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Unfolding of chaotic unimodal maps and the parameter dependence of natural measures

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Abstract

We consider one-parameter families f_a of interval maps, and discuss the structure in parameter space and the (dis)continuity properties of the natural measure as a function of the parameter near certain strongly chaotic maps (post-critically finite Misiurewicz maps and Benedicks–Carleson maps). In particular, it is shown that the mapping $a \mapsto \mu_a$ (the natural measure of f_a) is severely discontinuous at these strongly chaotic maps and is not continuous on any full measure set of parameters in full, generic families. Going in the other direction, it is also shown that if such a chaotic map has a measure for which the critical point is generic, then this measure can be approximated with measures supported on periodic attractors of nearby maps. The main idea is to construct cascades of post-critically finite Misiurewicz map and cascades of maps with periodic attractors, whose critical orbits reproduce various invariant sets of the unperturbed map. In the special case of the quadratic family, generalizations can be made to any non-renormalizable maps.

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1. Definitions, statements and related results

1.1. Introduction

We will discuss how the dynamics, as described by so-called natural measures, of typical chaotic interval maps is affected by perturbations within a one-parameter family. By a 'typical chaotic interval map' we mean a Misiurewicz map or a map of Benedicks–Carleson type. It will be shown that the natural measures of such maps have a sort of sensitive dependence on the parameter. In doing so we will have to consider the bifurcation structure in parameter space, and we will see that in every neighbourhood of such a chaotic map we find a large fauna of different types of dynamics, a situation reminiscent of the unfolding of diffeomorphisms exhibiting homoclinic tangencies. In the special case of the quadratic family we can generalize to any strictly finitely renormalizable map.

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An invariant measure μ for an interval map f is called a *natural* (*physical*, *observable*, *Sinai–Ruelle–Bowen*) measure, if μ describes the common asymptotic distribution of $\{f^n(x)\}_{n=0}^{\infty}$ for typical x, i.e. if

$$\mu \overset{\text{weak}^*}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}$$

holds for all x in a positive measure set. Those x for which this does hold, are said to be *generic* for μ.

If a map f has a periodic attractor $\{x_i\}_{i=1}^{p}$, then $\mu := \frac{1}{p}\sum_{i=1}^{p} \delta_{x_i}$ is of course a natural measure for f. In the other extreme we have maps admitting an *absolutely continuous invariant probability measure* (ACIP). For S-unimodal maps with non-flat critical point, almost all x are generic for a measure supported on a periodic attractor, and also every ACIP is a natural measure describing the dynamics of almost all points (see [BL91] and theorem V.1.5 in [MS93][1]).

Specializing further, it is known that almost all maps in the quadratic family have either a periodic attractor or a natural ACIP. This was proved by Lyubich in [Lyu97], see also [Lyu98]. In other words, almost all quadratic maps has a unique probability measure describing the asymptotic distribution of almost all orbits, and this measure is either a normalized sum of point masses or a measure with an integrable density. This is believed to also be the case in more general situations. A generalization to unimodal, real-analytic families with a quadratic critical point has recently been announced by Ávila, Lyubich and de Melo.

Therefore, it is natural to study the weak* continuity properties of the mapping

$$a \mapsto \mu_a =: \text{the natural measure of } f_a$$

for various one-parameter families f_a.

In what follows we discuss the structure in parameter space and the parameter dependence of μ_a near those chaotic maps hinted at above. In section 1.3 we consider the quadratic family and formulate our results in full detail for post-critically finite Misiurewicz maps and the maps whose existence is guaranteed by the theorem of Benedicks and Carleson (see below). Then in section 1.4 we explain how these results can be generalized to generic unimodal families. Finally, in section 1.5 we follow the suggestions given by an anonymous referee of an earlier version of this paper, and return to the quadratic family. Here we show that using quite different methods depending on rigidity, the results can be generalized to non-renormalizable maps. Section 1.6 reviews some related results, before we turn to the proofs.

1.2. Notation

f_a will always be a one-parameter family of unimodal interval maps. μ_a denotes a measure invariant under f_a and is, depending on f_a, either a measure supported on a periodic attractor or an ACIP. Typically these measures will be natural measures, often with a basin of full measure, but this is strictly speaking not necessary for the discussion.

