain, to ensure the large sample efficiency

$$\mathbb{V}ar(\hat{q}) \to 0$$
, as  $N \to \infty$ .

Choice of v: controls the variance, set to ensure rare-event efficiency

$$rac{\operatorname{Std}(\hat{q})}{1/
ho}=
ho\operatorname{Std}(\hat{q}) o 0, \quad ext{as } 
ho o 0.$$

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#### Controlling the variance

Estimator  $\hat{q} = \frac{1}{N} \sum_{k=1}^{N} u(X_k)$ , with  $u(X_k) = \frac{v(X_k)\mathbb{I}\{X_k \in A\}}{f(X_k)}$ . Goal is to show  $p \operatorname{Std}(\hat{q})$  tends to zero as  $p \to 0$ .

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#### Controlling the variance

Estimator  $\hat{q} = \frac{1}{N} \sum_{k=1}^{N} u(X_k)$ , with  $u(X_k) = \frac{v(X_k)\mathbb{I}\{X_k \in A\}}{f(X_k)}$ . Goal is to show  $p \operatorname{Std}(\hat{q})$  tends to zero as  $p \to 0$ .

Consider the term

$$p^{2} \mathbb{V}ar(u(X)) = p^{2} \left( \mathbb{E}[u(X)^{2}] - \mathbb{E}[u(X)]^{2} \right)$$
$$= p^{2} \left( \int_{A} \frac{v^{2}(x)}{f^{2}(x)} \frac{f(x)}{p} dx - 1 \right)$$
$$= p \int_{A} \frac{v^{2}(x)}{f(x)} dx - 1.$$

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## Controlling the variance continued

Choosing 
$$v(x) = f_A(x) = \frac{f(x)\mathbb{I}\{x \in A\}}{p}$$
 implies

$$p^2 \mathbb{V}ar(u(X)) = p \int_A \frac{f^2(x)/p^2}{f(x)} dx - 1 = \frac{1}{p} \int_A f(x) dx - 1 = 0.$$

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## Controlling the variance continued

Choosing 
$$v(x) = f_A(x) = \frac{f(x)\mathbb{I}\{x \in A\}}{p}$$
 implies

$$p^2 \mathbb{V}ar(u(X)) = p \int_A \frac{f^2(x)/p^2}{f(x)} dx - 1 = \frac{1}{p} \int_A f(x) dx - 1 = 0.$$

Choose v as an approximation of the zero-variance density!

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#### Sample $(X_k)_{k\geq 1}$ under $F_A$ via some MCMC sampler

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## Sample (X<sub>k</sub>)<sub>k≥1</sub> under F<sub>A</sub> via some MCMC sampler Show p<sup>2</sup> Var(u(X)) → 0 as p → 0

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Recipe			

Sample  $(X_k)_{k\geq 1}$  under  $F_A$  via some MCMC sampler

Show 
$$p^2 \mathbb{V}ar(u(X)) \to 0$$
 as  $p \to 0$ 

Show  $(X_k)_{k\geq 1}$  is geometric ergodic

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#### Setup

Consider a random walk S<sub>n</sub> = Y<sub>1</sub> + ··· + Y<sub>n</sub> with non-negative steps Y's with known heavy-tailed distribution F<sub>Y</sub> and objective of computing

$$p = \mathbb{P}\Big(rac{S_n}{n} > a\Big),$$

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where *a* is much larger than  $\mathbb{E}[Y]$ .

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#### Setup

Consider a random walk S<sub>n</sub> = Y<sub>1</sub> + ··· + Y<sub>n</sub> with non-negative steps Y's with known heavy-tailed distribution F<sub>Y</sub> and objective of computing

$$p = \mathbb{P}\Big(rac{S_n}{n} > a\Big),$$

where *a* is much larger than  $\mathbb{E}[Y]$ .

Construct  $(\mathbf{Y}_k)_{k>1}$  via MCMC with invariant density

$$f_{\mathcal{A}}(\mathbf{y}) = \frac{f_{\mathbf{Y}}(\mathbf{y})\mathbb{I}\{y_1 + \dots + y_n > an\}}{\mathbb{P}(S_n > an)}$$

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### Setup

Consider a random walk S<sub>n</sub> = Y<sub>1</sub> + ··· + Y<sub>n</sub> with non-negative steps Y's with known heavy-tailed distribution F<sub>Y</sub> and objective of computing

$$p = \mathbb{P}\Big(rac{S_n}{n} > a\Big),$$

where *a* is much larger than  $\mathbb{E}[Y]$ .

