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Positive exponent in families with flat critical point

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Abstract. It is known that in generic, full unimodal families with a critical point of finite order, there exists a set of positive measure in parameter space such that the corresponding maps have chaotic behaviour. In this paper we prove the corresponding statement for certain families of unimodal maps with flat critical point. One of the key-points is a large deviation argument for sums of 'almost' independent random variables with only finitely many moments.

1. *Statement of results*

Definition 1. For all $q > 0$ we define

$$f_{a,q}(x) = Q(1 - a\varphi_q(x)) \tag{1.1}$$

where

$$Q = \left(\frac{q}{1+q}\right)^{1/q} \quad \text{and} \quad \varphi_q(x) = \begin{cases} e^{(q+1)/q - 1/|x|^q}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases} \tag{1.2}$$

We will often drop the subscripts and just write φ and f_a (or f) when the meaning is obvious.

$f_{a,q}$ maps $I = I_q := [-Q, Q]$ into itself if $0 < a \leq 2$. Furthermore, for each q, $\{f_{a,q}\}_{0<a\leq 2}$ is a full unimodal family. Each f is a C^∞ flat top function with critical point $x = 0$, that is, for all $n > 0$, $d^n f/dx^n(0) = 0$. Flatness is increasing in q the sense that $\log|f_q'|$ is in $L^p(-\epsilon, \epsilon)$ exactly when $p < 1/q$. Note that Q is the critical value of $f_{a,q}$ for any a.

The families $\left\{f_{a,q}\right\}_{0<a\leq 2}$ are flat-topped versions of the standard quadratic family. Topologically they are the same; our main theorem says that they also share an important metric property, the analogue of the Benedicks–Carleson theorem, if the family is not too flat.

MAIN THEOREM. *Let $f_{a,q}$ and Q be defined as in (1.1) and (1.2). Then there exists a $q_0 > 0$ such that for all $0 < q < q_0$ the following holds. For all $\tilde{a} < 2$ there exists a constant $\lambda > 0$ and a set $A \subset (\tilde{a}, 2)$ of positive Lebesgue measure, with two as density point, such that if $a \in A$ then:*

(i) $|(d/dx) f^n_{a,q}(Q)| \geq e^{\lambda n}, \forall n \geq 1$;

(ii) *f_a admits an invariant probability measure, absolutely continuous with respect to Lebesgue measure;*

(iii) *$f_{a,q}$ has no periodic attractors.*

For a close to two, there is a slightly larger interval $[-z(a,q), z(a,q)] \supset I_q$ such that $f_{a,q}(\pm z(a,q)) = -z(a,q)$ and on which $f_{a,q}$ is a self-map with negative Schwarzian derivative. Thus, by Singer's theorem [**9**], (i) implies that f_a has no periodic attractors.

The main theorem has a series of ancestors in the non-flat category, the first of which is due to Yakobson [**15**], who proved that a positive measure set of maps in the quadratic family admits an absolutely continuous invariant probability measure (an acip). (See also [**8**].) Benedicks and Carleson in [**1**] proved sub-exponential growth of the derivative along the orbit of the critical value and existence of an acip for which the critical point is generic for a positive set of parameters in the quadratic family. (See also [**4**] and [**11**].) In [**2**] exponential growth of the derivative at the critical value was obtained for a positive measure set of quadratic maps. Another proof has been given by Tsujii [**14**]. This result was later generalized to generic families with non-flat critical point [**13, 10**].

Remark 1 (On the size of q). The main theorem holds with $q_0 = 1/8$, but this is certainly not optimal. In [**3**] Benedicks and Misiurewicz prove that if $f_{a,q}$ is a Misiurewicz-map, $f_{a,q}$ admits an acip if and only if $q < 1$. This fact suggests that our theorem fails for $q_0 > 1$. See also Remark 4 in §4. In fact we have arguments indicating that the theorem holds with $q_0 = 1$ (see §9). Restrictions on q_0 appear in the proof in several places. We will often consider high iterates f^k restricted to certain small intervals. In order to get a uniform bound on the distortion of f^k these intervals must not be too large. On the other hand, the images of these intervals must have grown to a certain size at so called free returns, if we want to obtain a parameter set A of positive measure. This trade-off is only possible if q is not too large:

- if the conclusions of Lemmas 4.1 and 4.2 are to hold simultaneously, we need $q < (\sqrt{5} - 1)/2$;
- the conclusions of Lemmas 4.2, 5.3 and 6.2 hold simultaneously if $q < 1/8$.

More efficient distortion estimates could allow for a somewhat larger q_0, but to get past $q_0 = 1/2$ does not seem possible at all with our methods.

We now list the properties of f_a and φ which are needed in the proof.

P1. $|f'_{a,q}(x)| \sim |\varphi'_q(x)| \sim |\varphi_q(x)|/|x|^{q+1}$.

P2. $f_{2,q}(\pm Q) = -Q$, that is, for $a = 2$ the critical orbit falls on the unstable fixed point $x = -Q$.

P3. $f'_{a,q}(\pm Q) = \mp a(1+q)$. Thus, there exists $a_0(q)$ such that $|f'_{a,q}(\pm Q)| > 2$ for all $a \in [a_0(q), 2]$.

P4. $f_{a,q}[0, x] = [Q(1 - a\varphi_q(x)), Q]$. Thus, $|f_{a,q}(0, x)| = aQ\varphi_q(x)$.

P5. $f_{a,q}$ is unimodal, has negative Schwarzian derivative on I_q, and is symmetric and concave.

Remark 2. In fact, you can pick any $\hat{a}$ such that the critical orbit of $f_{\hat{a}}$ falls on an unstable periodic orbit, and then find a positive measure set A in any neighbourhood of $\hat{a}$ as in the theorem. You can replace 'symmetric' with any condition that insures the non-existence of wandering intervals, e.g. that the involution τ defined by $f(x) = f(\tau(x))$ is Lipschitz. The concavity is only used to conclude that $\max_I |f'(x)| = f'(-Q)$.

It is essential that the nature of the critical point does not depend on the parameter a, since general (small) perturbations of a flat critical point could create infinitely many critical points.

2. *Strategy*

The basic idea follows that used in [**2**]. By the chain rule,

$$\frac{df^n}{dx}(Q) = \prod_{j=0}^{n-1} \frac{df}{dx}(f^j(Q)).$$

So we must select parameters which maximise the length of time the critical orbit spends far away from zero, where derivatives are small. The proof of the main theorem will consist of the following steps.

(i) By choosing a very close to two, the orbit of the critical value Q will have a long initial sequence, where the derivative grows exponentially fast, (Lemma 3.1).

(ii) As long as the orbit moves outside any fixed neighbourhood of the critical point, derivatives grow exponentially fast, (Lemma 3.2).

(iii) A return close to the critical point at time n, will always be followed by a (large) number of iterates (less than n) where the orbit shadows the initial part of the critical orbit closely. The condition is that, for some suitable β, $|f^{n+j}(Q) - f^j(Q)| < j^{-\beta}$ for all j considered. We can use inductive information about the size of the derivatives to make sure that the loss in the derivative that occurs at time n is compensated during this so called *bound period*. It is essential that there is even some net gain, although this will be polynomial rather than exponential in time. This idea works out only if the returns do not get too small too soon, so we will recursively delete parameters for which $|f_a^n(0)| < 1/n^\alpha$ for some suitable α, where $1 < \alpha < \beta < 1/q$.

To end up with exponential growth, we must also delete parameters with frequent small returns. For each n, we delete parameters who spend more than, say, $n/2$ iterates compensating returns to a small neighbourhood $(-\delta_1, \delta_1)$ of zero (§4).

(iv) Finally, we must show that what remains after making the exclusions in the previous step for all n is a set of parameters of positive measure (§6). We study the critical orbit for different parameters, a, using the mappings

$$\xi_n : a \mapsto f_a^n(0).$$

We consider the restrictions of ξ_n to certain intervals ω_n^k. Estimates on the size of the deleted sets at time n is performed on the images of the intervals ω_n^k under ξ_n. Thus, it is essential that the distortion of the mappings ξ_n is uniformly bounded. We show that x- and a-derivatives are comparable in size, as long as x-derivatives grow exponentially. This and inductive use of the expansion makes it possible to bound the distortion of ξ_n on intervals ω_n^k, if these are sufficiently small (§5).

On the other hand, $|\xi_n(\omega_n^k)|$ must be large compared to $1/n^\alpha$ so that the set deleted due to the $(1/n^\alpha)$-condition is small. This is carried out in §6.1.

In §6.2 we estimate the measure of the parameters with frequent small returns. This is the part of the proof requiring most work. We will consider two neighbourhoods $I_1^* \subset I_0^*$ of the critical point, where $I_i^* = (-\delta_i, \delta_i)$ and $\delta_1 \ll \delta_0$. The reason that we work with two nested neighbourhoods is as follows: we will show that for a typical a most returns to I_0^* are not close to zero in the sense that they return to $I_0^* \setminus I_1^*$. Even though a typical orbit spends most of its time in bound periods, most of this bound time comes from returns outside I_1^*. This enables us to conclude, using (ii), that we have exponential growth of derivatives for long stretches of the critical orbit for a large set of a-values. Special returns, called *escape returns*, when $|\xi_n(\omega_n^k)| > \delta_0$ are considered. Inductive use of the expansion shows that such returns do occur frequently. The length of bound periods following returns to $(-\delta_1, \delta_1)$ will be estimated with certain numbers E_k. By definition, at a return to $I_0^* \setminus I_1^*$, the corresponding $E_i = 0$. The total amount of bound time spent compensating returns to I_1^* for a up to time n is thus less than $T_s(a) := \sum_{k=1}^{s} E_k(a)$, for some suitable s. The value of $E_k(a)$ is essentially determined by the position of the corresponding return $\xi_n(a)$. At escape returns the images $\xi_n(\omega_n^k)$ are distributed over $(-\delta_0, \delta_0)$, and so the functions E_i may be considered as almost independent stochastic variables with small support and small expectation. Large deviation techniques for sums of random variables are used to show that the fraction of parameters deleted in this step at time n (those for which $T_s(a) > n/2$), decreases fast enough in n to leave us with a positive measure set in the limit. In contrast to the quadratic case, the functions E_i only have finitely many moments, and the techniques of [**2**], where exponential moments are considered, do not apply. Some recent ideas from the theory of large deviations are used to fix this up. Also some of the measure-estimates used as an input in the final large deviation argument are performed in a different manner than in [**2**].

(v) In §8 we prove the existence of a finite absolutely continuous invariant measure for each f_a, $a \in A$. We form averages of the push-forward of the Lebesgue measure restricted to certain intervals, and recycling some estimates from the previous step, we show that these averages accumulate on a measure with a finite, non-vanishing, absolutely continuous part. Since f_a and f_a^{-1} preserve zero Lebesgue measure, we may take this absolutely continuous part as our invariant measure, cf. Proposition 8.1.

3. *Preliminaries: expansion lemmas and notation*

For a close to two the critical orbit stays close to $-Q$ for a large number of iterates. More precisely we have the following.

LEMMA 3.1. *For any small δ the following holds. For all m sufficiently large, there exists $a_0 = a_0(m, q)$ such that for all $a \in (a_0, 2]$ and $k = 1, \dots, m-1$ the following holds:*
(i) $f_a^k(0) \in [-Q, 0)$;
(ii) $|(f_a^k)'(Q)| \geq 2^k$.
Furthermore, we can choose a_0 such that $f_{a_0}^m(0) = \delta$.

Proof. By continuity, this follows from properties P2 and P3. The details are left to the reader. □

Next we want to state that we have exponential growth of the derivative as long as we are moving outside any fixed I^*. In the quadratic case this can be done by an explicit change of coordinates. Here we need a more abstract formulation.

THEOREM 1. *Assume h is a unimodal, C^3 Misiurewicz map with all periodic points hyperbolic and repelling. Let c be the critical point of h. Also assume that h has negative Schwarzian derivative and that the involution, τ, defined by $h(x) = h(\tau(x))$, $\tau(x) \neq x$ for $x \neq c$, is Lipschitz continuous. Then the following holds. There exist constants $C > 0$ and $\theta > 1$ and a neighbourhood W of c such that for any neighbourhood $U \subset W$ of c there exists a neighbourhood $\mathcal{N}$ of h in the C^1 topology such that for all $g \in \mathcal{N}$ one has:*
(i) $x, \dots, g^k(x) \notin W \implies |Dg^k(x)| \geq C\theta^k$;
(ii) $x, \dots, g^{k-1}(x) \notin U$ *and* $g^k(x) \in W \implies |Dg^k(x)| \geq C\theta^k$;
(iii) $x, \dots, g^{k-1}(x) \notin U \implies |Dg^k(x)| \geq C\theta^k \inf_{j=0,1,\dots,k-1} |Dg(g^j(x))|$.

Proof. This is a modified version of Theorem 6.4 in Chapter III of [**6**], where we have substituted the condition 'C^2 with non-flat critical points' with 'C^3 with negative Schwarzian and Lipschitz continuous involution'. The proof in [**6**] is based on a series of results. The non-flatness condition is only used at one point to deduce the non-existence of wandering intervals (Theorems II, 6.2 and II, 6.3). But the proof of Theorem II, 6.3 (Guckenheimer's result on the non-existence of wandering intervals for unimodal maps with negative Schwarzian) uses the non-flatness condition only to conclude that the involution is Lipschitz! For reference, the path in Chapter III of [**6**] is as follows.

No wandering intervals and no periodic attractors $\implies$ Conclusion of Lemma 6.1. Negative Schwarzian $\implies$ Koebe principle. Koebe principle and Lemma 6.1 $\implies$ Theorem 6.1 $\implies$ Theorem 6.2 $\implies$ Theorem 6.3 $\implies$ Theorem 6.4.

(Theorem 6.1 in [**6**] contains a statement saying that iterates are quasi-polynomial on monotone branches. Of course this fails for our maps but the property needed in the proof of Theorem 6.2 remains true: there exists an $\bar{n}$, independent of n, such that if I_n is a monotone branch of h^n then

$$h^n | I_n = h^{\bar{n}} \circ \Psi_n \circ h \circ \Phi_n,$$

where Ψ_n and Φ_n are diffeomorphisms depending on n and I_n but with universally bounded distortion.) □

As a corollary we obtain the following lemma.

LEMMA 3.2. *There exists constants $K_0 > 0$ and $\lambda_0 > 0$ such that for any $I^* = (-\delta, \delta)$ sufficiently small, there exists $a_0 < 2$ such that for all $a \in (a_0, 2]$:*

(i) $x, \dots, f_a^{k-1}(x) \notin I^* \implies |Df_a^k(x)| \geq K_0 e^{\lambda_0 k} \inf_{j=0,1,\dots,k-1} |Df_a(f_a^j(x))|$;
(ii) $x, \dots, f_a^{k-1}(x) \notin I^*$, $f_a^k(x) \in I^* \implies |Df_a^k(x)| \geq K_0 e^{\lambda_0 k}$.

Remark 3. The constants K_0 and λ_0 do not depend on I^*. If we shrink I^*, we just choose a_0 closer to two.

Proof. By P2, the critical point of f_2 is strictly pre-periodic and so f_2 is a Misiurewicz map. Singer's theorem states that for maps with negative Schwarzian, all stable or neutral periodic orbits attract either a critical point or an end-point. Thus, P2 and P5 imply that all the periodic orbits of f_2 are hyperbolic and repelling. By P5, f_2 also fulfils the other conditions of Theorem 1 and the lemma follows. □

We now introduce some constants.

• First let ϵ_0 be a small positive number. ϵ_0 depends on q and controls the size of intervals $I_{\mu\nu}$ that will form a partition of a neighbourhood of zero. In §6 conditions on the size of ϵ_0 will be given.

• Let

$$\alpha = \frac{1 - 2q - \epsilon_0 q}{2q(q+1)} \quad \text{and} \quad \beta = \frac{1}{2q}.$$

α controls the speed with which the critical orbit is allowed to approach the critical point, and β controls the tightness of the bound period condition. To simplify notation later on we also introduce

$$\sigma := \frac{1}{q} - \beta - 1 - \epsilon_0 = \frac{1}{2q} - 1 - \epsilon_0$$

and

$$\sigma_1 := \frac{1}{2q} - 1 - 3\epsilon_0.$$

Note that we can choose ϵ_0 so that $\sigma_1 > N$ if and only if $q < 1/(2N+2)$.

• Let

$$\lambda = \min\left\{\frac{\lambda_0}{4}, \frac{\log 2}{2}\right\}.$$

With this λ, part (i) of the main theorem holds.

We will consider two neighbourhoods $I_1^* \subset I_0^*$ of the critical point, where $I_i^* = (-\delta_i, \delta_i)$ and

$$\delta_0 = \delta_1^{q+1/2+2q\epsilon_0}.$$

We will repeatedly shrink these neighbourhoods during the course of the proof. All estimates will continue to hold if we restrict ourselves to a-values close enough to two. We will always choose $\delta_0 = \Delta_0^{-1/q}$ for some $\Delta_0 \in \mathbb{N}$. Δ_1 will denote the largest integer such that $\delta_1 \leq \Delta_1^{-1/q}$. Depending on the situation we will say 'δ_0 (or δ_1) is small' or 'Δ_0 (or Δ_1) is large', and all will mean the same thing.

We will use C, possibly indexed by a natural number, to denote local constants that do not depend on the size of δ_0. Certain constants that will be inserted in estimates later on are indexed by letters indicating their origin.