Let c denote the critical point and let

$$\mu_a^c \overset{\text{weak}^*}{:=} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f_a^k(c)}$$

whenever this limit exists.

[1] There are also maps admitting other types of natural measures, for example measures with support on Cantor-attractors, as well as maps which do not admit any natural measure at all, but such maps will not play any role in this paper.

A *Misiurewicz map* is a unimodal map with no periodic attractors, and for which the forward critical orbit does not accumulate on the critical point. Such maps are known to admit an ACIP under some mild extra conditions [Mis81, BM89]. A *post-critically finite Misiurewicz map* is a Misiurewicz map for which the critical orbit is pre-periodic to a repelling periodic orbit.

A unimodal map is called a *Collet–Eckmann* (CE) map if it has a positive Lyapunov exponent at the critical value.

1.3. Misiurewicz maps and Benedicks–Carleson maps: the quadratic family

In this section we consider the quadratic family $f_a(x) = 1 - ax^2$, as a convenient example of a much more general phenomena. Theorem E in the next section gives a framework for generalizing these results to more general families. Our starting point is the following theorem:

Theorem (Benedicks and Carleson [BC85, BC91]). *Let $f_a(x) = 1 - ax^2$. There exists a set $\mathcal{A} \subset (0, 2]$ of positive Lebesgue measure, with $a_* = 2$ a Lebesgue density point of $\mathcal{A}$, such that if $a \in \mathcal{A}$, then f_a is a CE map, and also f_a admits an ACIP μ_a.*

Jakobson, in [Jak81], was the first to prove the existence of a positive measure set of parameters admitting ACIPs in the quadratic family. Other versions and can be found in for example [Ryc88, BY92, Tsu93b, Luz00].

In theorems A–C and corollary 1 below $\mathcal{A}$ refers to the set of parameters exhibited in Benedicks' and Carleson's theorem above.

Theorem A. *For each $a \in \mathcal{A}$, there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that f_{a_n} has a super-stable periodic attractor of length r_n, such that*

(a) $a_n \to a$, $n \to \infty$;
(b) $r_n \uparrow \infty$, $n \to \infty$;
(c) if μ_a^c exists, then $\mu_{a_n} \overset{\text{weak}*}{\longrightarrow} \mu_a^c$, $n \to \infty$.

Recall that μ_a^c is defined as the Birkhoff sum along the critical orbit. It was shown in [BC85] that for the quadratic family, the set $\mathcal{A}$ can be chosen with the additional property that $\mu_a^c = \mu_a$ (the ACIP of f_a) for all $a \in \mathcal{A}$. With this in mind, theorem A says that the ACIPs of the maps f_a, $a \in \mathcal{A}$, can be approximated by measures supported on superstable periodic attractors of nearby maps.

In [Thu96] a version of theorem A is proven, assuming only sub-exponential growth of the derivative along the orbit of the critical value. In [Ure95, Ure96], Ures proves corresponding statements for the Hénon family.

Theorem B. *Let $\Gamma = \Gamma_a \subset [f_a^2(c), f_a(c)]$ be a hyperbolic invariant set for f_a, $a \in \mathcal{A}$, let $z = z(a)$ be any point in Γ and let Γ_b and $z(b)$ be the continuations of Γ and z. Then*

$$a \in \mathrm{cl}\{b \mid f_b^N(c) = z(b) \text{ for some } N = N(b)\}.$$

Remark 1. The assumption that Γ belongs to the dynamical core is, in fact, always fulfilled for the family $f_a(x) = 1 - ax^2$. However, formulated in this way the theorem holds, as we will see, in other families where one might, for example, have an unstable fixed point on the boundary, outside the dynamical core.

Remark 2. An obvious choice is $\Gamma_a = \{z(a)\}$, where $z(a)$ is the interior unstable fixed point. This implies that (this type of) Misiurewicz points are dense in $\mathcal{A}$. This was proven independently and with different methods in [CW00].