Construct  $(\mathbf{Y}_k)_{k>1}$  via MCMC with invariant density

$$f_{\mathcal{A}}(\mathbf{y}) = \frac{f_{\mathbf{Y}}(\mathbf{y})\mathbb{I}\{y_1 + \dots + y_n > an\}}{\mathbb{P}(S_n > an)}$$

A typical such a random walk has a n – 1 number of "small" steps and one "large" step.

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#### Gibbs sampler

Initial state  $\mathbf{Y}_0 = (Y_{0,1}, \dots, Y_{0,n})$  such that  $Y_{0,1} > an$  and  $Y_{0,i} = 0$  for other indices. Given  $\mathbf{Y}_k = (Y_{k,1}, \dots, Y_{k,n})$ ,  $k = 0, 1, \dots$  the next state  $\mathbf{Y}_{k+1}$  is sampled as follows

Take a copy of the current state, let  $Y_{k+1,i} = Y_{k,i}$ ,
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Initial state  $\mathbf{Y}_0 = (Y_{0,1}, \dots, Y_{0,n})$  such that  $Y_{0,1} > an$  and  $Y_{0,i} = 0$  for other indices. Given  $\mathbf{Y}_k = (Y_{k,1}, \dots, Y_{k,n})$ ,  $k = 0, 1, \dots$  the next state  $\mathbf{Y}_{k+1}$  is sampled as follows

- Take a copy of the current state, let  $Y_{k+1,i} = Y_{k,i}$ ,
- **Draw a random index**  $j \in \{1, \ldots, n\}$ ,

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Initial state  $\mathbf{Y}_0 = (Y_{0,1}, \dots, Y_{0,n})$  such that  $Y_{0,1} > an$  and  $Y_{0,i} = 0$  for other indices. Given  $\mathbf{Y}_k = (Y_{k,1}, \dots, Y_{k,n})$ ,  $k = 0, 1, \dots$  the next state  $\mathbf{Y}_{k+1}$  is sampled as follows

- Take a copy of the current state, let  $Y_{k+1,i} = Y_{k,i}$ ,
- Draw a random index  $j \in \{1, \ldots, n\}$ ,
- Sample  $Y_{k+1,j}$  from the conditional distribution of Y given that the sum exceeds the threshold,

$$\mathbb{P}(Y_{k+1,j} \in \cdot) = \mathbb{P}(Y \in \cdot \mid Y + \sum_{i \neq j} Y_{k,i} > an).$$

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Initial state  $\mathbf{Y}_0 = (Y_{0,1}, \dots, Y_{0,n})$  such that  $Y_{0,1} > an$  and  $Y_{0,i} = 0$  for other indices. Given  $\mathbf{Y}_k = (Y_{k,1}, \dots, Y_{k,n})$ ,  $k = 0, 1, \dots$  the next state  $\mathbf{Y}_{k+1}$  is sampled as follows

- Take a copy of the current state, let  $Y_{k+1,i} = Y_{k,i}$ ,
- Draw a random index  $j \in \{1, \ldots, n\}$ ,
- Sample  $Y_{k+1,j}$  from the conditional distribution of *Y* given that the sum exceeds the threshold,

$$\mathbb{P}(Y_{k+1,j} \in \cdot) = \mathbb{P}(Y \in \cdot \mid Y + \sum_{i \neq j} Y_{k,i} > an).$$

#### Permutate the steps in $\mathbf{Y}_{k+1}$ .

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### Gibbs sampler continued

#### Proposition

The Markov chain  $(\mathbf{Y}_k)_{k\geq 1}$  constructed using the proposed Gibbs sampler has the conditional distribution  $F_A$  as its invariant distribution.

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### MCMC estimator

The MCMC estimator  $\hat{q} = \frac{1}{N} \sum_{k=1}^{N} \frac{v(\mathbf{y}_k) \mathbb{I}\{S_n > an\}}{f(\mathbf{y}_k)}$ . The steps are heavy-tailed in the sense that

$$\frac{\mathbb{P}(M_n > an)}{\mathbb{P}(S_n > an)} \to 1,$$

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where  $M_n = \max_i \{y_{k,i}\}$ .