To control the dynamics and the distortion of iterates following a return to I_0^*, we need a sufficiently fine partition of I_0^*. (On the other hand, the images of the intervals in the partition must grow to a certain size before the next return, so they must not be too small.) For $\mu \in \mathbb{Z}$, $|\mu| \geq \Delta_0$, let

$$I_\mu = ((\mu+1)^{-1/q}, \mu^{-1/q}] \quad \text{if } \mu > 0 \quad \text{and} \quad I_\mu = -I_{-\mu} \quad \text{for } \mu < 0.$$

Subdivide each I_μ into finitely many subintervals $I_{\mu\nu}$ of length

$$|I_{\mu\nu}| \sim \frac{|I_\mu|}{|\mu|^{1+\epsilon_0}} \sim \frac{1}{|\mu|^{2+1/q+\epsilon_0}}.$$

Thus,

$$I_0^* = \{0\} \cup \bigcup_{|\mu|\geq\Delta_0} I_\mu = \{0\} \cup \bigcup_{|\mu|\geq\Delta_0} \bigcup_{\nu} I_{\mu\nu}.$$

There is a bound, uniform in μ, on the one-step distortion of f_a restricted to an I_μ. This does not hold for partitions of any coarser type. Also, let $I_{\mu\nu}^+$ denote the union of $I_{\mu\nu}$ and its two neighbouring intervals and let

$$\hat{I}_\mu = [-\mu^{-1/q}, \mu^{-1/q}].$$

For future use we note that

$$|f_a(\hat{I}_\mu)| \sim |f_a(I_\mu)| \sim |\mu|^{1+\epsilon_0} |f_a(I_{\mu\nu})| \sim e^{-|\mu|}.$$

Finally, for any x and y, let $(x; y)$ denote the interval with x and y as endpoints.

4. *Bound periods for compensation, free periods for growth*

In this section we give two conditions on the parameters such if a close to two fulfils these conditions, then $|Df_a^n(Q)|$ grows exponentially in n.

Definition 2 (The approach rate). We say that

$$a \in BA_n \quad \text{if} \quad |f_{a,q}^j(0)| > \frac{1}{j^\alpha} \quad \text{for } 0 < j \leq n. \tag{4.1}$$

Remark 4 (How to choose α). Already at this point, we see that we have to choose an approach-rate of the form above, with $\alpha \in (1, 1/q)$. Suppose that $|Df^n(Q)| \sim e^n$ and that we have a small return $x_n := f^n(Q) \sim \pm n^{-\alpha}$ after n iterates. Then

$$|Df^{n+1}(Q)| = |Df(x_n)Df^n(Q)| \sim \frac{e^{-1/|x_n|^q}}{|x_n|^{q+1}} \cdot e^n \approx e^{n-1/|x_n|^q} \approx \exp(n - n^{\alpha q}).$$

Thus, at least for n large, we need $\alpha < 1/q$. Indeed, by the reasoning in [**12**], even one single return closer than $\sim 1/n^{1/q}$ appearing at a so-called S-time implies that the map under consideration has a periodic attractor. On the other hand, the set A will be something similar to a fat Cantor set, with a proportion $1/n^\alpha$ deleted at time n, so in order to end up with a positive measure set we need $\alpha > 1$.

A return $f^n(Q)$ close to the critical point will be followed by a number of iterates where $f^{n+j}(Q) \approx f^j(Q)$.

Definition 3 (Bound periods). Define $p = p(\mu) = p(\mu, a)$ to be the maximal positive integer such that

$$|f_a^j(|\mu|^{-1/q}) - f_a^j(0)| \leq \frac{1}{j^\beta} \quad \text{for all } 0 < j < p.$$

This is the *bound period* for $x \in I_\mu$.

Remark 5 (How to choose β). We want the orbit of $\hat{I}_\mu$ to stay away from the critical point during the bound period. All we know is that $|f_a^j(0)| > j^{-\alpha}$, so $\beta > \alpha$ is necessary.

When an orbit is not in a bound period following a return to some I^*, we say it is in a *free period*. A free period terminates when the orbit hits I^*, and a bound period follows. These always terminate in a finite number of iterates (see Lemma 4.2) so another free period follows. Thus, any orbit is the disjoint union of free and bound periods respectively (relative to I^*). During free periods the orbits stay outside I^*, and Lemma 3.2 tells us that derivatives then grow exponentially. During the bound periods (including the return) there will be no loss, in fact there is a small gain, according to Lemma 4.2.

Definition 4 (Free period assumption). Let

$$H(a, n) = \#\{j \leq n : f_a^j(Q) \text{ is in a bound period following a return to } I_1^*\}.$$

We say that $a \in FA_n = FA_n(\delta_1)$ if

$$H(a, n) \leq \frac{n}{2}.$$

Our good set will be a set $A = \bigcap_{n=1}^{\infty} A_n$ where $A_n \subset BA_n \cap FA_n$.

PROPOSITION 4.1. *Let $\lambda = \min\{\lambda_0/4, \log 2/2\}$. For each δ_1 sufficiently small there exists an $\hat{a}$ such that for all $n \geq 1$ and for all $a \in BA_n \cap FA_n(\delta_1) \cap (\hat{a}, 2)$, we have*

$$\left|Df_a^n(Q)\right| \geq e^{\lambda n}.$$

Before we prove this, we prove two lemmas relating to the dynamics during bound periods.

To use inductive information during bound periods, we need to know that distortion is uniformly bounded in the following sense.

LEMMA 4.1. *For δ_0 sufficiently small there exists a constant C_{bd}, independent of δ_0, such that if $a \in BA_n$ and $|\mu| \geq \Delta_0$, then*

$$\frac{|Df_a^j(x)|}{|Df_a^j(y)|} < C_{\text{bd}}, \quad \forall x, y \in f_a(\hat{I}_\mu), \forall j < \min\{p, n\}.$$

Proof. Write $x_i := f_a^i(x)$ and $y_i := f_a^i(y)$.

$$\frac{|Df_a^j(x)|}{|Df_a^j(y)|} = \prod_{i=0}^{j-1} \frac{|Df_a(x_i)|}{|Df_a(y_i)|} \le C_1(\delta_0) \prod_{i=1}^{j-1} \frac{|y_i|^{1+q}}{|x_i|^{1+q}} \exp\left\{\frac{1}{|y_i|^q} - \frac{1}{|x_i|^q}\right\}$$
$$= C_1 \exp\left\{\sum_{i=1}^{j-1}\left(\frac{1}{|y_i|^q} - \frac{1}{|x_i|^q} + (1+q)(\log|y_i| - \log|x_i|)\right)\right\}.$$

Here $C_1(\delta_0)$ takes care of the first factor. Obviously, $C_1 \to 1$ as $\delta_0 \to 0$. Since $j < \min\{p, n\}$ and $a \in BA_n$, it follows that $0 \ne (x_i; y_i)$ for $i \le j$ and so $||y_i| - |x_i|| = |y_i - x_i|$. By the mean value theorem, there exists $z_i \in (|x_i|; |y_i|)$ such that

$$\left|\frac{1}{|y_i|^q} - \frac{1}{|x_i|^q} + (1+q)(\log|y_i| - \log|x_i|)\right| = \left|\frac{-q}{z_i^{1+q}} + \frac{1+q}{z_i}\right| |y_i - x_i|$$

which is

$$\le \frac{1+2q}{z_i^{1+q}} |y_i - x_i|.$$

It follows that

$$\left|\sum_{i=1}^{j-1}\left(\frac{1}{|y_i|^q} - \frac{1}{|x_i|^q} + (1+q)(\log|y_i| - \log|x_i|)\right)\right| \le \sum_{i=1}^{j-1} \frac{1+2q}{z_i^{1+q}} |y_i - x_i|.$$

Now since

$$z_i \in (|x_i|; |y_i|) \subset \pm(f_a^{i+1}(0); f_a^{i+1}(1/\mu^{1/q})),$$

$i \le j < \min\{p(\mu), n\}$, $a \in BA_n$ and the definition of $p(\mu)$ implies that

$$|y_i - x_i| < \frac{1}{(i+1)^\beta}$$

and

$$z_i > \frac{1}{(i+1)^\alpha} - \frac{1}{(i+1)^\beta}.$$

This gives

$$\sum_{i=1}^{j-1} \frac{1+q}{z_i^{1+q}} |y_i - x_i| \le (1+2q) \sum_{i=2}^{\infty} \frac{1/i^\beta}{(1/i^\alpha - 1/i^\beta)^{1+q}}$$
$$= (1+2q) \sum_{i=2}^{\infty} i^{\alpha(1+q)-\beta} (1 - i^{\alpha-\beta})^{-(1+q)}$$
$$\le C(\alpha, \beta, q) \sum_{i=2}^{\infty} i^{\alpha(1+q)-\beta}.$$

This last sum is convergent since $\alpha\,(1+q) - \beta = -(1+\epsilon_0)$. Together this gives

$$\frac{|Df_a^j(x)|}{|Df_a^j(y)|} \le C_1 \exp\left\{C(\alpha, \beta, q) \sum_{i=2}^{\infty} i^{\alpha(1+q)-\beta}\right\} \le C_{\mathrm{bd}}(\alpha, \beta, q). \qquad \square$$

Remark 6 (On the choice of α and β and the size of C_{bd}). To make this distortion estimate work, it was necessary to choose α and β so that $\alpha(1+q) - \beta < -1$. If q is close to $1/8$ we need to choose ϵ_0 very small in a later step in the proof, and so this crude estimate leads to a huge distortion constant C_{bd} with our specific choice of α and β. This does not affect the rest of the proof of the main theorem.

The following lemma gives us control over the size of $p(\mu)$ and of derivatives and positions during bound periods.

LEMMA 4.2. *Let $a \in BA_n$ be sufficiently close to two and suppose that $|Df_a^j(Q)| \geq e^{\lambda j}$ for all $0 < j < n$, $\Delta_0 \leq |\mu| < n^{\alpha q}$, and that there is a constant C_{bd}, independent of μ, such that*

$$\frac{|Df_a^j(x)|}{|Df_a^j(y)|} \leq C_{\mathrm{bd}}, \quad \forall x, y \in f_a(\hat{I}_\mu), \forall j < \min\{p, n\}.$$

Then for Δ_0 and n large enough, the following holds.

(i) *There exists a constant C_{p}, independent of μ, such that*

$$\frac{1}{C_{\mathrm{p}}}|\mu| \leq p(\mu) \leq C_{\mathrm{p}}|\mu| \quad (< C_{\mathrm{p}} n^{\alpha q} < n).$$

(ii) *There exists a constant C_{be}, independent of μ, such that for all $x \in I_\mu$*

$$|Df_a^{p(\mu)}(x)| \geq C_{\mathrm{be}}|\mu|^{1+1/q-\beta}.$$

In fact $|Df_a^{p(\mu)}(x)| \sim |\mu|^{1+1/q-\beta} \sim p^{1+1/q-\beta}$.

(iii) *For all $j < p$ and all $x \in f(\hat{I}_\mu)$ we have*

$$|Df_a^j(x)| \geq 1.$$

(iv)

$$f_a^j(\hat{I}_\mu) \cap (-2n^{-\alpha}, 2n^{-\alpha}) = \emptyset, \quad \forall j \leq p.$$

Note that if $x = f_a^n(0) \in I_\mu$ and $a \in BA_n$, then $|\mu| < n^{\alpha q}$.

Proof. We first prove (i). As long as $j < \min\{p, n\}$, $|Df_a^j(x)| \geq C_{\mathrm{bd}} e^{\lambda j}$ for all $x \in f_a(\hat{I}_\mu)$. Let $l = \min\{p, n\}$. Then

$$1 > |f_a^{l-2}(f_a(\hat{I}_\mu))| > C_{\mathrm{bd}} e^{\lambda(l-2)} |f_a(\hat{I}_\mu)| > C e^{\lambda(l-2)} e^{-|\mu|}.$$

Now $|\mu| < n^{\alpha q} \ll n$ by assumption, so $l < n$ if n is large and thus $l = p < C'_{\mathrm{p}}|\mu|$.

By the definition of p, P3 and P5,

$$p^{-\beta} \sim |f_a^{p-1}(f_a(\hat{I}_\mu))| < C(a(q+1))^{p-1} e^{-|\mu|}$$

which implies $p > C''_{\mathrm{p}}|\mu|$.

We now prove (ii). Since

$$\frac{1}{p^\beta} \leq |f_a^{p-1}(f_a(\hat{I}_\mu))| \leq \frac{a(q+1)}{p^\beta},$$

the mean value theorem and bounded distortion imply

$$\frac{1}{C_{\mathrm{bd}}}\frac{1}{p^{\beta}} \le |Df_a^{p-1}(y)||f_a(\hat{I}_\mu)| \le 4C_{\mathrm{bd}}\frac{1}{p^{\beta}}$$

for all $y \in f_a(\hat{I}_\mu)$. Using the left inequality, P1, P4 and $p(\mu) \sim |\mu|$, for $x \in I_\mu$ we obtain

$$\begin{aligned}|Df_a^{p(\mu)}(x)| &= |Df_a(x)||Df_a^{p-1}(f_a(x))| \\ &\ge C\frac{\varphi_q(\mu^{-1/q})}{|\mu^{-1/q}|^{q+1}}\frac{1}{p^{\beta}\varphi_q(\mu^{-1/q})} \ge C_{\mathrm{be}}\frac{|\mu|^{(q+1)/q}}{|\mu|^{\beta}}.\end{aligned}$$

$|Df_a^{p(\mu)}(x)| \lesssim |\mu|^{1+1/q-\beta}$ is obtained in the same way.

To prove (iii), let j_0 be so large that $C_{\mathrm{bd}}e^{j_0\lambda} > 1$ and note that if a is sufficiently close to two and Δ_0 large, $f_a^j(x)$ stays in a neighbourhood of $-Q$ for $j = 2, \dots, j_0$.

Finally, we prove (iv). For any $x \in \hat{I}_\mu$ and any $j < p$

$$|f_a^j(x)| \ge |f_a^j(0)| - |f_a^j(\hat{I}_\mu)|.$$

If $a \in BA_n$ and $|f_a^j(\hat{I}_\mu)| \lesssim e^{j\log 4}e^{-|\mu|}$, then $|f_a^j(0)| > j^{-\alpha}$ for all $j \le n$, so for $j \le \sqrt{p}$,

$$|f_a^j(x)| \ge p^{-\alpha/2} - Ce^{\log 4\sqrt{p}-|\mu|},$$

and first using $p < C_{\mathrm{p}}|\mu|$ and $|\mu|$ large, and then $|\mu| < n^{\alpha q}$ and $\alpha q < 1$ this gives

$$|f_a^j(x)| \ge \frac{2}{n^{\alpha}}.$$

If $\sqrt{p} < j < p$ we use the fact that $|f_a^j(x) - f_a^j(0)| < 1/j^{\beta}$ for all $j < p$ to obtain

$$\begin{aligned}|f_a^j(x)| &\ge \frac{1}{j^{\alpha}} - \frac{1}{j^{\beta}} = \frac{1}{j^{\alpha}}\left(1 - \frac{1}{j^{\beta-\alpha}}\right) \\ &> \frac{1}{p^{\alpha}}\left(1 - \frac{1}{p^{(\beta-\alpha)/2}}\right) > \frac{1}{2p^{\alpha}} > \frac{2}{n^{\alpha}},\end{aligned}$$

where we once again use the facts that $p \sim |\mu|$ and $|\mu| \le n^{\alpha q}$. □

Remark 7 (More on the choice of α and β and the size of q). From Lemmas 4.1 and 4.2 we see that since

$$1 + \alpha(1+q) < \beta < 1 + 1/q, \tag{4.2}$$

a small return to I_μ is compensated for during the bound period. There is even some net-expansion, without which the measure-estimates in §6 break down completely. Thus, our choice of α and β was partially dictated by (4.2). In the remark on the choice of α following Definition 2, we saw that

$$1 < \alpha < 1/q \tag{4.3}$$

was an *a priori* condition on α. (4.2) and (4.3) are compatible, for $q > 0$, if and only if $2 + q < 1 + 1/q$, that is if and only if

$$0 < q < \frac{\sqrt{5}-1}{2}.$$

As we will see, sharper restrictions on the size of q will appear in other parts of the proof.

Proof of Proposition 4.1. Let $Q_i = f_a^i(Q)$. Then

$$Df_a^n(Q) = \prod_{i=0}^{n-1} Df_a(Q_i).$$

Let $\delta_1 > 0$. For any large positive integer M_0 we can find an $\hat{a}$ such that

$$|Df_a^n(Q)| \geq 2^n > e^{\lambda n}$$

for all $n \leq M_0$ and all $a \in (\hat{a}, 2]$. This follows from Lemma 3.1.

Let m_1 be the least positive integer such that $Q_{m_1} \in I_1^*$. For $M_0 < n < m_1$ we apply part (i) of Lemma 3.2 and the estimate above to obtain

$$\begin{aligned} |Df_a^n(Q)| &= |Df_a^{n-M_0}(Q_{M_0})||Df_a^{M_0}(Q)| \\ &\geq 2^{M_0} K_0 e^{\lambda_0(n-M_0)} \inf_{j=M_0,\ldots,n-1} |Df_a(Q_j)| \\ &\geq K_0 2^{M_0} e^{\lambda_0(n-M_0)} e^{-\Delta_1} \end{aligned}$$

Since M_0 is large when $\hat{a}$ is close to two we see that

$$|Df_a^n(Q)| \geq e^{\lambda n}$$

for $M_0 < n < m_1$ and all $a \in (\hat{a}, 2]$.

Recall the definition of m_1 above. If m_k is defined such that $Q_{m_k} \in I_1^*$, let $p_k = p(\mu_k)$ where $Q_{m_k} \in I_{\mu_k}$, and define m_{k+1} to be the least integer which is greater than or equal to $m_k + p_k$ such that $Q_{m_{k+1}} \in I_1^*$. Let $s \geq 1$ be maximal such that $m_s \leq n$. Also let $r_1 = m_1$ and $r_k = m_k - m_{k-1} - p_{k-1}$ for $k \leq s$. We distinguish two cases: (I) $n \geq m_s + p_s$ and (II) $m_s \leq n < m_s + p_s$.