Theorem C. *Let* $\{x_1, x_2, \dots, x_p\} \subset [f_a^2(c), f_a(c)]$ *be a hyperbolic periodic repeller for* f_a, $a \in \mathcal{A}$. *Then there exists a sequence of parameters* $\{a_n\}_{n=1}^{\infty}$ *converging to a such that* f_{a_n} *has a super-stable periodic attractor and such that*

$$\mu_{a_n} \stackrel{\text{weak}^*}{\longrightarrow} \mu_a^{\text{sing}} =: \frac{1}{p} \sum_{i=1}^{p} \delta_{x_i}.$$

We also state a theorem that holds for any post-critically finite Misiurewicz map in the quadratic family.

Theorem D. *Suppose that* f_a *is a map whose critical orbit is pre-periodic to an unstable periodic orbit* $\{x_1, x_2, \dots, x_p\}$. *Then there is a sequence of parameters* $\{a_n\}_{n=1}^{\infty}$ *accumulating an a such that* f_{a_n} *has a super-stable period attractor and*

$$\mu_{a_n} \stackrel{\text{weak}^*}{\longrightarrow} \mu_a^{\text{sing}} =: \frac{1}{p} \sum_{i=1}^{p} \delta_{x_i}.$$

Remark 3. The (dis)continuity properties of theorems A, C and D can be extended to open intervals (periodic windows) of parameters corresponding to periodic attractors. Let a and $\{a_n\}_{n=1}^{\infty}$ be as in theorem A, C or D, and let J_n be the periodic window containing a_n. For any sequence $\{b_n\}_{n=1}^{\infty}$ such that $J_n \ni b_n \to a$, the sequence $\{\mu_{b_n}\}_{n=1}^{\infty}$ will have the same limit as the sequence $\{\mu_{a_n}\}_{n=1}^{\infty}$. This follows immediately from the fact that $a \mapsto \mu_a$ is continuous when restricted to a periodic window.

From theorem C and remark 3, we obtain

Corollary 1. $a \mapsto \mu_a$ *is not continuous at any point in* $\mathcal{A}$, *and is not continuous on any full-measure subset of* (0, 2].

Since Misiurewicz maps admit an ACIP under some mild conditions, fulfilled in the quadratic family [Mis81, BM89], theorem D gives the following:

Corollary 2. $a \mapsto \mu_a$ *is discontinuous at every post-critically finite Misiurewicz map.*

1.4. Misiurewicz maps and Benedicks–Carleson maps: the generic case and the flat-topped case

The proofs of the results in the previous section are based solely on the construction of the set $\mathcal{A}$, and therefore they carry over to any setting where the same construction goes through. There are indeed generic situations where this can be carried out. Such generic conditions are given in [TTY94] and in [MS93]. The conditions given in these two papers differ slightly from each other, but they both include the following ingredients:

- f_a is a unimodal family passing through a Misiurewicz point $a = a_*$;
- some mild smoothness of $(x, a) \mapsto f_a(x)$;
- a non-flat/non-degenerate critical point;
- generic unfolding of the Misiurewicz map f_{a_*}.

The conclusion is then that you can construct, with methods essentially the same as those of Benedicks and Carleson, a positive measure set $\mathcal{A}$ of parameters with a_* as a Lebesgue density point, such that if $a \in \mathcal{A}$, then f_a is a CE map admitting an ACIP μ_a[2].

In [Thu99] it is proven that even for not-too-flat flat-topped unimodal families, one can with similar methods construct a positive measure set $\mathcal{A}$ of maps that are CE maps admitting ACIPs. In this case the set $\mathcal{A}$ is constructed by perturbing the full map.

[2] Similar conditions are also given in [Tsu93a], but the construction of the set of chaotic maps is rather different from those in the papers above, and does not fit so well into our framework.