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## MCMC estimator

The MCMC estimator  $\hat{q} = \frac{1}{N} \sum_{k=1}^{N} \frac{v(\mathbf{y}_k) \mathbb{I}\{S_n > an\}}{f(\mathbf{y}_k)}$ . The steps are heavy-tailed in the sense that

$$\frac{\mathbb{P}(M_n > an)}{\mathbb{P}(S_n > an)} \to 1,$$

where  $M_n = \max_i \{y_{k,i}\}$ .

Therefore seems smart to use

 $\mathbb{P}(\mathbf{Y} \in \cdot \mid M_n > an)$  as a proxy for  $\mathbb{P}(\mathbf{Y} \in \cdot \mid S_n > an)$ .

Propose

$$v(\mathbf{y}_k) = \frac{f(\mathbf{y}_k)\mathbb{I}\{M_n > an\}}{\mathbb{P}(M_n > an)}$$

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Markov Chain Monte Carlo for rare-event simulation in heavy-tailed settings

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### MCMC estimator continued

Choosing 
$$v(\mathbf{y}) = \frac{f(\mathbf{y})\mathbb{I}\{M_n > an\}}{\mathbb{P}(M_n > an)}$$
 yields  
$$u(\mathbf{y}) = \frac{v(\mathbf{y})\mathbb{I}\{S_n > an\}}{f(\mathbf{y})} = \frac{\mathbb{I}\{M_n > an\}}{\mathbb{P}(M_n > an)}.$$

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### MCMC estimator continued

Choosing 
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 yields  
$$u(\mathbf{y}) = \frac{v(\mathbf{y})\mathbb{I}\{S_n > an\}}{f(\mathbf{y})} = \frac{\mathbb{I}\{M_n > an\}}{\mathbb{P}(M_n > an)}.$$
$$\hat{q} = \mathbb{P}(M_n > an)^{-1}\frac{1}{N}\sum_{k=1}^{N}\mathbb{I}\{M_n(k) > an\}$$

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Since 
$$u(\mathbf{y}) = \frac{\mathbb{I}\{M_n > an\}}{\mathbb{P}(M_n > an)}$$
, we have:  

$$p^2 \mathbb{V}ar_{F_A}(u(\mathbf{Y})) = \frac{\mathbb{P}(S_n > an)^2}{\mathbb{P}(M_n > an)^2} \mathbb{V}ar_{F_A}(\mathbb{I}\{M_n > an\})$$

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$$= \frac{\mathbb{P}(S_n > an)^2}{\mathbb{P}(M_n > an)^2} \Big(\mathbb{E}_{F_A}[\mathbb{I}\{M_n > an\}] - \mathbb{E}_{F_A}[\mathbb{I}\{M_n > an\}]^2\Big)$$

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$$= \frac{\mathbb{P}(S_n > an)^2}{\mathbb{P}(M_n > an)^2} \left(\mathbb{E}_{F_A}[\mathbb{I}\{M_n > an\}] - \mathbb{E}_{F_A}[\mathbb{I}\{M_n > an\}]^2\right)$$

$$= \frac{\mathbb{P}(S_n > an)^2}{\mathbb{P}(M_n > an)^2} \left(\frac{\mathbb{P}(M_n > an)}{\mathbb{P}(S_n > an)} - \frac{\mathbb{P}(M_n > an)^2}{\mathbb{P}(S_n > an)^2}\right)$$

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Since 
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$$= \frac{\mathbb{P}(S_n > an)^2}{\mathbb{P}(M_n > an)^2} \left(\mathbb{E}_{F_A}[\mathbb{I}\{M_n > an\}] - \mathbb{E}_{F_A}[\mathbb{I}\{M_n > an\}]^2\right)$$

$$= \frac{\mathbb{P}(S_n > an)^2}{\mathbb{P}(M_n > an)^2} \left(\frac{\mathbb{P}(M_n > an)}{\mathbb{P}(S_n > an)} - \frac{\mathbb{P}(M_n > an)^2}{\mathbb{P}(S_n > an)^2}\right)$$

$$= \frac{\mathbb{P}(S_n > an)}{\mathbb{P}(M_n > an)} - 1 \to 0 \quad \text{as } p \to 0.$$

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- The design of the Gibbs sampler ensures that the Markov chain (Y<sub>k</sub>)<sub>k≥1</sub> is (uniformly) ergodic.
- This guarantees that the chain mixes sufficiently and hence that Var(p̂) → 0 as N → ∞ at same speed as 1/N.