(I) In this case let $r_{s+1} = n - m_s - p_s$. We have

$$\begin{aligned} Df_a^n(Q) = Df_a^{r_1}(Q) \cdot \left(\prod_{k=1}^{s-1} Df_a^{p_k}(Q_{m_k}) \cdot Df_a^{r_{k+1}}(Q_{m_k+p_k}) \right) \\ \cdot Df_a^{p_s}(Q_{m_s}) \cdot Df_a^{r_{s+1}}(Q_{m_s+p_s}). \end{aligned} \tag{4.4}$$

From (ii) of Lemma 3.2, we have that $\left|Df_a^{r_1}(Q)\right| > K_0 e^{\lambda_0 r_1}$. For a bound period followed by a full free period, the factors inside the bracket, (ii) of Lemma 4.2 and (ii) of Lemma 3.2 give us

$$|Df_a^{p_k}(Q_{m_k})||Df_a^{r_{k+1}}(Q_{m_k+p_k})| \geq C_{\text{be}}|\mu_k|^{1+1/q-\beta} \cdot K_0 e^{\lambda_0 r_{k+1}} \geq e^{\lambda_0 r_{k+1}},$$

since $1 + 1/q - \beta > 0$ and $|\mu_k| \geq \Delta_1$ is large. For the last two factors we have, using (ii) of Lemma 4.2, (i) of Lemma 3.2 and $|Q_n| \geq \delta_1$,

$$|Df_a^{p_s}(Q_{m_s})| \cdot |Df_a^{r_{s+1}}(Q_{m_s+p_s})| \geq C_{\text{be}}|\mu_s|^{1+1/q-\beta} \cdot K_0 e^{\lambda_0 r_{s+1}} e^{-\Delta_1} \geq e^{\lambda_0 r_{s+1} - \Delta_1},$$

once again because $|\mu_s| \geq \Delta_1$ is large.

Note that $\sum_{k=1}^{s+1} r_k$ is the total amount of free time (with respect to I_1^*), and so $\sum_{k=1}^{s+1} r_k > n/2$ since $a \in FA_n$. All together this implies

$$|Df_a^n(Q)| \geq K_0 e^{\lambda_0 \sum_{k=1}^{s+1} r_k - \Delta_1} \geq K_0 e^{n\lambda_0/2 - \Delta_1} \geq e^{\lambda n},$$

since $n > M_0$ is large.

(II) This is the case when the nth iterate is in a bound period. $Df_a^n(Q)$ is given by an expression similar to (4.4), where the last two factors are substituted by

$$Df_a(Q_{m_s}) \cdot Df_a^k(Q_{m_s+1})$$

where $k = n - m_s < p_s$. According to (iii) of Lemma 4.2, $|Df_a^k(Q_{m_s+1})| \geq 1$, and since $a \in BA_n \subset BA_{m_s}$, we have that

$$|Df_a(Q_{m_s})| \geq |Df_a(m_s^{-\alpha})| \geq |Df_a(n^{-\alpha})| \geq e^{-n^{\alpha q}}.$$

So in this case, using $a \in FA_n$, we have the estimate

$$|Df_a^n(Q)| \geq K_0 e^{\lambda_0 r_1} e^{\lambda_0 \sum_{k=1}^{s-1} r_{k+1}} e^{-n^{\alpha q}} \geq K_0 e^{\lambda_0 n/2 - n^{\alpha q}} \geq e^{\lambda n},$$

for our choice of λ if n is large, since $\alpha q < 1$. □

5. *Pseudo orbits of parameters*

Our task is to delete all parameters from a small interval $[a_0, 2]$ that does not belong to $\bigcap_{n=1}^{\infty} BA_n \cap FA_n$, and show that what remains has positive Lebesgue measure.

Definition 5. For $j \geq 1$ we define

$$\xi_j : a \mapsto f_a^j(0).$$

We will recursively construct sets $A_n \subset BA_n \cap FA_n$. The estimates of how much measure is deleted when constructing A_n from A_{n-1} will be based on the relative size of the images of subsets of A_{n-1} under ξ_n. Deleting parameters that are not in BA_n simply means deleting $\xi_n^{-1}(\xi_n(A_{n-1}) \cap (-n^{-\alpha}, n^{-\alpha}))$. Deleting parameters not in FA_n is more complicated, but measure estimates are still performed on the images. So we need an absolute bound on the distortion of ξ_j. This can only be obtained on small intervals ω, with the property that $\xi_j(\omega)$ is contained in one or two intervals $I_{\mu\nu}$ if $\xi_j(\omega) \cap I_0^* \neq \emptyset$. This calls for a partition $\mathcal{P}_n$ on A_n. If $\omega \in \mathcal{P}_{n-1}$, then $\mathcal{P}_n$ on ω will be the pullback by ξ_n of the partition $I = \left(I \setminus I_0^*\right) \cup \bigcup_{\mu,\nu} I_{\mu\nu}$. (This will carried out in detail in §6.1.)

We now define bound periods for parameter intervals ω.

Definition 6. Let $\omega \subset [a_0, 2]$. Then $p = p(\mu, \omega)$ is defined to be the maximal p such that

$$\left| \bigcup_{a \in \omega} f_a^j(\hat{I}_\mu) \right| < j^{-\beta}, \quad \forall j < p.$$

We have the following lemma.

LEMMA 5.1. *Let $\omega \subset BA_n$. Suppose $|(f_a^j)'(Q)| \geq e^{\lambda j}$ for all $j \leq N$ and all $a \in \omega$, and suppose that $\xi_N(\omega) \subset I_{\mu\nu}^+$ where $|\mu| \geq \Delta_0$ and Δ_0 is sufficiently large. Then there exists constants $C_{\mathrm{p}} > 0$ and $C_{\mathrm{pe}} > 0$, independent of Δ_0, such that:*

(i) $C_{\mathrm{p}}^{-1}|\mu| \leq p(\mu, \omega) \leq C_{\mathrm{p}}|\mu|$;

(ii) $|(f_a^p)'(x)| \geq C_{\mathrm{pe}}|\mu|^{1+1/q-\beta}$, $\forall x \in I_\mu$;

(iii) $|\xi_{N+p(\mu,\omega)}(\omega)| \geq C_{\mathrm{pe}}|\mu|^{1+1/q-\beta}|\xi_N(\omega)|$;

(iv) $\xi_{N+j}(\omega) \cap \left(-N^{-\alpha}, N^{-\alpha}\right) = \emptyset$, $\forall j \leq p(\mu, \omega)$.

So images $\xi_j(\omega)$ of small parameter intervals expand during their bound periods. As will be seen, when orbiting outside I_0^* they expand exponentially fast. So they will intersect I_0^* again, and the part very close to zero is a small fraction of the image. This will be stated precisely and exploited over and over again in §6.

Fix a positive integer N. For a small parameter interval ω we can split its 'orbit' under ξ_j into a disjoint union of bound and free periods up to the Nth 'iterate': there is minimal t_1 such that $\xi_{t_1}(\omega) \cap I_0^* \neq \emptyset$. This is the first free return and the end of the first free period. If ω is sufficiently small $\xi_{t_1}(\omega) \subset$ some $I_{\mu\nu}$. The first bound period of finite length $p_1 = p(\mu, \omega)$ follows. The second free period starts at time $t_1 + p_1$ and terminates at time t_2, the least integer which is greater than or equal to $t_1 + p_1$ such that $\xi_{t_2}(\omega) \cap I_0^* \neq \emptyset$, and so on.

The next lemma shows that images of an ω grows exponentially fast during free periods.

LEMMA 5.2. *Assume a_0 is sufficiently close to two, and suppose that $\omega \subset [a_0, 2]$ is such that $\xi_{\hat{n}}(\omega) \subset I_{\mu\nu}$ and $\xi_n(\omega)$ are two consecutive free returns with return times $\hat{n}$ and n, $\hat{n} < n$. Also assume that $|Df_a^j(Q)| \geq e^{\lambda j}$ for all $j < n$ and all $a \in \omega$. Then there is a constant C_{fp}, independent of δ_0, such that the following holds:*

(i) $|\xi_{n-k}(\omega)| \leq C_{\text{fp}} e^{-\lambda_0 k} |\xi_n(\omega)|, \forall\, 1 \leq k \leq n - \hat{n} - p(\mu, \omega)$;

(ii) $|\xi_n(\omega)| \geq 2|\xi_{\hat{n}}(\omega)|$.

Furthermore, there is a positive integer $N_0(\delta_0)$ such that for any ω close to two:

(iii) $\xi_{k+j}(\omega) \cap I_0^* = \emptyset,\ j = 0, 1, \ldots, N_0 \implies |\xi_{k+N_0}(\omega)| \geq e^{(\lambda_0/2)N_0} |\xi_k(\omega)|$.

As mentioned, measure estimates will be pulled back to parameter space. This is possible because of the next lemma.

LEMMA 5.3. *There exists a constant C_{ld}, independent of δ_0, such that for all $\epsilon_0 > 0$ the following holds. If $\omega = (b, c) \subset BA_N$ is a parameter interval such that $|Df_a^j(Q)| \geq e^{\lambda j}$ for all $a \in \omega$ and all $j \leq N$, $\xi_j(\omega) \subset$ some $I_{\mu\nu}^+$ at each free return time $j < N$ and such that $\xi_N(\omega)$ is a free return and $|\xi_N(\omega)| < 10\delta_0$, then*

$$\frac{1}{C_{\text{ld}}} \leq \left| \frac{\xi_N'(b)}{\xi_N'(c)} \right| \leq C_{\text{ld}}.$$

The proofs of Lemmas 5.1, 5.2 and 5.3 use the following two lemmas that relate the dynamics of $\xi_j|\omega$ to the dynamics of f_a, $a \in \omega$.

The first of these lemmas states that as long as the x-derivatives grow sufficiently fast, a- and x-derivatives will be comparable in size. This is a general fact; see Lemma 2.1 in [**2**]. For completeness we give a proof adapted to the maps under consideration. Let $F^n(x; a) = f_a^n(x)$. Then $\partial_x F^j(x; a) = df_a^j/dx$ and we want to compare $\partial_x F^j(Q; a)$ with $\partial_a F^j(Q; a) = \xi_{j+1}'(a)$.

LEMMA 5.4. *There is a constant C_{ax} such that for all a sufficiently close to two the following holds: if $|\partial_x F^j(Q; a)| \geq e^{\lambda j}$ for all $j \leq k$ then*

$$\frac{1}{C_{ax}} \leq \left| \frac{\partial_a F^k(Q; a)}{\partial_x F^k(Q; a)} \right| \leq C_{ax}.$$

Proof. Let

$$\mathcal{Q}_j = \mathcal{Q}_j(a) = \frac{\partial_a F^j(Q;a)}{\partial_x F^j(Q;a)}.$$

$F^j(x;a) = F(F^{j-1}(x;a);a) = Q(1 - a\varphi(F^{j-1}(x;a)))$, so by the chain-rule we have

$$\begin{aligned}\partial_x F^j(x;a) &= -aQ\varphi'(F^{j-1}(x;a)) \cdot \partial_x F^{j-1}(x;a)\\ \partial_a F^j(x;a) &= -aQ\varphi'(F^{j-1}(x;a)) \cdot \partial_a F^{j-1}(x;a) - Q\varphi(F^{j-1}(x;a)).\end{aligned}$$

This gives a recursion for $\mathcal{Q}_j$:

$$\mathcal{Q}_j = \mathcal{Q}_{j-1} - \frac{Q\varphi(F^{j-1}(Q;a))}{\partial_x F^j(Q;a)},$$

and since $0 \le \varphi(x) \le 1$ for all $x \in I$ we obtain

$$|\mathcal{Q}_{j-1}| - \frac{Q}{|\partial_x F^j(Q;a)|} \le |\mathcal{Q}_j| \le |\mathcal{Q}_{j-1}| + \frac{Q}{|\partial_x F^j(Q;a)|}.$$

Using $|\partial_x F^j(Q;a)| \ge e^{\lambda j}$ for all $j \le k$, a repeated use of the right inequality gives

$$|\mathcal{Q}_k| \le |\mathcal{Q}_1| + \sum_{i=2}^{k} \frac{Q}{e^{\lambda i}}$$

which is uniformly bounded in k.

Repeated use of the left inequality shows that for any $j < k$,

$$|\mathcal{Q}_k| \ge |\mathcal{Q}_j| - \sum_{i=j+1}^{k} \frac{Q}{|\partial_x F^i(Q;a)|} > |\mathcal{Q}_j| - \sum_{i=j+1}^{\infty} \frac{Q}{e^{\lambda i}},$$

so to obtain a positive lower bound on $|\mathcal{Q}_k(a)|$ it suffices to show that for some j_0, $|\mathcal{Q}_{j_0}(a)| - Q\sum_{i=j_0+1}^{\infty} e^{-\lambda i} > 0$. If this holds for $a = 2$, then it also holds for a sufficiently close to two with the same j_0. By explicit calculation, $\mathcal{Q}_1(2) = Q/2(1+q)$. Using this, the recursion formula for $\mathcal{Q}_j$ and the fact that $f_2(\pm Q) = -Q$ we get

$$\mathcal{Q}_j(2) = Q\sum_{i=2}^{j} \left(\frac{1}{2(1+q)}\right)^i.$$

So $\mathcal{Q}_j(2)$ is positive and increasing in j. The expression $Q\sum_{i=j+1}^{\infty} e^{-\lambda i}$ tends to zero as $j \to \infty$, and so the existence of the required j_0 is clear. □

Let HD-dist(J, K) denote the Hausdorff distance between the sets J and K.

LEMMA 5.5. *Suppose that $\omega \in BA_N \cap [a_0, 2]$ where a_0 is sufficiently close to two, $|Df_a^k(Q)| \ge e^{\lambda k}$ for all $a \in \omega$ and all $k \le N$, and assume that $\xi_N(\omega) \subset \hat{I}_\mu$. Then, for each $a, b \in \omega$ we have:*

(i) *HD-dist$(f_a^j(\hat{I}_\mu), f_b^j(\hat{I}_\mu)) < \frac{1}{1000}|f_a^j(\hat{I}_\mu)|$*

(ii) *HD-dist$(\xi_{N+j+1}(\omega), f_a^j(\xi_{N+1}(\omega))) < \frac{1}{1000}|f_a^j(\xi_{N+1}(\omega))|$ for all $j \le p(\mu,\omega)$.*

Proof of Lemma 5.5. Pick an $x \in \hat{I}_\mu$ and $a, b \in \omega$. We want to estimate

$$|f_a^j(x) - f_b^j(x)|$$

for all $j \leq p$. Let $\kappa = \sup_{a,x} |\partial_a F(x; a)|$. For $j = 1$ we have

$$|f_a(x) - f_b(x)| \leq \kappa |a - b|.$$

Let $\Lambda_1 = 1$, and for $0 < j \leq p$, let

$$\Lambda_j := d_{j-1}\Lambda_{j-1} + 1 = d_{j-1}(d_{j-2}(\dots(d_2(d_1 + 1) + 1)\dots) + 1) + 1,$$

where $d_i := |f_a'(y_i)|$ for some $y_i \in (f_a^i(x), f_b^i(x))$. For $1 < j \leq p$ we now verify that

$$|f_a^j(x) - f_b^j(x)| \leq \Lambda_j \kappa |a - b|. \tag{5.1}$$

This follows by induction on j:

$$\begin{aligned} |f_a^j(x) - f_b^j(x)| &= |F(f_a^{j-1}(x); a) - F(f_b^{j-1}(x); b)| \\ &\leq |\partial_x F(y_{j-1}, a)||f_a^{j-1}(x) - f_b^{j-1}(x)| + \kappa |a - b| \\ &\leq d_{j-1}\Lambda_{j-1}\kappa |a - b| + \kappa |a - b| = \Lambda_j \kappa |a - b|. \end{aligned} \tag{5.2}$$

From the definition of Λ_k it follows that

$$\Lambda_k = \left(\prod_{i=1}^{k-1} d_i\right)\left(1 + \frac{1}{d_1} + \frac{1}{d_1 d_2} + \cdots + \frac{1}{d_1 d_2 \dots d_{k-1}}\right). \tag{5.3}$$

We claim that

$$\prod_{i=1}^{j-1} d_i \sim |Df_a^{j-1}(Q)|. \tag{5.4}$$

This can be proved with an argument similar to the proof of Lemma 4.1 since

$$\frac{\prod_{i=1}^{j-1} d_i}{|Df_a^{j-1}(Q)|} = \frac{\prod_{i=1}^{j-1} |Df_a(y_i)|}{\prod_{i=1}^{j-1} |Df_a(f_a^i(0))|},$$

$|y_i - f_a^i(0)| < \beta^{-i}$ for all i considered, and $\omega \subset BA_N$. Combining (5.3) and (5.4), we obtain

$$\Lambda_j \leq \left(\prod_{i=1}^{j-1} d_i\right) \sum_{k=1}^{\infty} C e^{-\lambda k} \lesssim \prod_{i=1}^{j-1} d_i.$$

Inserting this estimate and (5.4) in (5.1), we find that

$$|f_a^j(x) - f_b^j(x)| < C|Df_a^{j-1}(Q)||a - b|,$$

which is

$$\leq C \frac{|f_a^j(\hat{I}_\mu)|}{|f_a(\hat{I}_\mu)|} |\omega|.$$

Note that $|\omega| \lesssim e^{-\lambda N}$, because of the assumption on $|Df_a^k(Q)|$ and Lemma 5.4. Using $|f_a(\hat{I}_\mu)| \sim e^{-|\mu|}$ and $|\mu| < N^{\alpha q}$ gives

$$|f_a^j(x) - f_b^j(x)| < C|f_a^j(\hat{I}_\mu)| e^{N^{\alpha q} - \lambda N} < \tfrac{1}{1000} |f_a^j(\hat{I}_\mu)|$$

for large N since $\alpha q < 1$. This proves the first statement of Lemma 5.5. The second is proved in a similar manner: Pick $b \in \omega$, let $x = \xi_{N+1}(b)$ and estimate the distance between $\xi_{N+j+1}(b) = f_b^j(x)$ and $f_a^j(x)$ as above. □