Theorem E. *Let f_a be a generic family with generic unfolding of a Misiurewicz map f_{a_*} as specified in [MS93] or [TTY94], or let f_a be those flat-topped families considered in [Thu99]. Also let $\mathcal{A}$ be the corresponding positive measure set of parameters constructed, for which f_a is CE and admits an ACIP μ_a. Then theorems A and B holds for f_a, $a \in \mathcal{A}$. Theorem D and corollary 2 hold at every post-critically finite Misiurewicz map with generic unfolding. If every post-critically finite Misiurewicz map within the family unfolds in a generic way, then theorem C and corollary 1 also hold.*

Remark 4. In order to obtain the generalized theorem C and corollary 1 it suffices that (sufficiently many of) the Misiurewicz maps appearing in the approximation procedure in the proof of theorem C can be used as a starting point for a Benedicks–Carleson construction. We can allow for some Misiurewicz maps in the family to have a non-generic unfolding. This suggests that in typical one-parameter families, $a \mapsto \mu_a$ is not continuous on any full measure set.

Remark 5. It may happen that the last statement of theorem A becomes irrelevant to natural measures; it is not known whether the critical point typically is generic for the ACIP in these cases.

1.5. Non-renormalizable quadratic maps

In the case of the quadratic family, the results, in fact, hold for any finitely renormalizable map, not just for parameters in $\mathcal{A}$. This result and its elegant proof was suggested by an anonymous referee of an earlier version of this paper. The proof uses the rigidity property of the quadratic family, and is therefore not available for generalizations at the present time. We give an outline of the proof at the end of the paper. The author would like to thank Professor Graczyk for helpful discussions on this matter. Of course it may happen that our non-renormalizable map f_{a_*} has no natural measure at all, but we can still discuss the nature of the singularity of $a \mapsto \mu_a$ at $a = a_*$.

Theorem F. *The statements of theorems A–C hold for every strictly finitely renormalizable quadratic map f_{a_*}, and the map $a \mapsto \mu_a$ has a non-removable singularity at $a = a_*$.*

1.6. Related results for the quadratic family

The measures μ_a, $a \in \mathcal{A}$, are known to be stable under random perturbations of the iterations of f_a [BY92, BV96]. This also holds in the generalized non-flat setting.

In [Ryc88], Rychlik gave a new proof of Jakobson's theorem. A positive measure set $\mathcal{A}'$ of parameters is constructed such that for $a \in \mathcal{A}'$, f_a admits an ACIP μ_a with a density $\nu_a \in L^p$ for $1 \leqslant p < 2$. Then in [RS92], Rychlik and Sorret among other things proved that $a \mapsto \nu_a$, defined on $\mathcal{A}'$, is continuous in L^p, $1 \leqslant p < 2$, with a Hölder estimate at Misiurewicz points. The set $\mathcal{A}'$ also has full density at $a = 2$, and so it has a fat intersection with the set $\mathcal{A}$ considered in this paper.

As mentioned, Tsujii has a proof of the Benedicks–Carleson–Jakobson theorem (leading to an *a priori* different positive measure set $\mathcal{A}''$ of Collet–Eckmann maps), and some generalizations thereof [Tsu93b, Tsu93a]. In [Tsu96] he also discusses weak*-continuity properties of the invariant measures, and proves continuity at Misiurewicz points of the restriction of $a \mapsto \mu_a$ to $\mathcal{A}''$. Also in this case, 2 is a Lebesgue density point of the good set $\mathcal{A}''$. He also constructs a set F of hyperbolic attracting maps accumulating on $a = 2$, whose natural measures converge to a point mass at the unstable fixed point at $x = -1$ as a tends to 2, and shows that $|F \cap [2 - \epsilon, 2]| \gtrsim \epsilon^2$.

The following is also relevant to this discussion, even though it is a measure 0 phenomena. In [HK90] one constructs:

- an uncountable set of parameters, accumulating on $a = 2$, such that the corresponding maps do not have any natural measure at all;
- an uncountable set of parameters, accumulating on $a = 2$, such that the corresponding maps have natural measures $\mu_a = \delta_{z(a)}$, where $z(a)$ is the interior unstable fixed point of f_a.

The first examples of maps with no natural measures was given in [Joh87].

2. Properties of the set $\mathcal{A}$

In sections 2.1–2.3 we recall some basic facts and definitions from the construction of the set $\mathcal{A}$ that will be used in the proofs. We just state the results that we need in the following, and statements do not appear in their logical order. For proofs and more details we refer to [BC85, BC91] and the expositions in [MS93, Luz00]. Similar statements with similar proofs for the flat-topped case can be found in [Thu99]. For simplicity we stick to the quadratic family $f_a(x) = 1 - ax^2$ (with critical point $c = 0$) and consider the Misiurewicz point $a_* = 2$, that is, the set $\mathcal{A}$ built by perturbing the full map f_2.