Geometric ergodicity

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Numerical experiments

- The MCMC estimator q<sup>-1</sup> of the probability p tested against importance sampling and standard Monte Carlo.
- Steps are Pareto(2) distributed.
- Number of batches: 25, simulations per batch: 10,000.

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n	а	MCMC	IS	MC	
5	10	3.40e-3	2.91e-3	2.83e-3	Avg. est.
		(0.81e-4)	(1.77e-4)	(4.74e-4)	(Std. dev.)
		[4.1]	[3.4]	[0.7]	[Avg. time (ms)]
10	20	3.34e-4	3.02e-4	2.68e-4	Avg. est.
		(5.83e-6)	(2.02e-6)	(162.58e-6)	(Std. dev.)

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#### 10,000 simulations for m = 10 and a = 20



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Random Sum with Heavy-tails			



Consider a random walk  $S_{N_n} = Y_1 + \cdots + Y_{N_n}$  with non-negative heavy-tailed steps *Y*, discrete random variable  $N_n$ and the objective of computing

$$p = \mathbb{P}(S_{N_n} > a\mathbb{E}[N_n]),$$

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where *a* is much larger than  $\mathbb{E}[Y]$ .

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# The challenge

How to design a Gibbs sampler to construct a Markov chain with the following invariant distribution

$$F_{\mathcal{A}}(\cdot) = \mathbb{P}((\mathcal{N}, Y_1, \ldots, Y_N) \in \cdot \mid S_{\mathcal{N}_n} > a_n).$$

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# The challenge

How to design a Gibbs sampler to construct a Markov chain with the following invariant distribution

$$F_{\mathcal{A}}(\cdot) = \mathbb{P}((\mathcal{N}, Y_1, \ldots, Y_N) \in \cdot \mid S_{\mathcal{N}_n} > a_n).$$

The trick was to sample N from  $\mathbb{P}(N = k \mid N \ge k^*)$  where  $k^* = min\{k : Y_1 + \ldots + Y_k > a_n\}.$ 

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Numerical experiments

- The MCMC estimator <sup>^1</sup> of the probability p tested against importance sampling and standard Monte Carlo.
- Steps are Pareto(1) distributed.
- Number of steps is Geometric(0.2) distributed
- Number of batches: 25, simulations per batch: 10,000.

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# Numerical experiments

а	MCMC	IS	MC	
100	1.149e-2	1.087e-2	1.089-2	Avg. est.
	(4e-5)	(6e-5)	(35e-5)	(Std. dev.)
	[25]	[11]	[1.2]	[Avg. time (ms)]
$5 \cdot 10^{7}$	2.000003e-8	1.999325e-8		Avg. est.
	(6e-14)	(1114e-14)		(Std. dev.)

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### Setup

Consider the following setup for the risk reserve  $U_k$ , for positive claim size *B*:

$$U_k = R_k(U_{k-1} - B_k), \text{ for } k \ge 1,$$
  
 $U_0 = u.$ 

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### Setup

Consider the following setup for the risk reserve  $U_k$ , for positive claim size *B*:

$$U_k = R_k(U_{k-1} - B_k), \text{ for } k \ge 1,$$
  
 $U_0 = u.$ 

Iteration gives:  $U_n = R_n \cdots R_1 u - (R_n \cdots R_1 B_1 + \cdots + R_N B_n)$ .

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### Setup

Consider the following setup for the risk reserve  $U_k$ , for positive claim size *B*:

$$U_k = R_k(U_{k-1} - B_k), \text{ for } k \ge 1,$$
  
 $U_0 = u.$ 

Iteration gives:  $U_n = R_n \cdots R_1 u - (R_n \cdots R_1 B_1 + \cdots + R_N B_n)$ . Writing  $A_k = 1/R_k$  then

$$A_1 \cdots A_n U_n = u - W_n$$
, where  
 $W_n = B_1 + A_1 B_2 + \cdots + A_1 \cdots A_{n-1} B_n$ .