Proof of Lemma 5.1. The right inequality in (i) follows from part (i) of Lemma 4.2 and the definition of $p(\mu, \omega)$. The first statement of Lemma 5.5 implies that $\left|\bigcup_{b\in\omega} f_b^j(\hat{I}_\mu)\right|$ and $|f_a^j(\hat{I}_\mu)|$ are, up to a small constant independent of a, of the same size for $j \le p$. Thus (ii) and the left inequality of (i) follows from the proof of the corresponding statements in Lemma 4.2. In the light of part (ii) of Lemma 5.5, (iii) follows from (ii), and (iv) follows from part (iv) of Lemma 4.2. □

Proof of Lemma 5.2. Let $\omega = (b, c)$. For any $a \in \omega$ and $x = \xi_{n-k}(a) \in \xi_{n-k}(\omega)$ we have that $f_a^j(x) \notin I_0^*$ for $j = 0, 1, \dots, k-1$, and so $|Df_a^j(x)| \ge C(\delta_0)e^{\lambda_0 j}$, $j = 1, 2, \dots, k-1$ and $|Df_a^k(x)| \ge K_0 e^{\lambda_0 k}$ holds by Lemma 3.2. Copying the proof of the right inequality in Lemma 5.4 we see that

$$|\partial_a F^k(x, a)| \le C_1(\delta_0)|\partial_x F^k(x, a)|.$$

Introduce the auxiliary function $\psi_k(a) = F^k(\xi_{n-k}(a), a)$. From Cauchy's mean value theorem it follows that

$$\frac{|\xi_n(\omega)|}{|\xi_{n-k}(\omega)|} = \frac{|F^k(\xi_{n-k}(b), b) - F^k(\xi_{n-k}(c), c)|}{|\xi_{n-k}(b) - \xi_{n-k}(c)|} = \frac{|\psi_k(b) - \psi_k(c)|}{|\xi_{n-k}(b) - \xi_{n-k}(c)|} = \frac{|\psi_k'(a^*)|}{|\xi_{n-k}'(a^*)|},$$

for some $a^* \in \omega$. We calculate ψ_k' and obtain

$$\frac{|\xi_n(\omega)|}{|\xi_{n-k}(\omega)|} = \frac{1}{|\xi_{n-k}'(a^*)|}|\partial_x F^k(\xi_{n-k}(a^*), a^*) \cdot \xi_{n-k}'(a^*) + \partial_a F^k(\xi_{n-k}(a^*), a^*)|$$
$$\ge |\partial_x F^k(\xi_{n-k}(a^*), a^*)| - \frac{|\partial_a F^k(\xi_{n-k}(a^*), a^*)|}{|\xi_{n-k}'(a^*)|},$$

and, invoking the estimate on $\partial_a F^k$, this is

$$\ge |\partial_x F^k(\xi_{n-k}(a^*), a^*)| \left(1 - \frac{C_1(\delta_0)}{|\xi_{n-k}'(a^*)|}\right).$$

By the assumption and Lemma 5.4, $|\xi_{n-k}'(a^*)| \sim |Df_{a^*}^{n-k-1}(Q)| \ge e^{\lambda(n-k-1)}$ which is greater than $2C_1(\delta_0)$, since $n - k > m_0$ and m_0 is large if a_0 is close enough to two. This proves part (i). (ii) follows immediately from what we have just proved and from part (iii) of Lemma 5.1. (iii) is proved in the same way as (i), using part (i) of Lemma 3.2. □

Proof of Lemma 5.3. Using Lemma 5.4 we get

$$\left|\frac{\xi_N'(b)}{\xi_N'(c)}\right| \le C_{ax}^2 \left|\frac{Df_b^{N-1}(Q)}{Df_c^{N-1}(Q)}\right| = C_{ax}^2 \left|\frac{\prod_{i=0}^{N-2} f_b'(f_b^i(Q))}{\prod_{i=0}^{N-2} f_c'(f_c^i(Q))}\right|$$
$$= \left(\frac{b}{c}\right)^{N-1} C_{ax}^2 \left|\frac{\prod_{i=1}^{N-1} \varphi'(\xi_i(b))}{\prod_{i=1}^{N-1} \varphi'(\xi_i(c))}\right| \le C \left|\frac{\prod_{i=2}^{N-1} \varphi'(\xi_i(b))}{\prod_{i=2}^{N-1} \varphi'(\xi_i(c))}\right|$$

since $|b - c| \lesssim e^{-\lambda N}$, and just as in the proof of Lemma 4.1, we see that this is

$$\le C_1 \exp\left\{(1 + 2q) \sum_{i=2}^{N-1} \frac{|\xi_i(c) - \xi_i(b)|}{|z_i|^{1+q}}\right\},$$

for some z_i between $\xi_i(b)$ and $\xi_i(c)$.

Let $\Omega_j = (\xi_j(b); \xi_j(c))$ and let $t_1 < \cdots < t_k = N$ denote the free return times of ω. Then, by assumption, there exists $\{(\mu_i, \nu_i)\}$ such that $\Omega_{t_i} \subset I^+_{\mu_i \nu_i}$ for $i < k$. Let $p_i = p(\mu_i, \omega)$ for $0 < i < k$ and let $t_0 = p_0 = 0$. We now split the sum $S = \sum_{i=2}^{N-1} |\xi_i(c) - \xi_i(b)| \,/\, |z_i|^{1+q}$ into sub-sums corresponding to free and bound periods respectively:

$$S = \sum_{i=0}^{k-1} (S_i^{\mathrm{bp}} + S_i^{\mathrm{fp}}),$$

where

$$S_i^{\mathrm{bp}} = \sum_{l=t_i}^{t_i+p_i-1} \frac{|\Omega_l|}{|z_l|^{1+q}}$$

and

$$S_i^{\mathrm{fp}} = \sum_{l=t_i+p_i}^{t_{i+1}-1} \frac{|\Omega_l|}{|z_l|^{1+q}}.$$

With this notation, S_0^{fp} is the contribution up until the first return to I^*, where the two first terms in S_0^{fp} equal zero by definition. S_0^{bp} is an empty sum and is equal to zero by definition.

First we estimate $\sum_{i=0}^{k-2} S_i^{\mathrm{fp}}$, using (i) and (ii) of Lemma 5.2, and $|z_l| > \delta_0$.

$$\begin{aligned} \sum_{i=0}^{k-2} S_i^{\mathrm{fp}} &= \sum_{i=0}^{k-2} \sum_{l=t_i+p_i}^{t_{i+1}-1} \frac{|\Omega_l|}{|z_l|^{1+q}} \le \frac{1}{\delta_0^{1+q}} \sum_{i=0}^{k-2} C_1 |\Omega_{t_{i+1}}| \\ &\le \frac{C_2}{\delta_0^{1+q}} |\Omega_{t_{k-1}}| \le C_3 \frac{\Delta_0^{1+1/q}}{\Delta_0^{2+1/q+\epsilon_0}} = C_3 \Delta_0^{-(1+\epsilon_0)}. \end{aligned} \tag{5.5}$$

So we have a δ_0-independent estimate of the distortion-contribution during all but the last free period.

We now turn to the last free period and estimate S_{k-1}^{fp}. Let $\hat{N}$ be the largest integer such that $t_{k-1} + p_{k-1} \le \hat{N} < N$ and such that $\Omega_{\hat{N}} \cap (-\sqrt{\delta_0}, \sqrt{\delta_0}) \ne \emptyset$. From the argument below one sees that if no such $\hat{N}$ exists, this part of the proof reduces to a simpler case. We split S_{k-1}^{fp} into two sub-sums:

$$S_{k-1}^{\mathrm{fp}} = \sum_{l=t_{k-1}+p_{k-1}}^{N-1} \frac{|\Omega_l|}{|z_l|^{1+q}} = \sum_{l=t_{k-1}+p_{k-1}}^{\hat{N}} \frac{|\Omega_l|}{|z_l|^{1+q}} + \sum_{l=\hat{N}+1}^{N-1} \frac{|\Omega_l|}{|z_l|^{1+q}}. \tag{5.6}$$

Estimating the second sub-sum is easy:

$$\sum_{l=\hat{N}+1}^{N-1} \frac{|\Omega_l|}{|z_l|^{1+q}} \le \frac{1}{(\sqrt{\delta_0})^{1+q}} \sum_{l=\hat{N}+1}^{N-1} C_{\mathrm{fp}} e^{-\lambda_0 (N-l)} |\Omega_N| \le C \frac{10\delta_0}{\delta_0^{(1+q)/2}} \le C_1, \tag{5.7}$$

where C_1 can be chosen independently of δ_0 since we only consider $q < 1$.

To estimate the first sub-sum we need to know that $N - \hat{N}$ is large. We make the following two claims.

CLAIM 1. $\Omega_{\hat{N}} \subset (-2\sqrt{\delta_0}, 2\sqrt{\delta_0})$

CLAIM 2. *If* $\Omega_{\hat{N}} \subset (-2\sqrt{\delta_0}, 2\sqrt{\delta_0})$, *then* $N - \hat{N} \gtrsim \sqrt{\Delta_0}$.

Proof of Claim 1. Assume that $\Omega_{\hat{N}} \setminus \left(-2\sqrt{\delta_0}, 2\sqrt{\delta_0}\right) \neq \emptyset$. We will show that this contradicts the assumption that $|\Omega_N| \leq 10\delta_0$. Clearly, we may suppose $\Omega_{\hat{N}} = [\sqrt{\delta_0}, 2\sqrt{\delta_0}]$, so that $|\Omega_{\hat{N}}| = \sqrt{\delta_0}$. Now $\Omega_{\hat{N}+j}$ is close to $\pm Q$ as long as $j \leq T_0$, where T_0 can be made arbitrarily large by choosing δ_0 small and ω close to two. It follows, using Lemma 5.2, that

$$|\Omega_N| \geq C 2^{T_0} e^{\lambda_0 (N - \hat{N} - T_0)} |\Omega_{\hat{N}}|,$$

where C is independent of δ_0. For δ_0 sufficiently small and ω close to two we have that $C 2^{T_0} \geq 1$, so $|\Omega_N| \geq |\hat{\Omega}_N| = \sqrt{\delta_0}$, and we have reached a contradiction. This proves Claim 1. □

Proof of Claim 2. Let $\delta = \Delta^{-1/q}$ be a small positive number. Then $f_{2,q}$ maps $J_\delta = [0, \delta]$ onto an interval of length $C(q)e^{-\Delta}$ with Q as an end-point. Since $f_{2,q}(\pm Q) = -Q$ and $\sup |f'_{2,q}| \leq 4$ for all q considered, $f^j_{2,q}(J_\delta)$, $j \geq 2$, is contained in $[-Q, -Q/2]$ as long as

$$4^j e^{-\Delta} \leq C_1,$$

for some suitable $C_1 = C_1(q)$. If Δ is large, this holds when $j \leq \Delta/2$. Thus, it takes at least $\Delta/2$ iterates under $f_{2,q}$ for J_δ to intersect a small neighbourhood of zero. Given q and δ, the same holds for $f_{a,q}$ if a is sufficiently close to two. We now apply this with $\delta = 2\sqrt{\delta_0}$. This corresponds to taking $\Delta = 1/2^q \sqrt{\Delta_0}$. From this the claim follows. □

We proceed to estimate the first sub-sum in (5.6), using Claim 2, Lemma 5.2 and the assumption $|\Omega_N| \leq 10\delta_0$.

$$\begin{aligned}
\sum_{l=t_{k-1}+p_{k-1}}^{\hat{N}} \frac{|\Omega_l|}{|z_l|^{1+q}} &\leq \frac{1}{\delta_0^{1+q}} \sum_{l=t_{k-1}+p_{k-1}}^{\hat{N}} |\Omega_l| \\
&\leq \frac{1}{\delta_0^{1+q}} \sum_{l=t_{k-1}+p_{k-1}}^{\hat{N}} C_{\mathrm{fp}} e^{-\lambda_0 (N-l)} |\Omega_N|, \\
&\leq \frac{C}{\delta_0^q} \sum_{l=t_{k-1}+p_{k-1}}^{\hat{N}} e^{-\lambda_0 (N-l)} \\
&\leq \frac{C}{\delta_0^q} \sum_{j=N-\hat{N}}^{\infty} e^{-\lambda_0 j} \\
&\leq \frac{C_1}{\delta_0^q} e^{-\lambda_0 (N-\hat{N})} \leq \frac{C_1}{\delta_0^q} e^{-c\sqrt{\Delta_0}} \leq C_1 \frac{\Delta_0}{e^{c\sqrt{\Delta_0}}} \leq C_2. \qquad (5.8)
\end{aligned}$$

Combining (5.5)–(5.8) we get the desired bound on the total distortion during the free periods.

Now we estimate S_i^{bp}. First, we estimate the individual terms in S_i^{bp} for $l > t_i$:

$$\frac{|\Omega_l|}{|z_l|^{1+q}} \lesssim \frac{|f_a^{l-t_i}(\xi_{t_i}(\omega))|}{|z_l|^{1+q}} = \frac{|f_a^{l-t_i}(\xi_{t_i}(\omega))|}{|f_a^{l-t_i}(\hat{I}_{\mu_i})|} \cdot \frac{|f_a^{l-t_i}(\hat{I}_{\mu_i})|}{|z_l|^{1+q}}$$
$$\sim \frac{|f_a(\xi_{t_i}(\omega))|}{|f_a(\hat{I}_{\mu_i})|} \cdot \frac{|f_a^{l-t_i}(\hat{I}_{\mu_i})|}{|z_l|^{1+q}} \lesssim \frac{|f_a(\xi_{t_i}(\omega))|}{|f_a(\hat{I}_{\mu_i})|} \cdot \frac{1/(l-t_i)^{\beta}}{1/(l-t_i)^{\alpha(1+q)}}.$$

This holds for any $a \in \omega$ because of Lemma 5.5, Lemma 4.1, Definition 6 and $a \in BA_n$.

Since $\beta > 1 + \alpha(1+q)$ we can, just as in the proof of Lemma 4.1, sum over l to obtain an estimate for $S_i^{\mathrm{bp}} - |\Omega_{t_i}|/|z_{t_i}|^{1+q}$:

$$\sum_{l=t_i+1}^{t_i+p_i-1} \frac{|\Omega_l|}{|z_l|^{1+q}} \le \sum_{l=t_i+1}^{\infty} \frac{|f_a(\xi_{t_i}(\omega))|}{|f_a(\hat{I}_{\mu_i})|} \cdot \frac{1/(l-t_i)^{\beta}}{1/(l-t_i)^{\alpha(1+q)}} \lesssim \frac{|f_a(\xi_{t_i}(\omega))|}{|f_a(\hat{I}_{\mu_i})|}$$
$$= \frac{|f_a(\xi_{t_i}(\omega))|}{|f_a(I_{\mu_i})|} \cdot \frac{|f_a(I_{\mu_i})|}{|f_a(\hat{I}_{\mu_i})|} \sim \frac{|f_a(\xi_{t_i}(\omega))|}{|f_a(I_{\mu_i})|} \sim \frac{|\xi_{t_i}(\omega)|}{|I_{\mu_i}|}.$$

In the last two steps we used the fact that $|f_a(I_{\mu_i})| \sim |f_a(\hat{I}_{\mu_i})|$ and that the distortion for f_a restricted to I_μ is bounded uniformly in μ. It is easily seen that the first term in S_i^{bp} is also $\sim |\xi_{t_i}(\omega)|/|I_{\mu_i}|$. Finally, we estimate $\sum_{i=0}^{k-1} S_i^{\mathrm{bp}}$:

$$\sum_{i=0}^{k-1} S_i^{\mathrm{bp}} \lesssim \sum_{\mu_i} \sum_{\substack{\text{returns} \\ \text{to } I_{\mu_i}}} \frac{|\xi_{t_i}(\omega)|}{|I_{\mu_i}|} \lesssim \sum_{\mu_i} \sum_{\substack{\text{last return} \\ \text{to } I_{\mu_i}}} \frac{|\xi_{t_i}(\omega)|}{|I_{\mu_i}|}$$
$$\le \sum_{\mu_i} \sum_{\substack{\text{last return} \\ \text{to } I_{\mu_i}}} \frac{|I^+_{\mu_i \nu_i}|}{|I_{\mu_i}|} \lesssim \sum_{|\mu| > \Delta_0} \frac{1}{|\mu|^{1+\epsilon_0}} < C\Delta_0^{-\epsilon_0}.$$

The second '$\lesssim$' holds because of part (ii) of Lemma 5.2. □

6. *The good set A*

In this section we construct a decreasing sequence of sets A_n in parameter space such that $A_n \subset BA_n \cap FA_n \cap [\hat{a}, 2)$, where $\hat{a} = \hat{a}(\delta_0)$ is so close to two that Proposition 4.1 holds. We show that $A_{n-1} \setminus A_n$ is so small compared to A_{n-1} that

$$A = \bigcap_{n=1}^{\infty} A_n$$

has positive Lebesgue measure.

In §5, in the discussion following Lemma 5.1, bound periods and free returns for parameter intervals were defined. We now distinguish two types of free returns. If n is a free return for ω and $\xi_n(\omega) \supset I_{\mu\nu}$, for some $I_{\mu\nu}$ with $|\mu| \ge \Delta_0$, then n is an *essential* free return. Free returns that are not essential are called *inessential*. Note that in this case, $\xi_n(\omega) \subset$ some $I^+_{\mu\nu}$.

PROPOSITION 6.1. *Let* $\lambda = \min\{\lambda_0/4, \log\sqrt{2}\}$. *For any* $0 < q < 1/8$, *any* $\tilde{a} < 2$ *and any* $\delta_0 > 0$ *sufficiently small, there exists a sequence of sets* $\{A_n\}_{n=1}^{\infty}$ *and a* $\gamma > 1$ *such that:*

(i) $[\tilde{a}, 2) \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \ldots;$
(ii) $A_n \subset BA_n \cap FA_n(\delta_1)$;
(iii) $|Df^j_{a,q}(Q)| \geq e^{\lambda j}$ *for all* $j \leq n$ *and all* $a \in A_n$;
(iv) $|A_n| \geq (1 - n^{-\gamma})|A_{n-1}|$.