2.1. A partition on the interval

A small neighbourhood $I^* = (-\delta, \delta)$ of the critical point $c = 0$ is chosen. I^* is partitioned into subintervals

$$I^* \setminus \{0\} = \bigcup_{|\mu| \geqslant -\log\delta} I_\mu = \bigcup_{\substack{|\mu| \geqslant -\log\delta \\ 1 \leqslant \nu \leqslant \mu^2}} I_{\mu,\nu}$$

where $I_\mu = \left[\mathrm{e}^{-(\mu+1)}, \mathrm{e}^{-\mu}\right)$ for $\mu > 0$, $I_\mu = -I_{-\mu}$ for $\mu < 0$, and

$$I_\mu = \bigcup_{1 \leqslant \nu \leqslant \mu^2} I_{\mu,\nu}$$

is a subdivision of I_μ into μ^2 intervals of equal length. $I^+_{\mu,\nu}$ denotes the union of $I_{\mu,\nu}$ and its two nearest neighbours. We also define $\hat{I}_\mu = \left(-\mathrm{e}^{-\mu}, \mathrm{e}^{-\mu}\right)$.

2.2. Partitions in parameter space and the mappings ξ_n

The set $\mathcal{A}$ is given as

$$\mathcal{A} =: \bigcap_{n \geqslant 0} A_n$$

where A_n is a decreasing sequence of sets in a small one-sided neighbourhood of $a = 2$. A_n is constructed by deleting from A_{n-1} according to two principles, which together guarantees that for $a \in A_n$,

$$\left|Df_a^j\,(f_a(0))\right| \geqslant \mathrm{e}^{\lambda j} \qquad \forall j \leqslant n \tag{2.1}$$

for some $\lambda > 0$ independent of n and a. One of the exclusion principles simply requires that

$$\left|f_a^n(0)\right| \geqslant \mathrm{e}^{-\alpha n} \tag{2.2}$$

for some suitable, small $\alpha > 0$.

The mappings ξ_n from parameter space to dynamical space are defined via

$$\xi_n(a) := f_a^n(0).$$

These mappings are expanding as long as f_a^n is, in the following sense:

Lemma 1. *There is a constant C such that for all a sufficiently close to 2 the following holds: if $\left|D_x f_a^j(1)\right| \geqslant \mathrm{e}^{\lambda j}$ for all $j \leqslant k$ then*

$$\frac{1}{C} \leqslant \left|\frac{D_a \xi_{k+1}(a)}{D_x f_a^k(1)}\right| \leqslant C.$$

On each A_n there is a partition $\mathcal{P}_n$ into intervals; each $a \in \mathcal{A}$ is given as $a = \bigcap_{n \geqslant 0} \omega_n(a)$, where $\omega_n(a)$ is the element of $\mathcal{P}_n$ containing a. For each a there is a sequence of times $n_k(a)$, *the essential free return times*, with the following properties:

- $n_k \leqslant n < n_{k+1} \Longrightarrow \omega_n = \omega_{n_k}$;
- $n \leqslant n_k, \xi_n(\omega_{n_k}) \cap I^* \neq \emptyset \Longrightarrow \xi_n(\omega_{n_k}) \subset$ some $I^+_{\mu,\nu}$;
- $I_{\mu,\nu} \subset \xi_{n_k}(\omega_{n_k}) \subset I^+_{\mu,\nu}$, for some $I_{\mu,\nu}$.

A free return is followed by a so-called *bound period*, when $\xi_{n_k+j}(a) = f_a^{n_k+j}(0)$ shadows an initial segment $f_a^j(0)$ of the critical orbit closely. More precisely, if $I_{\mu,\nu} \subset \xi_{n_k}(\omega_{n_k}) \subset I^+_{\mu,\nu}$, then $\xi_{n_k+j}(\omega)$ is in a bound period as long as

$$\left|\bigcup_{a\in\omega} f_a^j\left(-\mathrm{e}^{-\mu}, \mathrm{e}^{-\mu}\right)\right| \leqslant \mathrm{e}^{-2\alpha j}. \tag{2.3}$$

We use $p = p(\mu, \omega)$ to denote the length of a bound period.