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#### Problem

#### Thus the event of ruin can be expressed as follows

$$\{\inf_{k} U_{k} < 0\} = \{\sup_{k} W_{k} > u\}.$$

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#### Problem

Thus the event of ruin can be expressed as follows

$$\{\inf_{k} U_{k} < 0\} = \{\sup_{k} W_{k} > u\}.$$

Goal: Construct an MCMC estimator for computing

$$p = \mathbb{P}(\sup_{k} W_{k} > u).$$

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Construct a Markov chain  $(\mathbf{A}_t, \mathbf{B}_t)_{t \ge 0}$  with the invariant distribution

$$\mathbb{P}\big((\mathbf{A},\mathbf{B})\in\cdot\mid\sup_{k}W_{k}>u\big).$$

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Construct a Markov chain  $(\mathbf{A}_t, \mathbf{B}_t)_{t \ge 0}$  with the invariant distribution

$$\mathbb{P}\big((\mathbf{A},\mathbf{B})\in\cdot\mid\sup_{k}W_{k}>u\big).$$

Carried out by updating one of  $(A_1, \ldots, A_n, B_1, \ldots, B_n)$  at a time, conditioned so that

$$\max_{1 \le k \le n} W_k = \max_{1 \le k \le n} B_1 + A_1 B_2 + \dots + A_1 \dots A_{k-1} B_k > u.$$

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Assume that

- The claim size B is Pareto( $\alpha$ ) distributed
- The stochastic return *R* fulfills  $\mathbb{E}[R^{-\alpha-\epsilon}] < \infty$  for some  $\epsilon > 0$

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#### Assume that

- The claim size B is Pareto( $\alpha$ ) distributed
- The stochastic return *R* fulfills  $\mathbb{E}[R^{-\alpha-\epsilon}] < \infty$  for some  $\epsilon > 0$

Then we have the asymptotic result

$$\frac{\mathbb{P}(\sup_{1 \le k \le n} W_k > u)}{\mathbb{P}(B > u) \sum_{k=0}^{n-1} \mathbb{E}[A^{\alpha}]^k} \to 1, \text{ as } n \to \infty.$$

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#### Efficiency continued

Now  $W_n = B_1 + A_1B_2 + \cdots + A_1 \cdots A_{n-1}B_n$ . Based on the existing asymptotic results we propose the following choice for *V* 

$$V(\cdot) = \mathbb{P}ig((\mathsf{A},\mathsf{B}) \in \cdot \mid (\mathsf{A},\mathsf{B}) \in R),$$

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where

$$R = \{B_1 > u\}$$

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### Efficiency continued

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where

$$R = \{B_1 > u\} \cup \{A_1 > a, B_2 > u/a\}$$

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#### Efficiency continued

Now  $W_n = B_1 + A_1B_2 + \cdots + A_1 \cdots A_{n-1}B_n$ . Based on the existing asymptotic results we propose the following choice for *V* 

$$V(\cdot) = \mathbb{P}ig((\mathsf{A},\mathsf{B}) \in \cdot \mid (\mathsf{A},\mathsf{B}) \in R),$$

#### where

$$\begin{array}{rcl} R & = & \{B_1 > u\} \cup \{A_1 > a, B_2 > u/a\} \cup \ldots \\ & \cup & \{A_1 > a, \ldots, A_{n-1} > a, B_n > u/a^{n-1}\}. \end{array}$$

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# 10,000 simulations for n = 10 and $u = 10^5$



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# Conclusion

Established a framework for new and simple method within stochastic simulation: Markov chain Monte Carlo methodology.

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# Conclusion

Established a framework for new and simple method within stochastic simulation: Markov chain Monte Carlo methodology. Applied the framework and proved efficiency on four concrete examples:

- Random walk with heavy-tails
- Random sum with heavy-tails
- Solution to stochastic recurrent equations with heavy-tailed innovations
- Insurance model with risky investments and Pareto distributed claim size
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| Insurance Model with Risky Investments |                     |   |                |  |  |
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## Conclusion

Possibilities for future work:

- Extension to random walk with light-tails
- Perfect simulation / coupling form the past
- Solution to stochastic recurrent equations where the ruin event is controlled by the stochastic returns rather than the claim size

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## Thank you for your attention!

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