Furthermore, on each A_n*, there is a partition* $\mathcal{P}_n$ *such that for any* $\omega \in \mathcal{P}_n$ *the following holds:*

(v) *if* $j \leq n$ *is a free return of* ω*, then* $\xi_j(\omega) \subset$ *some* $I^+_{\mu\nu}$*. If* n *is an essential free return of* ω*, then* $I_{\mu\nu} \subset \xi_n(\omega) \subset I^+_{\mu\nu}$ *for some* $I_{\mu\nu} \subset I^*_0$*. If* n *is not an essential free return and* $\hat{m}$ *is the largest integer which is less than* n *such that* $\hat{m}$ *is an essential free return for* ω*, then* $\omega \in \mathcal{P}_j$ *for* $j = \hat{m}, \hat{m}+1, \ldots, n$.

The proof of Proposition 6.1 will occupy the rest of §6. Throughout this section we will assume that δ_0 is small enough, and only consider a sufficiently close to two. We will not always state this explicitly. For any $\delta_0 > 0$, we choose an $\hat{a}$ so close to two that all lemmas and propositions from the previous sections hold. This also means that we have chosen a sufficiently large integer m_0, and an $\hat{a} = a_0(m_0, q)$ according to Lemma 3.1. Clearly we may assume that $\hat{a} > \tilde{a}$. By construction $A_n \subset [\hat{a}, 2]$, so (iii) will follow from (ii) by Proposition 4.1. Note that if A_{n-1} is as above, then ξ_n is a diffeomorphism on elements of $\mathcal{P}_{n-1}$.

Let

$$A_1 = A_2 = \cdots = A_{m_0} = \xi^{-1}_{m_0}(I_{-\Delta_1} \cup I_{\Delta_1}).$$

For $1 \leq n \leq m_0$ define

$$\mathcal{P}_n = \{\xi^{-1}_{m_0}(I_{\Delta_0,\nu})\}.$$

It is clear that (i)–(v) of Proposition 6.1 hold for $1 \leq n \leq m_0$ and that any subset of A_{m_0} is contained in $[\hat{a}, 2]$.

We proceed to construct the sets A_n recursively. This is done in two steps. In §6.1 we construct a set $A'_n \subset A_{n-1} \cap BA_n$, by deleting those parameters a for which $|\xi_n(a)| \leq 1/n^\alpha$. We show that the deleted set is small compared to A_{n-1} if n is large. Also, $\mathcal{P}_n$ is constructed as a refinement of $\mathcal{P}_{n-1}$ on A'_n. In §6.2 we delete and estimate the measure of those parameters in A_{n-1} for which the free period assumption does not hold.

6.1. *Construction of* A'_n. In this section we construct A'_n from A_{n-1} and a refinement $\mathcal{P}_n$ of $\mathcal{P}_{n-1}$. For intervals $\omega \in \mathcal{P}_n$ we verify (v) of Proposition 6.1, and we show that $\omega \subset BA_n$. We also show that there is a $\gamma > 1$ such that $|A'_n| > (1 - n^{-\gamma})|A_{n-1}|$.

Pick an $\omega \in \mathcal{P}_{n-1}$. Let $\hat{m}$ be the last free return of ω before n, and let $\hat{p}$ be the length of the corresponding bound period. If $\xi_n(\omega) \cap I^*_0 = \emptyset$, or $n < \hat{m} + \hat{p}$, we leave ω as it is: $\omega \subset A'_n$ and $\omega \in \mathcal{P}_n$. (v) of Proposition 6.1 then holds by induction (n is not a free return). Also $\omega \subset BA_n$: by induction we only have to consider the case of a return to I^*_0 during a bound period at time n. In this case, (iv) of Lemma 5.1 shows that $|\xi_n(a)| > 1/n^\alpha$ for all $a \in \omega$.

Now assume that $\xi_n(\omega) \cap I^*_0 \neq \emptyset$ and $n \geq \hat{m} + \hat{p}$. If $\xi_n(\omega)$ does not cover any $I_{\mu\nu}$, an inessential return, we still leave ω unchanged. (v) of Proposition 6.1 once again holds trivially, and we have to show that $\omega \subset BA_n$. This follows by induction from the next lemma.

LEMMA 6.1. *For Δ_0 sufficiently large, the following holds. Suppose $\xi_n(\omega) \subset I_{\mu\nu}$ is an inessential return, and assume that the preceding essential return took place in $I_{\hat{\mu}\hat{\nu}}$ at time $\hat{m}$. Then $|\mu| < |\hat{\mu}|$. More precisely, there exists $\kappa = \kappa(q) < 1$ such that*

$$|\mu| < |\hat{\mu}|^{\kappa}.$$

Proof. Since the return is assumed to be inessential and because of Lemmas 5.1 and 5.2, we have

$$|\hat{\mu}|^{-(1+\beta+\epsilon_0)} \lesssim |\xi_{\hat{m}+p(\hat{\mu})}(\omega)| \lesssim |\xi_n(\omega)| \lesssim |I_{\mu\nu}| \lesssim |\mu^{-(2+1/q+\epsilon_0)}|.$$

Thus,

$$|\mu| \lesssim |\hat{\mu}|^{(1+\beta+\epsilon_0)/(2+1/q+\epsilon_0)},$$

and the lemma follows since $\beta = 1/(2q)$. □

Finally, consider an essential free return: $n \geq \hat{m} + \hat{p}$ and $\xi_n(\omega) \supset$ some $I_{\mu\nu}$, $|\mu| \geq \Delta_0$. Let

$$\omega' = \omega \setminus \xi_n^{-1}(\xi_n(\omega) \cap (-n^{-\alpha}, n^{-\alpha})).$$

The remaining part ω' will be in A'_n. $\mathcal{P}_n$ on ω' is defined so that $\omega'' \in \mathcal{P}_n$ implies that $\xi_n(\omega'')$ is either a component of $\xi_n(\omega) \setminus I_0^*$ or $I_{\mu\nu} \subset \xi_n(\omega'') \subset I_{\mu\nu}^+$ for some $I_{\mu\nu} \subset I_0^*$. By construction, $\omega' \subset BA_n$ and (v) holds for n. This finishes the construction of A'_n and $\mathcal{P}_n$.

We now estimate the measure of the deleted set. Note that it was only in the case of an essential free return that we deleted anything at all.

LEMMA 6.2. *For each $0 < q < 1/8$ there exists a $\gamma > 1$ such that*

$$|A'_n| > (1 - n^{-\gamma})|A_{n-1}|.$$

Proof. Suppose $\omega \in \mathcal{P}_{n-1}$ has an essential free return at time n. Clearly, we may assume that $|\xi_n(\omega)| \leq 10\delta_0$. There is a maximal $m < n$ such that $\xi_m(\omega)$ is either (I) an essential free return and $I_{\mu\nu} \subset \xi_m(\omega) \subset I_{\mu\nu}^+$ for some $I_{\mu\nu} \subset I_0^*$ or (II) $\xi_m(\omega) \supset I_{\Delta_0-1,\nu}$ where $I_{\Delta_0-1,\nu}$ is a neighbouring interval of I_0^*. (II) happens if $\omega \subset \hat{\omega}$ is a component mapped outside I_0^* by ξ_m and m is an essential return for $\hat{\omega}$.

Consider case I. The return at time m is followed by a bound period of length $p = p(\mu, \omega)$ and $m + p \leq n$ since n is a free return. Let ω' be as above. We apply Lemma 5.3 to conclude that

$$\frac{|\omega \setminus \omega'|}{|\omega|} \leq C_{\mathrm{ld}} \frac{|\xi_n(\omega) \cap (-1/n^{\alpha}, 1/n^{\alpha})|}{|\xi_n(\omega)|} \lesssim \frac{1/n^{\alpha}}{|\xi_n(\omega)|}.$$

Using Lemmas 5.1 and 5.2 we see that

$$\begin{aligned} |\xi_n(\omega)| &\gtrsim |\xi_{m+p}(\omega)| \geq C|\mu|^{1+1/q-\beta}|\xi_m(\omega)| \\ &\sim |\mu|^{1+1/q-\beta}|I_{\mu\nu}| \sim |\mu|^{(1+1/q-\beta)-(2+1/q+\epsilon_0)}, \end{aligned}$$

so

$$|\xi_n(\omega)| \geq C_1 |\mu|^{-(1+\beta+\epsilon_0)}.$$

Since $\omega \subset BA_{n-1}$, $|\mu| \leq n^{\alpha q}$ and thus

$$\frac{|\omega \setminus \omega'|}{|\omega|} \leq C_2 \frac{n^{-\alpha}}{|\mu|^{-(1+\beta+\epsilon_0)}} \leq C_2 n^{\alpha q(1+\beta+\epsilon_0)-\alpha}.$$

From the definitions of α and β in §3, it follows that

$$-\gamma_1 := \alpha q(1+\beta+\epsilon_0) - \alpha < -1,$$

if $\epsilon_0 > 0$ is sufficiently small. We only delete at essential returns, and they do not appear until after time m_0. If m_0 is sufficiently large (which is the same as requiring that $\hat{a}$ is sufficiently close to two), we can find γ, $1 < \gamma < \gamma_1$, such that the statement of the lemma holds for all $n > m_0$.

Now consider case II. Then $|\xi_n(\omega)| > \delta_0$ (see Lemma 6.3) and we may proceed as in case I. Since this holds for all $\omega \in \mathcal{P}_{n-1}$, the lemma follows. □

Remark 8 (On the choice of α *and* β *and the size of* q_0*).* Consider a fixed q and suppose we had not yet specified the values of α and β. From Remarks 4 and 7 in §4, we know that our choice is limited to an open triangle T in the $\alpha\beta$-plane given by the inequalities

$$1 < \alpha < 1/q, \quad 1 + \alpha(1+q) < \beta < 1 + 1/q.$$

In the light of the proof of Lemma 6.2 we want to choose α and β in T such that for sufficiently small ϵ_0,

$$h_q(\alpha, \beta) := \alpha q(1+\beta+\epsilon_0) - \alpha < -1.$$

To find out for which q this is possible, we minimize h_q over the given region of $\alpha\beta$-space. One can easily prove the following claim.

CLAIM.

(i) *For* $q \geq 1/8$ *and for any* $\epsilon_0 > 0$, $h_q \geq -1$ *in* T.

(ii) *For* $0 < q < 1/8$ *we can find an* $\epsilon_0 > 0$ *such that the function*

$$h_q(\alpha, \beta) = \alpha q(1+\beta+\epsilon_0) - \alpha$$

attains values which are less than -1 *inside the triangle* T. *More precisely this happens in a neighbourhood of*

$$\alpha = \frac{1-(2+\epsilon_0)q}{2q(q+1)},$$
$$\beta = 1 + \alpha(1+q).$$

In fact, $\alpha = (1-(2+\epsilon_0)q)/(2q(q+1))$ *and* $\beta = 1/2q$ *will do for* ϵ_0 *sufficiently small.*

6.2. *Deleting* $A_{n-1} \setminus FA_n$. In this section we delete those parameters that are in some $B_n \supset A_{n-1} \setminus FA_n(\delta_1)$, and prove that $|B_n|$ is small.

Consider $a \in A_n'$. Then there exists parameter intervals $\omega_j = \omega_j(a) \in \mathcal{P}_j$, $j = 1, \ldots, n$, such that $a \in \bigcap_{j=1}^{n} \omega_j$. There also exists integers $n_i(a)$,

$$n_1 = m_0 < n_2 < \cdots < n_k \leq n,$$

such that for each $1 < j \leq k$:

- n_j is an essential free return for $\omega_{n_{j-1}}$;
- ω_{n_j} is either (I) returning inside I_0^* under ξ_{n_j}, and then

$$I_{\mu_j \nu_j} \subset \xi_{n_j}(\omega_{n_j}) \subset I^+_{\mu_j \nu_j}$$

  for some (μ_j, ν_j), or (II) a component of

$$\xi_{n_j}^{-1}(\xi_{n_j}(\omega_{n_{j-1}}) \setminus I_0^*).$$

  In this case $\xi_{n_j}(\omega_{n_j}) \supset I_{\pm\Delta_0 - 1, \nu}$, a neighbouring interval of I_0^*. In case (I) we call $I_{\mu_j \nu_j}$ the host interval, and in case (II) we say that $I_{\Delta_0 - 1}$ is the host interval.
- $a \in \omega_m \in \mathcal{P}_m$ and $n_j < m < n_{j+1}$ implies $\omega_m = \omega_{n_j}$.

$n_j(a)$ is called the jth essential return of a.

Definition 7. A free return n_j of $a \in \omega_{n_j - 1}$ is an *escape situation* if

$$|\xi_{n_j}(\omega_{n_j - 1})| > \delta_0.$$

Also $n_1 = m_0$, the first return time, is considered as an escape. The corresponding *escape time* $n_k(a)$ for $a \in \omega_{n_j}$ is the smallest $n_k > n_j$ such that $|\xi_{n_k}(\omega_{n_k - 1}(a))| > \delta_0$. The time interval (n_j, n_k) is the called *the escape period*. Also define $e_j(a) = n_k - n_j$, the length of the escape period.

Returns outside $(-\delta_1, \delta_1)$ will be followed by an escape situation at the next free return. This is stated in the following lemma. Recall that $\delta_0 = \Delta_0^{-1/q}$, $\delta_0 = \delta_1^{q+1/2+2q\epsilon_0}$ and that $(\Delta_1 + 1)^{-1/q} < \delta_1 \leq \Delta_1^{-1/q}$.

LEMMA 6.3. *For all* $\epsilon_0 > 0$, *a sufficiently close to two,* Δ_0 *sufficiently large and for* $|\mu| \leq \Delta_1$ *the following holds:*

$$\xi_{n_j}(\omega_{n_j}) \textit{ has host interval } I_{\mu\nu} \textit{ or } I_{\Delta_0 - 1} \implies |\xi_{n_{j+1}}(\omega_{n_j})| > \delta_0.$$

Proof. From (iii) of Lemma 5.1 and (i) of Lemma 5.2, we have that

$$|\xi_{n_{j+1}}(\omega_{n_j})| \geq C|\mu|^{1+1/q-\beta}|I_{\mu\nu}| \geq C_1|\mu|^{-(1+\beta+\epsilon_0)} \geq C_1\Delta_1^{-(1+\beta+\epsilon_0)},$$

where C_1 does not depend on Δ_0. Since $\beta = 1/2q$ and $\Delta_1 \leq \Delta_0^{1/(q+1/2+2q\epsilon_0)}$ by definition, the lemma follows for Δ_0 sufficiently large. □

We want to count the amount of bound time originating from returns inside $(-\delta_1, \delta_1)$.

Definition 8. Let n_j be the escape times which are less than or equal to n for a given a. Define

$$E_j(a) = \begin{cases} 0, & \text{if } \xi_{n_j}(a) \notin (-\delta_1, \delta_1), \\ e_j(a), & \text{if } \xi_{n_j}(a) \in (-\delta_1, \delta_1), \end{cases} \tag{6.1}$$

and

$$T(a, n) = \sum_{n_j \le n} E_j(a).$$

Clearly, each bound period following a return to $(-\delta_1, \delta_1)$ is contained in an escape period corresponding to a non-zero E_i. Let

$$B_n = \{a \in A_{n-1} \mid T(a, n) > n/2\}$$

To prove (iv) in Proposition 6.1 it now suffices to prove that there exists a $\gamma > 1$ such that

$$|B_n| < n^{-\gamma} |A_{n-1}|.$$

This will be proved in several steps. When going from one escape to another, small sets $I_{\mu\nu}$ expand to size δ_0. This makes the functions E_i almost independent, and the sum $T(a, n)$ can be estimated with a large deviation argument. To accomplish this we must estimate the distribution functions of the functions E_i. Consider an ω at an escape situation at time n_j. We may use inductive information about behaviour during bound periods for another n_j iterates. For $0 < t < n_j$, we want to estimate

$$m\{a \in \omega \mid E_j(a) \ge t\}.$$

Subdivide ω into sets $\omega_{\mu\nu}$ corresponding to different host intervals in I_1^*. Consider each $\omega_{\mu\nu}$ separately, and for any s, consider a subset $\omega_s \subset \omega_{\mu\nu}$ such that $n_{j+i}(\omega_s)$ is well-defined, is less than $n_j + t$ and is not an escape situation for $i = 1, \ldots, s$. Then we can estimate $|\omega_s|/|\omega_{\mu\nu}|$ in terms of μ, and the indices of the host intervals at the return times n_{j+i}. This is done in §6.2.1.

In §6.2.2 we give an upper bound for the time gap between two subsequent essential returns, in terms of the host interval index.

In §6.2.3 we consider an ω_s as above going through host intervals $\{I_{\mu_i \nu_i}\}_{i \le s}$ and such that $E_j(\omega_s) \ge t$ for some $t \le n_j$. Using the estimates from §6.2.2, we prove a relation between s, t, and $\{\mu_i\}_{i \le s}$. From this and the estimates from §6.2.1, we obtain estimates of $m\{a \in \omega_s \mid E_j(a) \ge t\}$, and after summation we obtain estimates of the distribution function restricted to one $\omega_{\mu\nu}$, and then to ω.

Finally, in §6.2.4, we first delete and estimate the measure of those parameters that have an escape situation at time n and for which the length of the following escape period is greater than n. This has been done for all n, so in the last step we may assume that an escape period starting at time $n_j \le n$ terminates before time $2n_j \le n_j + n$. In the last step, $m\{a \in A_{n-1} \mid T(a, n) > n/2\}$ is estimated using the information about the distribution functions for the functions E_i.

6.2.1. *The relative size of ω_s.* Consider an $\omega_0 \in \mathcal{P}_{n_j}$, $n_j \le n$ an essential return time for $a \in \omega_0$ with host interval I_{μ_0,ν_0}, and a subset $\omega_s \subset \omega_0$ such that for $i = 1, 2, \ldots, s$ the following holds:

- $n_{j+i}(a)$ is constant on ω_s;
- $n_{j+i}(a)$ is not an escape situation for any $a \in \omega_s$;
- $n_{j+s}(\omega_s) \le 2n_j$ $(\le n_j + n)$.