As long as it is orbiting outside I^*, $\xi_n(\omega)$ grows exponentially (lemma 3). The small derivative picked up at a return to I^* is compensated for during the bound period by an inductive argument, the net effect is, in fact, a weaker exponential growth. In particular, bound periods are always of finite length (lemma 2). Let p_k temporally denote the length of the bound period following a return at time n_k. By definition, $n_{k+1}(a)$ is the smallest integer $j \geqslant n_k(a) + p_k(a)$ such that $\xi_j(\omega_{n_k}(a)) \supset$ some $I_{\mu,\nu}$ in I^*. A return to I^* at some time j, $n_k + p_k \leqslant j < n_{k+1}$, when no $I_{\mu,\nu}$ is covered, is called an *inessential free return*. Such returns are also followed by a bound period, after which comes a free period terminating in a new free return. It can be shown that an essential free return always occur after finitely many steps. The 'dynamics' of $\{\xi_n\}$ is described more precisely in the following lemmas.

Lemma 2. *Suppose the inequality (2.1) and condition (2.2) hold for all $j \leqslant n$ and all $a \in \omega$, and suppose that $\xi_n(\omega) \subset I^+_{\mu,\nu}$ where $|\mu| \geqslant -\log\delta$ and δ is sufficiently small. Then there exist positive constants C_0 and C, independent of δ, such that*

(a) for all $a \in \omega$, all $y \in f_a(\hat{I}_\mu)$ and all $j \leqslant p$,

$$\frac{1}{C_0} \leqslant \frac{\left|D_x f_a^j(y)\right|}{\left|D_x f_a^j(1)\right|} \leqslant C_0$$

(b) $C|\mu| \leqslant p(\mu,\omega) \leqslant 3|\mu|/\lambda \leqslant 3\alpha n/\lambda < n/100$*;*
(c) $\left|\left(f_a^p\right)'(x)\right| \geqslant C e^{\lambda p/4} \quad \forall x \in I_\mu$*;*
(d) $\left|\xi_{n+p(\mu,\omega)}(\omega)\right| \geqslant C e^{\lambda p/4}|\xi_n(\omega)|$.

Lemma 3. *There is a $\lambda_0 > 0$ such that for any small $\delta > 0$ and a_0 sufficiently close to 2 the following holds: suppose that $\omega \subset [a_0, 2]$ is such that $\xi_{\hat{n}}(\omega) \subset I^+_{\mu,\nu}$ and $\xi_n(\omega)$ are two consecutive free returns with return times $\hat{n}$ and n, $\hat{n} < n$. Also assume that $\left|Df_a^j(1)\right| \geqslant e^{\lambda j}$ for all $j < n$ and all $a \in \omega$. Then there is a constant C, independent of δ, such that the following holds:*

(a) $$|\xi_{n-k}(\omega)| \leqslant C e^{-\lambda_0 k}|\xi_n(\omega)| \qquad \forall\, 1 \leqslant k \leqslant n - \hat{n} - p(\mu,\omega)$$

(b) $$|\xi_n(\omega)| \geqslant 2|\xi_{\hat{n}}(\omega)|.$$

Furthermore, there is a positive integer $N_0(\delta)$ such that for any ω close to 2,

(c) $$\xi_{k+j}(\omega) \cap I^* = \emptyset \qquad j = 0, 1, \ldots, N_0 \quad \Longrightarrow \quad \left|\xi_{k+N_0}(\omega)\right| \geqslant e^{\frac{1}{2}\lambda_0 N_0}|\xi_k(\omega)|.$$

Let HD-dist(J, K) denote the Hausdorff distance between the sets J and K.

Lemma 4. *Suppose inequality (2.1) and condition (2.2) hold for all $j \leqslant n$ and all $a \in \omega$, and suppose that $\xi_n(\omega) \subset I_\mu$ where $|\mu| \geqslant -\log\delta$. If ω is sufficiently close to 2, then for each $a, b \in \omega$ we have*

$$\text{HD-dist}\left(f_a^j(\hat{I}_\mu), f_b^j(\hat{I}_\mu)\right) < \frac{1}{1000}\left|f_a^j(\hat{I}_\mu)\right|$$

$$\text{HD-dist}\left(\xi_{n+j+1}(\omega), f_a^j(\xi_{n+1}(\omega))\right) < \frac{1}{1000}\left|f_a^j(\xi_{n+1}(\omega))\right|$$

for all $j \leqslant p(\mu,\omega)$.

2.3. Escape times

An important role is played by the *escape times* of a. They are an infinite subsequence of $\{n_k(a)\}$, defined via the condition

$$\begin{aligned} &n_k \text{ is an escape time} \\ &\Leftrightarrow \\ &\xi_{n_k}(\omega_{n_{k-1}}) \text{ intersects } (-\delta^2, \delta^2) \text{ and } \left|\xi_{n_k}(\omega_{n_{k-1}})\right| \geqslant \delta. \end{aligned} \tag{2.4}$$

It will be important that each $a \in \mathcal{A}$ experiences infinitely many escape times. This is a consequence of the second parameter selection principle, which roughly speaking discards parameters that on the average have to wait too long for their escape situations.

3. Escaping

We prepare the proofs of theorems A and B by showing that any escaping parameter interval $\omega_n(a)$ contains a super-stable parameter as well as parameters for which the critical orbit lands on any prescribed point in any hyperbolic set of f_a. Once again we do the proof for $\mathcal{A}$ constructed by perturbing the full quadratic map.

Lemma 5. *Pick an $a \in \mathcal{A}$ and a hyperbolic set $\Gamma_a \subset [f_a^2(c), f_a(c)]$ for f_a and a point $z(a) \in \Gamma_a$. Let n_k be a sufficiently large escape time for a. Then there are two parameters $\tilde{a}, \hat{a} \in \omega_{n_{k-1}}(a)$ and two integers $\tilde{r}$ and $\hat{r}$, $0 < \tilde{r}, \hat{r} \lesssim -\log\delta$ such that*

(a) $f_{\tilde{a}}$ *has a super-stable attractor of period* $n_k + \tilde{r}$*;*
(b) $f_{\hat{a}}^{j}(0) \neq z(\hat{a})$ *for* $j < n_k + \hat{r}$ *and* $f_{\hat{a}}^{n_k+\hat{r}}(0) = z(\hat{a})$.

Proof. Let $\omega_{n_{k-1}}(a) = (b, d)$. By taking n_k large enough, we may assume that Γ_α is contained in the dynamical core $[f_\beta^2(c), f_\beta(c)]$ of f_β for all $\alpha, \beta \in (b, d)$. We may also assume that $\xi_{n_k}(b) = \delta^2$ and $\xi_{n_k}(d) = \delta$. The idea is that the distance between $\xi_{n_k+j}(b)$ and -1 will be $\leqslant 4^j\delta^4$, while the distance between -1 and $\xi_{n_k+j}(d)$ will be $\geqslant 3^j\delta^2$, for $j > 1$ as long as $\xi_{n_k+j}(d) < -\frac{3}{4}$. From this it follows that for some $j_0 \lesssim -\log\delta$, $\xi_{n_k+j_0}(\omega_{n_{k-1}})$ will grow to length $\sim \frac{1}{4}$, with its left end $\xi_{n_k+j_0}(b)$ still within a distance $o(\delta)$ from -1.

Thus $\xi_{n_k+\tilde{r}}$ maps (b, d) across $x = 0$ for $\tilde{r} = j_0 + 1$ or $\tilde{r} = j_0 + 2$, and since $0 \notin \xi_i(b, d)$ for $i \leqslant n_k + j_0$ by (2.2) and the definition of j_0, the required $\tilde{a}$ has been found.