This means that there are parameter intervals $\{\omega_i\}_{i=0}^s$ and intervals $I_{\mu_i \nu_i}$, $|\mu_i| \ge \Delta_0 - 1$, such that

$$\omega_s \subset \omega_{s-1} \subset \cdots \subset \omega_0$$

and such that $I_{\mu_i \nu_i}$ is the host interval of $\xi_{n_{j+i}}(\omega_i)$. None of these returns are escape returns so we have

$$|\xi_{n_{j+i}}(\omega_{i-1})| < \delta_0 \quad \text{for } i = 1, 2, \ldots, s.$$

(Because of Lemma 6.3 this actually forces $|\mu_i| > \Delta_1$ for $i < s$. We will not use this fact.)

We estimate the relative size of ω_s, using the results of §5. Let $p_i = p(\mu_i)$.

$$\begin{aligned}
\frac{|\omega_s|}{|\omega_0|} &\le C_{\mathrm{ld}} \frac{|\xi_{n_j}(\omega_s)|}{|\xi_{n_j}(\omega_0)|} \le C_{\mathrm{ld}} \frac{|I^+_{\mu_s \nu_s}|}{|I_{\mu_0 \nu_0}|} \prod_{i=0}^{s-1} |Df_a^{p_i}(\mu_i^{-1/q})|^{-1} \\
&\lesssim C_0^s \frac{|\mu_0|^{2+1/q+\epsilon_0}}{|\mu_s|^{2+1/q+\epsilon_0}} \prod_{i=0}^{s-1} \frac{1}{|\mu_i|^{1+1/q-\beta}} \\
&\le C_0^s \frac{1}{\Delta_0^{1+\beta+\epsilon_0}} \frac{1}{|\mu_s|^{1+1/q-\beta}} \frac{|\mu_0|^{2+1/q+\epsilon_0}}{|\mu_0|^{1+1/q-\beta}} \prod_{i=1}^{s-1} \frac{1}{|\mu_i|^{1+1/q-\beta}} \\
&= \frac{C_0^s}{\Delta_0^{1+\beta+\epsilon_0}} |\mu_0|^{1+\beta+\epsilon_0} \prod_{i=1}^{s} \frac{1}{|\mu_i|^{1+1/q-\beta}},
\end{aligned} \tag{6.2}$$

where the second inequality follows since

$$\prod_{i=0}^{s-1} |Df_a^{p_i}(\mu_i^{-1/q})| |\xi_{n_j}(\omega_s)| \le |I^+_{\mu_s \nu_s}|,$$

and third inequality follows from part (ii) of Lemma 5.1 with $C_0 = C_{\mathrm{pe}}^{-1}$.

6.2.2. *The time between two essential returns.* Suppose $\omega \in \mathcal{P}_m$ has an essential return at time m with host interval $I_{\mu\nu}$. The return is followed by a bound period of length $p = p(\mu, \omega) \sim |\mu|$, and then a free period follows. Let $r = r(\mu, \omega)$ be the length of this free period. From the results of §5 it follows that

$$2Q \ge |\xi_{m+p+r}(\omega)| \ge \frac{1}{C_{\mathrm{fp}}} e^{\lambda_0 r} |\xi_{m+p}(\omega)| \ge C e^{\lambda_0 r} |\mu|^{-(1+\beta+\epsilon_0)},$$

and we conclude that

$$r \le C_1 \log |\mu|. \tag{6.3}$$

An essential free return of an ω at time m to a host interval $I_{\mu_0\nu_0}$ might be followed by a sequence of inessential free returns. Suppose they appear in host intervals $I_{\mu_i\nu_i}$, at times m_i, $m = m_0 < m_1 < m_2 < \cdots$. Let $\Omega_i = \xi_{m_i}(\omega)$. From Lemma 5.2 we know that

$$|\Omega_k| \geq 2|\Omega_{k-1}|.$$

Since $\Omega_k \subset I^+_{\mu_k\nu_k}$, for $k \geq 0$, and $I_{\mu_0\nu_0} \supset \Omega_0$, it follows that

$$|\mu_k|^{-(2+1/q+\epsilon_0)} \gtrsim |\Omega_k| \geq 2^k |I_{\mu_0\nu_0}| \gtrsim 2^k |\mu_0|^{-(2+1/q+\epsilon_0)}.$$

So $|\mu_k| \lesssim \hat{\kappa}^k|\mu_0|$ for some $\hat{\kappa} < 1$. In particular, a new essential return appears in finitely many steps. Using this estimate, (6.3) and part (i) of Lemma 5.1, we can estimate the total time u until the next essential free return of ω_0. Let p_k and r_k denote the length of the bound and free period respectively following the return to $I_{\mu_k\nu_k}$.

$$\begin{aligned} u = \sum_{k=0}^{\hat{k}(\mu_0)} (p_k + r_k) &\leq C_0 \sum_{k=0}^{\hat{k}(\mu_0)} (|\mu_k| + \log|\mu_k|) \\ &\leq C_1 \sum_{k=0}^{\hat{k}(\mu_0)} |\mu_k| \leq C_2 \sum_{k=0}^{\infty} \hat{\kappa}^k |\mu_0| \leq C|\mu_0|. \end{aligned}$$

We record this as follows.

LEMMA 6.4. *If $\xi_m(\omega)$ is an essential return to host interval $I_{\mu\nu}$, and if u is the time to the next essential free return we have*

$$u(\mu) < C|\mu|,$$

if δ_0 is small.

6.2.3. *Estimates on the distribution functions of E_i.* Let ω_0 be as in §6.2.1, that is, $\xi_{n_j}(\omega_0)$ is an essential free return with host interval $I_{\mu_0\nu_0}$. Let $\{n_{j+i}(a)\}$ denote the essential free return times which are greater than n_j. $e_j(a)$ is the time it takes for $a \in \omega_0$ to be in an escape situation, thus taking place at time $n_j + e_j(a)$. For any $t \leq n_j$, let

$$\mathcal{A}_t = \{a \in \omega_0 \mid e_j(a) \geq t\},$$

and let

$$\mathcal{B}_{t,s} = \{a \in \omega_0 \mid e_j(a) \geq t, n_{j+s} < n_j + t \leq n_{j+s+1}\}.$$

$\mathcal{B}_{t,s}$ is the set of parameters in ω_0 that have an escape period of length at least t, and which experience exactly s essential returns before time $n_j + t$. Clearly

$$\mathcal{A}_t \subset \bigcup_{s\geq 0} \mathcal{B}_{t,s}.$$

Finally, with $\hat{\mu} = ((\mu_1, \nu_1), \ldots, (\mu_s, \nu_s))$, for $s > 0$, let

$$\mathcal{B}_{t,s,\hat{\mu}} = \{a \in \mathcal{B}_{t,s} \mid \xi_{n_{j+i}}(a) \in I_{\mu_i\nu_i}, i = 1, 2, \ldots, s\}.$$

Note that each $\mathcal{B}_{t,s,\hat{\mu}}$ is a subset of the type ω_s defined in §6.2.1.

From all this it follows that

$$m(\mathcal{A}_t) \le m(\mathcal{B}_{t,0}) + \sum_{s\ge 1} 2^s \sum_{\bar{\mu}\in\Omega} m(\mathcal{B}_{t,s,\hat{\mu}}) \prod_{i=1}^{s} |\mu_i|^{1+\epsilon_0}, \tag{6.4}$$

where $\Omega = \Omega(t,\mu_0) \subset \mathbb{N}^s$ is specified, $\bar{\mu}$ is the projection onto the first component of each double-index in $\hat{\mu}$, and $2^s \prod_{i=1}^{s} |\mu_i|^{1+\epsilon_0}$ represents the number of possible choices of double-indices $((\mu_1,\nu_1),\dots,(\mu_s,\nu_s))$ with $\{|\mu_i|\}$ fixed.

Let $u_i = n_{j+i+1} - n_{j+i}$. Then for $a \in \mathcal{B}_{t,s,\hat{\mu}}$, $t \le n_{j+s+1} - n_j = \sum_{i=0}^{s} u_i$. Because of Lemma 6.4 there is a positive constant κ_0, independent of s, t, $\hat{\mu}$ and ω_0, such that

$$t \le \sum_{i=0}^{s} u_i \le \frac{1}{\kappa_0} \sum_{i=0}^{s} |\mu_i|,$$

and so

$$Y = Y(t) := \kappa_0 t - |\mu_0| \le \sum_{i=1}^{s} |\mu_i|, \quad \forall a \in \mathcal{B}_{t,s,\hat{\mu}}.$$

This means that $\mathcal{B}_{t,s,\hat{\mu}} = \emptyset$ if $\sum_{i=1}^{s} |\mu_i| < Y$. For $s = 0$ this holds with the empty sum equal to zero.

This, and $\mu_i > \Delta_0$, defines Ω in (6.4):

$$\Omega(t,\mu_0) = \left\{ \bar{\mu} = (\mu_1,\dots,\mu_s) \in \mathbb{N}^s \;\middle|\; \sum_{i=1}^{s} |\mu_i| \ge Y, \quad \mu_i > \Delta_0 \right\}.$$

Using (6.2), we can now estimate $m(\mathcal{B}_{t,s})$ for $s > 0$.

$$\begin{aligned} 2^s \sum_{\bar{\mu}\in\Omega} m(\mathcal{B}_{t,s,\hat{\mu}}) \prod_{i=1}^{s} |\mu_i|^{1+\epsilon_0} &\le \sum_{\bar{\mu}\in\Omega} \frac{2^s C_0^s |\mu_0|^{1+\beta+\epsilon_0} |\omega_0|}{\Delta_0^{1+\beta+\epsilon_0}} \prod_{i=1}^{s} \frac{|\mu_i|^{1+\epsilon_0}}{|\mu_i|^{1+1/q-\beta}} \\ &\le \frac{C_1^s |\mu_0|^{1+\beta+\epsilon_0} |\omega_0|}{\Delta_0^{1+\beta+\epsilon_0}} \int_{\substack{\sum x_i \ge Y \\ x_i \ge \Delta_0}} \prod_{i=1}^{s} \frac{1}{x_i^{1/q-\beta-\epsilon_0}} \, dV. \end{aligned} \tag{6.5}$$

To estimate the integral, note that

$$\begin{aligned} &\left\{ (x_1,\dots,x_s) \;\middle|\; \sum_{i=1}^{s} x_i \ge Y \text{ and } x_i \ge \Delta_0,\ i = 1,\dots,s \right\} \\ &\quad \subset \bigcup_{j=1}^{s} \left\{ (x_1,\dots,x_s) \;\middle|\; x_j \ge \frac{Y}{s} \text{ and } x_i \ge \Delta_0,\ i \ne j, 1 \le i \le s \right\}, \end{aligned}$$

so that

$$\int_{\substack{\sum x_i \ge Y \\ x_i \ge \Delta_0}} \prod dV \le \sum_{j=1}^{s} \int_{\substack{x_j \ge Y/s \\ x_i \ge \Delta_0, i \ne j}} \prod dV = s \int_{\substack{x_1 \ge Y/s \\ x_i \ge \Delta_0, i \ne 1}} \prod dV.$$

Thus, by Fubini's Theorem,

$$\int_{\substack{\sum x_i \geq Y \\ x_i \geq \Delta_0}} \prod_{i=1}^{s} \frac{1}{x_i^{1/q-\beta-\epsilon_0}}\, dV \leq s\Bigg(\int_{x\geq Y/s} \frac{1}{x^{1/q-\beta-\epsilon_0}}\, dx \Bigg)\Bigg(\int_{x\geq \Delta_0} \frac{1}{x^{1/q-\beta-\epsilon_0}}\, dx \Bigg)^{s-1},$$

and this is, writing $\sigma := 1/q - \beta - 1 - \epsilon_0 = 1/(2q) - 1 - \epsilon_0$,

$$\leq C_2(\beta, q, \epsilon_0) \frac{s^{\sigma+1}}{B^{s-1}} \frac{1}{Y^{\sigma}},$$

where $B = \Delta_0^{\sigma} \to \infty$ when $\delta_0 \to 0$. Using this, (6.4) and (6.5) we obtain

$$\begin{aligned} m(\mathcal{A}_t) &\leq m(\mathcal{B}_{t,0}) + \frac{C_2|\mu_0|^{1+\beta+\epsilon_0}}{\Delta_0^{1+\beta+\epsilon_0}} \sum_{s=1}^{\infty} \frac{C_1^s s^{\sigma+1}}{B^{s-1}} \frac{1}{Y^{\sigma}} |\omega_0| \\ &\leq m(\mathcal{B}_{t,0}) + \frac{C_3}{\Delta_0^{1+\beta+\epsilon_0}} |\mu_0|^{1+\beta+\epsilon_0} \frac{1}{Y^{\sigma}} |\omega_0|, \end{aligned}$$

with $Y = \kappa_0 t - |\mu_0|$ for some positive constant κ_0.

We now consider the case $s = 0$. $\mathcal{B}_{t,0}$ are those parameters that do not have any free return before time $n_j + t$, and this set is empty for $t > |\mu_0|/\kappa_0$. This proves the following.

LEMMA 6.5. *There are positive constants κ_0 and C such that if $\omega_0 \in \mathcal{P}_k$ has an essential free return $\xi_k(\omega_0)$ with host interval $I_{\mu\nu}$, and if $e(a)$ is the time until the next escape situation for $a \in \omega_0$ then for all $t > |\mu|/\kappa_0$*

$$m\{a \in \omega_0 \mid e(a) \geq t\} \leq \frac{C}{\Delta_0^{1+\beta+\epsilon_0}} \frac{|\mu|^{1+\beta+\epsilon_0}}{(\kappa_0 t - |\mu|)^{\sigma}} |\omega_0|, \tag{6.6}$$

where $\sigma := 1/q - \beta - 1 - \epsilon_0 = 1/2q - 1 - \epsilon_0$.

Our goal is to estimate the distribution functions of E_i on an ω which is at its ith escape situation. Using the previous lemma, we can prove the following.

LEMMA 6.6. *There exists a constant $K(\delta_0)$, $\lim_{\delta_0 \to 0} K(\delta_0) = 0$, such that if $\omega^* \in \mathcal{P}_{k-1}$ has its ith escape return at time k, then for all $0 < t \leq k$, we have*

$$m\{a \in \omega^* \mid E_i(a) \geq t\} \leq K(\delta_0) \frac{1}{t^{\sigma_1}} |\omega^*|,$$

where $\sigma_1 := 1/2q - 1 - 3\epsilon_0$.

Proof. Fix t. Remember that I_μ is subdivided into $\sim |\mu|^{1+\epsilon_0}$ subsets $I_{\mu\nu}$ and that $E_i(a) = 0$ if $\xi_k(a) \notin I_1^*$, where k is the ith escape return for a. We partition the set of all possible index pairs with $|\mu| \geq \Delta_1$:

$$\Omega_1 = \{(\mu, \nu) \mid |\mu| \geq \kappa_0 t/2\}, \quad \Omega_2 = \{(\mu, \nu) \mid |\mu| < \kappa_0 t/2\}.$$

Let $\omega^*_{\mu\nu} = \xi_k^{-1}(I_{\mu\nu} \cap \xi_k(\omega^*))$ (or $\omega^*_{\mu\nu} = \xi_k^{-1}(I^+_{\mu\nu} \cap \xi_k(\omega^*))$ if $I_{\mu\nu}$ is one of the end-intervals of $\xi_k(\omega^*)$). Then

$$\begin{aligned} m\{a \in \omega^* \mid E_i \geq t\} &= \sum_{\mu,\nu \text{ s.t. } |\mu| \geq \Delta_1} m\{a \in \omega^*_{\mu\nu} \mid E_i \geq t\} \\ &= \sum_{\Omega_1} m\{a \in \omega^*_{\mu\nu} \mid E_i \geq t\} + \sum_{\Omega_2} m\{a \in \omega^*_{\mu\nu} \mid E_i \geq t\}, \end{aligned}$$

and, applying Lemma 6.5 to the second sum, this is

$$\leq \sum_{\Omega_1} |\omega^*_{\mu\nu}| + \frac{C}{\Delta_0^{1+\beta+\epsilon_0}} \sum_{\Omega_2} \frac{|\mu|^{1+\beta+\epsilon_0}}{(\kappa_0 t - |\mu|)^\sigma} |\omega^*_{\mu\nu}|.$$

Using Lemma 5.3, saying that $|\omega^*_{\mu\nu}|/|\omega^*| \sim |I_{\mu\nu}|/|\xi_k(\omega^*)|$, and the definition of Ω_2, we see that this is

$$\begin{aligned}
&\lesssim \sum_{|\mu|\geq \max\{\Delta_1, \kappa_0 t/2\}} |\mu|^{1+\epsilon_0} \frac{|\mu|^{-(2+1/q+\epsilon_0)}}{\delta_0} |\omega^*| \\
&\qquad + \frac{1}{\Delta_0^{1+\beta+\epsilon_0}} \sum_{\Delta_1 \leq |\mu| < \kappa_0 t/2} |\mu|^{1+\epsilon_0} \frac{|\mu|^{1+\beta+\epsilon_0}}{t^\sigma} \frac{|\mu|^{-(2+1/q+\epsilon_0)}}{\delta_0} |\omega^*|,
\end{aligned}$$

and since $\beta = 1/2q$ and $\delta_0 = \Delta_0^{-1/q}$, this is

$$\begin{aligned}
&\leq \Delta_0^{1/q} \sum_{|\mu|\geq \max\{\Delta_1, \kappa_0 t/2\}} |\mu|^{-(1+1/q)} |\omega^*| + \frac{\Delta_0^{1/2q-1-\epsilon_0}}{t^\sigma} \sum_{|\mu|\geq \Delta_1} |\mu|^{-(1/2q-\epsilon_0)} |\omega^*| \\
&\leq \frac{\Delta_0^{1/q}}{\Delta_1^{1+1/2q+3\epsilon_0}} \sum_{|\mu|\geq \kappa_0 t/2} |\mu|^{-(1/2q-3\epsilon_0)} |\omega^*| + \frac{\Delta_0^{1/2q-1-\epsilon_0}}{t^\sigma} \sum_{|\mu|\geq \Delta_1} |\mu|^{-(1/2q-\epsilon_0)} |\omega^*| \\
&\lesssim \left(\frac{\Delta_0^{1/q}}{\Delta_1^{1+1/2q+3\epsilon_0}} + \frac{\Delta_0^{1/2q-1-\epsilon_0}}{\Delta_1^{1/2q-1-\epsilon_0}} \right) t^{-\sigma_1} |\omega^*|.
\end{aligned}$$

This proves the lemma, since for each ϵ_0, we have that $\Delta_0^{1/q}$ is of smaller order of magnitude than $\Delta_1^{1+1/2q+3\epsilon_0}$. □

6.2.4. *Adjusting to the free period assumption.* In this section we estimate the measure of the set deleted from A_{n-1} in order to fulfil the FA_n condition. First we delete the set B'_n consisting of those $a \in A_{n-1}$ such that a belongs to an $\omega \in \mathcal{P}_{n-1}$ having an escape at time $n_k = n$ and such that $E_k(a) \geq n$. Let

$$A''_n = A_{n-1} \setminus B'_n. \tag{6.7}$$

From Lemma 6.6 it follows that

$$|B'_n| \leq K(\delta_0) n^{-\sigma_1} |A_{n-1}|, \tag{6.8}$$

where $\lim_{\delta_0 \to 0} K(\delta_0) = 0$.

We may assume that we have performed this operation at all previous times, so in what follows we are allowed to assume that an escape period staring at some time m, always terminates before time $2m$. All previous estimates of this section then hold, and we can apply Lemma 6.6 at all escape situations up until time n.

Notation. In the following, $\{n_i\}$ will denote a sequence of escape times (not free return times, as earlier in the text) and s will denote the number of escapes considered.

PROPOSITION 6.2. *For each* $q < 1/8$ *and for* δ_0 *small enough, there is a* $K'(\delta_0)$*,* $K'(\delta_0) \to 0$ *as* $\delta_0 \to 0$*, such that for all* n*,*

$$m\{a \in A''_n \mid a \notin FA_n(\delta_1)\} < K'(\delta_0)n^{-(\sigma_1-1)}|A_{n-1}|.$$

Proof. Consider an $a \in A''_n$. For such an a a sequence of escape times $n_i \leq n$ and integers $E_i(a) \geq 0$ are defined for $1 \leq i \leq \bar{s}(a) \ll n$ according to Definition 7 and Definition 8. $\bar{s}(a)$ is the number of escape situations for a up until time n. Positive E_i's are the length of escape periods following escape returns to $(-\delta_1, \delta_1)$. Since each bound period originating from a return in I_1^* is contained in an escape period corresponding to a positive E_i, it suffices to prove that

$$m\{a \in A''_n \mid T(a,n) \geq n/2\} < K'(\delta_0)n^{-(\sigma_1-1)}|A_{n-1}|,$$

where

$$T(a,n) = \sum_{n_i(a) \leq n} E_i(a).$$

Let $M = M(a_0, \delta_1)$ be the shortest possible value of a bound period for $a \in [a_0, 2]$ following a return to $(-\delta_1, \delta_1)$. It is clear that we have a uniform bound on $\bar{s}(a)$: there exists a positive integer $\hat{s} \leq n/M$, independent of a, such that

$$\bar{s}(a) \leq \hat{s}.$$

For simplicity we define $E_i(a)$ also for $i > \bar{s}(a)$:

$$E_i(a) = \begin{cases} E_i(a), & \text{for } i \leq \bar{s}(a) \\ 0, & \text{for } i > \bar{s}(a). \end{cases}$$

Then $T(a,n) = T_{\hat{s}}(a) = \sum_{i=1}^{\hat{s}} E_i(a)$ is defined for all a. We may also consider T_s for $s < \hat{s}$.

For each $s \leq \hat{s}$ we form a partition on A''_n. Write

$$A''_n = \{a \in A''_n \mid \bar{s}(a) \geq s\} \cup \{a \in A''_n \mid \bar{s}(a) < s\}.$$

It is not hard to see that we can partition $\{a \in A''_n \mid \bar{s}(a) \geq s\}$ into subsets ω^* such that:

- $n_i(a)$ is constant for $i = 1, 2, \ldots, s$ on each ω^*;
- for each ω^* and each $i < s$, $\xi_{n_i}(\omega^*)$ is either outside I_0^* or contained in some host interval $I_{\mu\nu}$;
- $|\xi_{n_s}(\omega^*)| \geq \delta_0$.

Likewise we partition $\{a \in A''_n \mid \bar{s}(a) < s\}$ into sets ω^* such that:

- $\bar{s}(a)$ is constant on each ω^*;
- $n_i(a)$ is constant for $i = 1, 2, \ldots, \bar{s}(a)$ on each ω^*;
- for each ω^* and each $i \leq \bar{s}(a)$, $\xi_{n_i}(\omega^*)$ is either outside I_0^* or contained in some host interval $I_{\mu\nu}$.

This gives us a partition $\hat{\mathcal{P}}_s$, $A''_n = \bigcup \omega^*$, with the following additional properties:

- each $\omega^* \in \mathcal{P}_k$ for some $k \leq n$;
- $T_{s-1}(a) = \sum_{j=1}^{s-1} E_i(a)$ is constant on each ω^*.

Also note that:

- $T_1(a) = E_1(a) = 0$ for all a since the first escape situation, by definition, is the first return at time m_0 to $I_{-\Delta_0} \cup I_{\Delta_0}$.

We are now in position to carry out our large deviation estimate. The idea of how to handle sums of random variables that do not have exponential moments is inspired by Nagaev's article [**7**].

Fix a positive integer τ (which we will take to be $n/2$ in the end), and a sufficiently small positive real number $\theta = \theta(q)$. Let

$$\tilde{E}_i(a) = \begin{cases} E_i(a), & \text{if } E_i(a) \le \theta\tau \\ 0, & \text{if } E_i(a) > \theta\tau. \end{cases}$$

Define

$$\tilde{T}_s(a) = \sum_{i=1}^{s} \tilde{E}_i(a).$$

Then, with $\{a \mid \dots\}$ as a shorthand for $\{a \in A_n'' \mid \dots\}$,

$$\{a \mid T_s > \tau\} \subset \{a \mid \tilde{T}_s > \tau\} \cup \bigcup_{i=1}^{s} \{a \mid E_i > \theta\tau\}$$

so

$$m\{a \mid T_s > \tau\} \le m\{a \mid \tilde{T}_s > \tau\} + \sum_{i=1}^{s} m\{a \mid E_i > \theta\tau\}. \tag{6.9}$$

We will show that the right-hand side is dominated by the second term, and is therefore $\lesssim s n^{-\sigma_1} |A_n''| < n^{-(\sigma_1 - 1)} |A_n''|$, according to Lemma 6.6.

For all $h > 0$ we have, by Chebychev's inequality, that

$$m\{a \in A_n'' \mid \tilde{T}_s > \tau\} \le e^{-h\tau} \int_{A_n''} e^{h \sum_1^s \tilde{E}_i(a)} \, da. \tag{6.10}$$

Using the partition $A_n'' = \bigcup \omega^*$ and using the fact that $\tilde{T}_{s-1}$ is constant on each ω^* we have

$$\int_{A_n''} e^{h \sum_1^s \tilde{E}_i(a)} \, da = \sum_{\omega^*} (e^{h \tilde{T}_{s-1}(\omega^*)} |\omega^*|) \left(\frac{1}{|\omega^*|} \int_{\omega^*} e^{h \tilde{E}_s(a)} \, da \right). \tag{6.11}$$

To estimate $\int_{\omega^*} e^{h\tilde{E}_s(a)} \, da$ we need a lemma.

LEMMA 6.7. *Let J be a compact, measurable set and let $g : J \longrightarrow \mathbb{R}$ be a positive, measurable function with $\|g\|_\infty < \infty$. Let $b > 0$, $r \ge 2$, $B > 0$, μ and A_r be numbers such that:*

(i) $m\{a \in J \mid g(a) > b\} = 0$;

(ii) $(1/|J|) \int_J g(a) \, da \le \mu$;

(iii) $(1/|J|) \int_J (g(a))^2 \, da \le B$;

(iv) $(1/|J|) \int_J (g(a))^r \, da \le A_r$.

Then the following holds:

(i) *for any* h, $0 < h \le r/b$,

$$\frac{1}{|J|}\int_J e^{hg(a)}\,da \le \exp\left[h\mu + e^r B\frac{h^2}{2}\right];$$

(ii) *for any* $h > r/b$,

$$\frac{1}{|J|}\int_J e^{hg(a)}\,da \le \exp\left[h\mu + e^r B\frac{h^2}{2} + \frac{A_r(e^{hb}-1-hb)}{b^r}\right].$$

This is a reformulation of Lemma 1.4 in [**7**]. The proof is not very difficult.

We now want to apply this lemma to the functions E_i. For $1 \le u < \sigma_1 - 1$, define

$$M_u = \sum_{t=1}^{\infty} K(\delta_0)t^{u-\sigma_1},$$

where $K(\delta_0)$ is the constant from Lemma 6.6.

LEMMA 6.8. *For all* $q < 1/8$, *for all* $\omega^* \in \hat{\mathcal{P}}_i$, *and for all* r, $2 \le r < \sigma_1 - 1$, *the following is true for sufficiently small* δ_0 *and all* $h > 0$*:*

$$\frac{1}{|\omega^*|}\int_{\omega^*} e^{h\tilde{E}_i(a)}\,da \le e^{\Psi(h)},$$

where

$$\Psi(h) = \begin{cases} hM_1 + e^r\dfrac{h^2}{2}M_2, & \text{if } h \le \dfrac{r}{\theta\tau} \\[2ex] hM_1 + e^r\dfrac{h^2}{2}M_2 + \dfrac{e^{h\theta\tau}-1-h\theta\tau}{(\theta\tau)^r}M_r, & \text{if } h > \dfrac{r}{\theta\tau}. \end{cases}$$

Proof of Lemma 6.8. According to the definition of $\tilde{E}_i$ and Lemma 6.6,

$$\frac{1}{|\omega^*|}\int_{\omega^*}\tilde{E}_i^r(a)\,da \le \frac{1}{|\omega^*|}\sum_{t=0}^{n_i} t^r m\{a \mid E_i \ge t\} \le \sum_{t=1}^{\infty} K(\delta_0)t^r\frac{1}{t^{\sigma_1}} = M_r.$$

Since $\tilde{E}_i(a) \le \theta\tau$ for all a, we may apply Lemma 6.7 with $J = \omega^*$, $b = \theta\tau$, $\mu = M_1$, $B = M_2$ and $A_r = M_r$. □

Because of (6.10), by repeated applications of (6.11) and Lemma 6.8 it follows that for any $s > 1$:

$$\begin{aligned} m\{a \in A_n'' \mid \tilde{T}_s > \tau\} &\le e^{-h\tau}\sum_{\omega^*\in\hat{\mathcal{P}}_s}(e^{h\tilde{T}_{s-1}(\omega^*)}|\omega^*|)\left(\frac{1}{|\omega^*|}\int_{\omega^*}e^{h\tilde{E}_s(a)}\,da\right) \\ &\le e^{-h\tau}e^{\Psi(h)}\int_{A_n''}e^{h\tilde{T}_{s-1}(a)}\,da \\ &\le e^{-h\tau}\cdot e^{s\Psi(h)}|A_n''| = e^{s\Psi(h)-h\tau}|A_n''|. \end{aligned}$$

We also used the fact that $T_1 \equiv 0$.

For

$$h = \hat{h} := \frac{1}{\theta\tau} \log\left(\frac{(\theta\tau)^r}{sM_r} + 1\right)$$

we have

$$s\Psi(\hat{h}) - \hat{h}\tau \le \hat{h} s M_1 + e^r \frac{\hat{h}^2}{2} s M_2 + \frac{e^{\hat{h}\theta\tau} - 1 - \hat{h}\theta\tau}{(\theta\tau)^r} s M_r - \hat{h}\tau$$

and since $((e^{\hat{h}\theta\tau} - 1 - \hat{h}\theta\tau)/(\theta\tau)^r) s M_r < 1$, this is

$$\begin{aligned} &< -\hat{h}\left(\tau - sM_1 - e^r \frac{\hat{h}}{2} s M_2\right) + 1 \\ &= -\frac{1}{\theta}\log\left(\frac{(\theta\tau)^r}{sM_r} + 1\right)\left(1 - \frac{s}{\tau}\left(M_1 + e^r \frac{\hat{h}}{2} M_2\right)\right) + 1. \end{aligned}$$

M_1, M_2 and M_r are absolute constants, and for fixed θ with $\tau = n/2$ we see that $\hat{h}$ is uniformly small for large n. Thus, the expression $(M_1 + e^r(\hat{h}/2)M_2)$ is bounded. Choosing $\tau = n/2$ and remembering that $s \le n/M$, we see that we can get

$$1 - \frac{s}{\tau}\left(M_1 + e^r \frac{\hat{h}}{2} M_2\right) \ge 1 - \frac{2}{M}\left(M_1 + e^r \frac{\hat{h}}{2} M_2\right)$$

arbitrarily close to one by taking δ_0 small enough, since $M \to \infty$ as $\delta_0 \to 0$. Thus,

$$s\Psi(\hat{h}) - \hat{h}\tau \le -\frac{1}{2\theta}\log\left(\frac{(\theta\tau)^r}{sM_r} + 1\right) + 1$$

for sufficiently small δ_0. With $\tau = n/2$ and $s \le n/M$ we have

$$e^{s\Psi(\hat{h}) - \hat{h}\tau} \le e\left(\frac{sM_r}{(\theta\tau)^r + sM_r}\right)^{1/2\theta} \le e\left(\frac{2^r M_r}{M\theta^r}\right)^{1/2\theta} \frac{1}{n^{(r-1)/2\theta}} = K(\delta_0) n^{-(r-1)/2\theta},$$

where $K(\delta_0) \to 0$ as $\delta_0 \to 0$. Since $r \ge 2$, this is

$$\ll n^{-\sigma_1 + 1},$$

if $\theta(q)$ is sufficiently small, and so

$$m\{a \in A_n'' \mid \tilde{T}_s > n/2\} \ll n^{-\sigma_1+1} |A_n''|.$$

From (6.9) and Lemma 6.6 we therefore have

$$\begin{aligned} m\left\{a \in A_n'' \mid T_s > \frac{n}{2}\right\} &\le 2sK(\delta_0)\left(\frac{2}{\theta}\right)^{\sigma_1} n^{-\sigma_1} |A_n''| \\ &\le K'(\delta_0) n^{-\sigma_1+1} |A_{n-1}|, \end{aligned}$$

since θ is chosen independently of δ_0. In particular, this holds for $s = \hat{s}$. This completes the proof of Proposition 6.2. □

Let $A_n''' = \{a \in A_{n-1} \mid T(a, n) \le n/2\}$. Then $A_n'' \cap A_n''' \subset FA_n \cap A_{n-1}$ and we have already constructed $A_n' \subset BA_n \cap A_{n-1}$. Finally let

$$A_n = A_n' \cap A_n'' \cap A_n'''.$$

From Lemma 6.2, the estimate (6.8) and Proposition 6.2, part (iv) of Proposition 6.1 follows, and this finishes the proof of Proposition 6.1.

7. *The properties of the set A and the first part of the main theorem*

Let

$$A := \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=m_0}^{\infty} A_n.$$

From part (iv) of Proposition (6.1) it follows that

$$|A| \geq \prod_{n=m_0}^{\infty} (1 - n^{-\gamma})|A_1|,$$

for some $\gamma > 1$. So $|A| > 0$, and the first statement of the main theorem follows from parts (i)–(iii) of Proposition 6.1. Choosing a close to two means choosing m_0 large. From this, it is not hard to see that two is a density point of A.

8. *Invariant measures for f_a, $a \in A$*

In this section we prove that if $f = f_{a,q}$ for some $a \in A$, then f admits a finite invariant measure that is absolutely continuous with respect to Lebesgue measure. We assume throughout this section that $q < 1/8$.

Let m_0 denote normalized Lebesgue measure on $J_0 := I_{-\Delta_0,1} \cup I_{\Delta_0,1}$, the union of the two end-intervals of our partition on I_0^*. Construct the averages of the push-forward of m_0 under f:

$$m_n := \frac{1}{n} \sum_{k=0}^{n-1} f_*^k(m_0).$$

Then any infinite subsequence of $\{m_n\}$ has a convergent subsequence (in the weak* topology), converging to an invariant Borel probability measure m^*, which in turn can be decomposed as

$$m^* = m^*_{\text{ac}} + m^*_{\text{sing}}, \quad m^*_{\text{ac}} \ll \text{Leb.}, \quad m^*_{\text{sing}} \perp \text{Leb.}$$

It is a standard fact that if f and f^{-1} map sets of measure zero to sets of measure zero, which is true for our f, then the absolutely continuous part of an invariant measure is itself invariant. For the convenience of the reader, we formulate this as a proposition below, and give a short proof. Thus, it suffices to find a subsequence of $\{m_n\}$ converging to an invariant measure with non-zero absolutely continuous part. We will choose a subsequence such that the contributions from the escape situations (defined for iterates of f) give us a non-vanishing absolutely continuous part.

PROPOSITION 8.1. *Let m be a probability measure on a measure space X and let $g : X \to X$ be measurable such that $m(E) = 0$ implies $m(g(E)) = m(g^{-1}(E)) = 0$. If m^* is a finite invariant measure for g with Lebesgue decomposition $m^* = m^*_{\text{ac}} + m^*_{\text{sing}}$, $m^*_{\text{ac}} \ll m$ and $m^*_{\text{sing}} \perp m$, then m^*_{ac} and m^*_{sing} are invariant under g.*

Proof. There exists a set $Y \subset X$ such that $m(Y) = 0$ and $m^*_{\text{sing}}(Y) = m^*_{\text{sing}}(X)$. Let

$$Y' = \bigcup_{i=0}^{\infty} g^{-i}\left(\bigcup_{j=0}^{\infty} g^j(Y)\right).$$

Then Y' is a totally invariant set and so is $Y'' = X \setminus Y'$. Note that $m^*_{\mathrm{ac}}(Y') = 0$. Using this, it is easily checked that $m^*_{\mathrm{ac}}(g^{-1}(E)) = m^*_{\mathrm{ac}}(E)$ for any measurable set E. It follows that m^*_{sing} is also invariant. □

For an interval $J \subset \hat{I}_\mu$, $f^i(J)$ is in a bound period for $i = 1, \dots, p(\mu)-1$ according to Definition 3. If $t \geq p(\mu)$ is minimal such that $f^t(J) \cap I_0^* \neq \emptyset$, we call the iterates $f^i(J)$, $i = p, \dots, t-1$, free. t is called a free return time. This could be essential or inessential, according to whether $f^i(J)$ covers some $I_{\mu'\nu'}$ or not.

We now form a sequence $\{\mathcal{R}_n\}$ of partitions of I_0^* and a sequence of free return times $\{t_k(x)\}$ with the following properties:

- t_k is constant on elements of $\mathcal{R}_m$ if $m \geq k$;
- if $\omega \in \mathcal{R}_n$ and $f^{t_k}(\omega)$, $t_k < n$, is a free return, then either $f^{t_k}(\omega) \subset$ some $I^+_{\mu\nu}$, or $f^{t_k}(\omega) \cap I_0^* = \emptyset$. In the latter case, $f^{t_k}(\omega)$ contains an interval of the form $I_{\pm\Delta_0-1,\nu}$, adjacent to I_0^*;
- if $\omega \in \mathcal{R}_n$ and $t_k = n$ is a free return for ω, then $I_{\mu\nu} \subset f^{t_k}(\omega) \subset I^+_{\mu\nu}$ for some $I_{\mu\nu} \subset I_0^*$.

Let

$$\mathcal{R}_0 = \{I_{\mu\nu}\}_{|\mu| \geq \Delta_0}.$$

We consider $t_0 = 0$ a free return for each element of $\mathcal{R}_0$. Suppose $\mathcal{R}_{n-1}$ is defined with the above properties, and pick an $\omega \in \mathcal{R}_{n-1}$. If $f^n(\omega)$ is in a bound period, does not intersect I_0^* or has an inessential return at time n, ω becomes an element of $\mathcal{R}_n$. In case of a free return $\mathcal{R}_n$ refines $\mathcal{R}_{n-1}$ on ω according to host intervals at time n. We omit the details.

We now define the free return times $t_k(x)$ for $k > 0$. Suppose $t_{k-1}(x)$ is defined and let $\omega_n(x)$ denote the element of $\mathcal{R}_n$ containing x. If $f^{t_{k-1}}(\omega_{t_{k-1}}(x))$ has host interval $I_{\mu\nu}$, $t_k(x)$ is the smallest positive integer which is greater than or equal to $t_{k-1} + p(\mu)$ such that t_k is a free return for $\omega_{t_{k-1}}(x)$.

A free return time is an escape situation for x if

$$|f^{t_i}(\omega_{t_{i-1}}(x))| \geq \delta_0.$$

This gives a sequence of escape times $\{n_s(x)\}_{s=0}^\infty$, a subsequence of $\{t_k(x)\}$, where $n_0 = t_0 = 0$ by definition. Also define

$$e(x) := \begin{cases} n_1(x), & \text{for } x \in I_0^* \\ M, & \text{for } x \in I \setminus I_0^*, \end{cases}$$

where M is the first return time to I_0^* for $I_{\pm(\Delta_0-1),\nu}$ a neighbouring interval of I_0^*. It is easily shown that $|f^M(I_{\pm(\Delta_0-1),\nu})| > \delta_0$ (cf. Lemma 6.3). Note that $n_s(x) - n_{s-1}(x) = e(f^{n_{s-1}}(x))$.

Let $\hat{\mathcal{R}}_s$ be the refinement of $\mathcal{R}_0$ into escaping intervals on the sth level:

$$\omega \in \hat{\mathcal{R}}_s, \quad n_s|\omega = k \iff \omega \in \mathcal{R}_{k-1}, \quad k = n_s(\omega).$$

As we will see, $n_s(x)$ is finite for almost all x, so $\hat{\mathcal{R}}_s$ is well-defined up to measure zero. Now define $\hat{m}_n^+$ and $\hat{m}_n^-$ to be the restrictions to $I_{\Delta_0,1}$ and $I_{-\Delta_0,1}$ respectively of

$$\frac{1}{n} \sum_{s=0}^{n-1} \sum_{\substack{\omega \in \hat{\mathcal{R}}_s \\ n_s(\omega) < n}} f_*^{n_s}(m_0|\omega).$$

Following [**5**] we will show that in the limit, at least one of $\hat{m}_n^+$ and $\hat{m}_n^-$ gives a non-vanishing, absolutely continuous contribution to m^*. First we state a phase space analogue of the distortion estimate in Lemma 5.3.

LEMMA 8.1. *There is constant C such that if $\omega \subset I_0^*$ is an interval such that $f^{t_j}(\omega)$ is contained in some $I^+_{\mu_j\nu_j}$ at each free return $t_j < N$, then*

$$\frac{1}{C} \le \left|\frac{Df^N(x)}{Df^N(y)}\right| \le C, \quad \forall x, y \in \omega.$$

Proof. This is proved in the same way as Lemma 5.3. The proof of Lemma 5.3 is in fact a perturbed version of the proof of this lemma. □

Using the previous lemma we show that the measures $\hat{m}_n^\pm$ are uniformly dominated by Lebesgue measure. Let E be a measurable subset of I, and define $E^+ := E \cap I_{\Delta_0,1}$. Then

$$\begin{aligned}
\hat{m}_n^+(E) &= \frac{1}{n}\sum_{s=0}^{n-1} \sum_{\substack{\omega\in\hat{\mathcal{R}}_s \\ n_s(\omega)<n}} f_*^{n_s}(m_0|\omega)(E^+) \\
&\le C\frac{1}{n}\sum_{s=0}^{n-1} \sum_{\substack{J_0\supset\omega\in\hat{\mathcal{R}}_s \\ n_s(\omega)<n}} |f^{-n_s}(E^+)\cap\omega| \\
&\le C_1\frac{1}{n}\sum_{s=0}^{n-1} \sum_{\substack{J_0\supset\omega\in\hat{\mathcal{R}}_s \\ n_s(\omega)<n}} \frac{|E^+\cap f^{n_s}(\omega)|}{|f^{n_s}(\omega)|}|\omega| \\
&\le \frac{C_2}{\delta_0}|I_{\Delta_0,1}||E^+| \le C_3|E|.
\end{aligned}$$

The constants do not depend on n. Therefore, we have proved the following lemma.

LEMMA 8.2. *Each weak* accumulation point of the sequences $\{\hat{m}_n^+\}$ and $\{\hat{m}_n^-\}$ is absolutely continuous with respect to Lebesgue measure.*

We proceed to show that $\hat{m}_n^+ + m_n^-$ has total mass larger than some fixed positive constant for infinitely many n. First, we state the following lemma, remembering that $\sigma := 1/q - \beta - 1 - \epsilon_0 = (1/2q) - 1 - \epsilon_0$.

LEMMA 8.3. *There are positive constants κ and C such that if $I_{\mu\nu} \subset J \subset I^+_{\mu\nu}$, then*

$$m\{x \in J \mid e(x) \ge t\} \le C\frac{|\mu|^{1+\beta+\epsilon_0}}{(\kappa t - |\mu|)^\sigma} m(J), \quad \forall t > \frac{|\mu|}{\kappa}.$$

Proof. This is proved in the same manner as Lemma 6.5, see §§6.2.1–6.2.3. Since we are now considering a fixed map $f = f_a$ with $\log|Df^n(Q)| \ge \lambda n$ for all n, the statement holds for all $t > |\mu|/\kappa$. □

The next lemma gives a bound on the average length of escape periods.

LEMMA 8.4. *There is a constant K such that for all $s \geq 0$,*

$$\int_{I_0^*} (n_{s+1}(x) - n_s(x))\, dx \leq K.$$

Proof. It suffices to give a uniform upper bound on

$$\frac{1}{|\omega|} \int_{\omega} (n_{s+1}(x) - n_s(x))\, dx,$$

where $\omega \in \hat{\mathcal{R}}_s$. Let $J = f^{n_s}(\omega)$. Using Lemma 8.1, remembering that $n_{s+1} - n_s = e \circ f^{n_s}$, and taking $y = f^{n_s}(x)$ as a new variable, we get

$$\frac{1}{|\omega|} \int_{\omega} (n_{s+1}(x) - n_s(x))\, dx \leq C \frac{1}{|J|} \int_J e(y)\, dy.$$

Now $\int_J \leq \int_{I_0^*} + \int_{J \setminus I_0^*}$ and

$$\int_{J \setminus I_0^*} e\, dy = M|J \setminus I_0^*| < M.$$

Let us estimate $\int_{I_0^*} e\, dy$:

$$\begin{aligned} \int_{I_0^*} e(y)\, dy &= \sum_{t=1}^{\infty} t \cdot m\{x \in I_0^* \mid e(x) = t\} \\ &\leq \sum_{t=1}^{\infty} t \sum_{\substack{\mu,\nu \\ \Delta_0 \leq |\mu| \leq \kappa t}} m\{x \in I_{\mu\nu} \mid e(x) \geq t\}. \end{aligned}$$

We split the inner sum into two sub-sums,

$$\sum_{\substack{\mu,\nu \\ \Delta_0 \leq |\mu| \leq \kappa t}} = \sum_{\substack{\mu,\nu \\ \Delta_0 \leq |\mu| \leq \kappa t/2}} + \sum_{\substack{\mu,\nu \\ |\mu| \geq \kappa t/2}},$$

and estimate them separately. Using Lemma 8.3 we obtain

$$\begin{aligned} \sum_{\substack{\mu,\nu \\ \Delta_0 \leq |\mu| \leq \kappa t/2}} m\{x \in I_{\mu\nu} \mid e(x) \geq t\} &\lesssim \sum_{\Delta_0 \leq |\mu| \leq \kappa t/2} |\mu|^{1+\epsilon_0} \frac{|\mu|^{1+\beta+\epsilon_0}}{t^{\sigma}} |I_{\mu\nu}| \\ &\leq t^{-\sigma} \sum_{|\mu| \geq \Delta_0} |\mu|^{\beta - 1/q + \epsilon_0} \\ &\leq C t^{-\sigma}. \end{aligned}$$

In the second sub-sum we apply the trivial estimate

$$m\{x \in I_{\mu\nu} \mid e(x) \geq t\} \leq |I_{\mu\nu}|,$$

and so

$$\sum_{\substack{\mu,\nu \\ |\mu| \geq \kappa t/2}} m\{x \in I_{\mu\nu} \mid e(x) \geq t\} \leq \sum_{|\mu| > \kappa t/2} |\mu|^{-1-1/q} \lesssim t^{-1/q}.$$

It follows that $\int_{I_0^*} e\, dy$ is finite, and this gives us a uniform bound on $\int_J e\, dy$ when J is as above. Since $|J| \geq \delta_0$, this proves the lemma. □

Recall that we want to conclude that either $\{\hat{m}_n^+\}$ or $\{\hat{m}_n^-\}$ has a subsequence converging to a measure with positive total mass. This is the content of the next lemma. Let

$$\mathcal{R}_{k,J_0} = \bigcup \{\omega \in \mathcal{R}_k \mid f^k(\omega) = I_{\Delta_0,1} \text{ or } I_{-\Delta_0,1}, k = n_i|\omega, \text{ for some } i\}.$$

LEMMA 8.5. *There exists a constant $K_1 > 0$ such that*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} m(\mathcal{R}_{k,J_0}) \geq K_1.$$

Proof. From Lemma 8.4 we know that

$$\int_{I_0^*} n_s \, dx \leq Ks.$$

This implies that

$$m\{x \in I_0^* \mid n_1 < n_2 < \cdots < n_s \leq 2Ks\} \geq \frac{|I_0^*|}{2},$$

and it follows that

$$\sum_{k=1}^{2Ks} m\{x \in I_0^* \mid k = n_i(x) \text{ for some } i\} \geq s\frac{|I_0^*|}{2}. \tag{8.1}$$

If k is an escape situation for ω', there exists an $\omega \subset \omega'$ such that $f^k(\omega) = I_{\Delta_0,1}$ or $I_{-\Delta_0,1}$. Because of Lemma 8.1,

$$\frac{|\omega|}{|\omega'|} \gtrsim |I_{\Delta_0,1}|,$$

and this combined with (8.1) gives us the proof of the lemma. □

Proof of part (ii) of the main theorem. From the previous lemma it follows that

$$(\hat{m}_n^+ + \hat{m}_n^-)(I) \geq \frac{K_1}{2},$$

for infinitely many n. Combined with Lemma 8.2 and the discussion preceding Proposition 8.1, this proves the second statement in the main theorem. □

9. *A toy model for $q \in (0, 1)$*

We consider a simplified model of the free returns of parameters for hypothetical unimodal families $f_{a,q}$ with all the properties of our explicit maps, (1.1) and (1.2), and with some of the extra simplifying features. Given $q < 1$, we choose α and β such that $1 < \alpha < \beta < 1/q$. To begin with, we assume the following.

A1. Distortion is negligible in the situations considered.

Thus, there is no need for the subdivision of each I_μ into $I_{\mu\nu}$-intervals. Let $\omega \in \mathcal{P}_m$ have a free return at time m, $\xi_m(\omega) = I_\mu$, and suppose n is the next free return for $a \in \omega$. We then assume that:

A2. $n - m = p(\mu, \omega) = |\mu|$;

A3. If $\xi_m(\omega) = I_\mu$ and n is the next free return for $a \in \omega$, then $|\xi_n(\omega)| \sim 1$.

Assumptions A2 and A3 imply that each free return is essential, and even an escape return. Let $\{n_s(a)\}$ denote the set of free return times for a, let $\mu_s = \mu_s(a)$ be the host-interval index for a at time n_s. The length of the sth escape-period is then just

$$e_s = n_{s+1} - n_s = p(\mu_s) = |\mu_s|. \tag{9.1}$$

CLAIM. *With these extra assumptions, the statements of the main theorem hold with* $q_0 = 1$.

Recall the definitions of BA_n (Definition 2) and FA_n (Definition 4) in §4. If $A = \bigcap_{n=1}^{\infty} A_n$, where $A_n = [a_0, 2) \cap BA_n \cap FA_n$, for some a_0 sufficiently close to two, then parts (i)–(iii) of the main theorem hold for all $a \in A$. We will argue that under assumptions A1–A3, A will have positive Lebesgue measure, provided $q < 1$. The assumptions hold, modulo constants, in our explicit families if intervals $\xi_n(\omega)$ have length approximately one at each essential free return. This is not true in general, but we conjecture that it is generic in measure sense, that is, it holds for all parameter-intervals except for a small exceptional set that should be excluded according to a third parameter-selection rule.

9.1. *BA_n-exclusion.* If $\omega \in \mathcal{P}_{n-1}$ has a free return at time n, then $|\xi_n(\omega)| \sim 1$ by assumption A3. From each such ω we delete

$$\xi_n^{-1}(\xi_n(\omega) \cap (-n^{-\alpha}, n^{-\alpha})).$$

It follows that at time n we delete at most a fraction $1/n^{\alpha}$ of the measure in parameter space because of the approach-rate condition.

9.2. *FA_n-exclusion.* Let e_s, E_s and T_s be defined as in Defintions 7 and 8 in §6.2. If $\omega \in \mathcal{P}_{n_s-1}$ has its sth free return at time n_s, then this is an escape situation by assumption, and from (9.1) it follows that

$$m\{a \in \omega \mid e_s(a) = t\} = |I_t||\omega| \sim \frac{1}{t^{1+1/q}}|\omega|, \tag{9.2}$$

and that

$$m\{a \in \omega \mid e_s(a) \geq t\} = |\hat{I}_t||\omega| \sim \frac{1}{t^{1/q}}|\omega|. \tag{9.3}$$

We estimate the fraction of parameters deleted at time n because of the FA_n condition with the same type of large deviation argument as in §6.2.4. Because of (9.2), the functions E_i have moments of order r iff $r < 1/q$. In the argument in §6.2.4 we used the existence of second moments, but it suffices to have first order moments. This only effects the explicit form of the 'error-term' $m\{a \mid \tilde{T}_s > \tau\}$ in (6.9). See Theorem 1.2 and Theorem 1.3 in [**7**]. We want to estimate

$$m\{a \in A_{n-1} \mid T_s(a) > n/2\},$$

where $T_s(a) = \sum_{i=1}^{s} E_i(a)$. Using Theorem 1.2 in [**7**], suitably adapted to our situation, we can show that

$$m\{a \in A_{n-1} \mid T_s(a) > \tau\} \le \sum_{i=1}^{s} m\{a \in A_{n-1} \mid E_i(a) > \theta\tau\} + R(\tau; \theta, q), \tag{9.4}$$

where the error term R is of smaller order in τ than the sum, if $\theta = \theta(q)$ is sufficiently small. Combining (9.3) and (9.4) we see that

$$m\{a \in A_{n-1} \mid T_s(a) > n/2\} \le sC(\theta)\frac{1}{n^{1/q}} \le C(\theta)\frac{1}{n^{1/q-1}}.$$

Finally, we note that it is enough to delete according to the free period assumption FA_n at dyadic times $n = 2^k m_0$, since we are then guaranteed that at least $1/4$ of the iterates are free up to an arbitrary time. The remaining parameter set A will then have positive measure if and only if

$$\sum_{k=0}^{\infty} \frac{1}{(2^k m_0)^{1/q-1}} = \frac{1}{m_0^{1/q-1}} \sum_{k=0}^{\infty} \frac{1}{(2^{1/q-1})^k}$$

is finite, which of course happens exactly when $2^{1/q-1} > 1$, that is, when $q < 1$.

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