By assumption Γ_a is hyperbolic, so $z(a) \in \Gamma_a$ moves continuously with a, and $(z(b), z(d))$ will be a very small interval. From the discussion above it is easily seen that $\xi_{n_k+\hat{r}}(b, d) \supset (z(b), z(d))$ for some $\hat{r} \leqslant \tilde{r} + 1$. Thus $\xi_{n_k+\hat{r}}(a) - z(a)$ changes sign on (b, d), and so $\xi_{n_k+\hat{r}}(\hat{a}) = z(\hat{a})$ for some $\hat{a} \in (b, d)$. □

4. Proof of theorems A and B

Each $a \in \mathcal{A}$ is given as

$$a = \bigcap_{n=1}^{\infty} \omega_n \qquad \omega_n \in \mathcal{P}_n$$

with infinitely many escape situations. For any $b \in \omega_n$, inequality (2.1) holds. Thus we may use lemma 1 to conclude that $|\omega_n| \lesssim \mathrm{e}^{-n\lambda}$. Combining this with lemma 5, the first two statements of theorem A and also theorem B follows.

We now prove the last part of theorem A. Let $a \in \mathcal{A}$ be such that μ_a^c exists, and let $\{a_n\}$ be the sequence constructed above of super-stable parameters converging to a.

We have to show that

$$\lim_{n\to\infty} \int \varphi \, \mathrm{d}\mu_{a_n} = \int \varphi \, \mathrm{d}\mu_a^c$$

for any continuous function φ. It is enough to consider Lipschitz continuous test functions. Let κ be the Lipschitz constant of φ, and let r_k denote the length of the super-stable attractor of f_{a_k}. We have that for any a_k,

$$\left|\int \varphi \, \mathrm{d}\mu_a^c - \int \varphi \, \mathrm{d}\mu_{a_k}\right| \leqslant \left|\int \varphi \, \mathrm{d}\mu_a^c - \frac{1}{r_k}\sum_{j=0}^{r_k-1} \varphi\left(\xi_j(a)\right)\right| + \left|\frac{1}{r_k}\sum_{j=0}^{r_k-1}\left(\varphi\left(\xi_j(a)\right) - \varphi\left(\xi_j(a_k)\right)\right)\right|$$
$$+ \left|\frac{1}{r_k}\sum_{j=0}^{r_k-1}\varphi\left(\xi_j(a_k)\right) - \int \varphi \, \mathrm{d}\mu_{a_k}\right|$$

where the last term equals 0 by definition, and the first one is $< \epsilon$ for all $k > N_0(\epsilon)$ since c is generic for μ_a^c and since $r_k \uparrow \infty$ when $k \to \infty$. The second term finally is

$$\leqslant \frac{\kappa}{r_k}\sum_{j=0}^{r_k-1}\left|\xi_j(a) - \xi_j(a_k)\right|$$

so it suffices to prove that $S := \sum_{j=0}^{r_k-1} \left|\xi_j(a) - \xi_j(a_k)\right|$ is bounded by some constant independent of a_k. Let

$$\Lambda_j = \left|\xi_j(a) - \xi_j(a_k)\right|$$

and let n_k be the escape time preceding the creation of the super-stable orbit at time r_k; then $r_k - n_k \lesssim \log 1/\delta$ (cf lemma 5). Remember that $a_k \in \omega_{n_{k-1}}(a) \in \mathcal{P}_{n_{k-1}}$, and that for the 'orbit' of $\omega_{n_{k-1}}$ under the family $\{\xi_i\}_{i=1}^{n_k}$ certain free return times $\{t_i\}_{i=1}^{T}$ are defined, $t_T = n_k$, and that each free return $t_i < t_T$ is followed by a well defined bound period of finite length, which we denote p_i. For the sake of notation, we define $t_0 = p_0 = 0$. We split the sum S into sub-sums:

$$S := \sum_{i=0}^{T-1} \left(S_i^{\mathrm{bp}} + S_i^{\mathrm{fp}}\right) + S^{\mathrm{tail}}$$

where

$$S_i^{\mathrm{bp}} = \sum_{l=t_i}^{t_i+p_i-1} \Lambda_l \qquad S_i^{\mathrm{fp}} = \sum_{l=t_i+p_i}^{t_{i+1}-1} \Lambda_l$$

and

$$S^{\mathrm{tail}} = \sum_{l=t_T}^{r_k} \Lambda_l.$$

With this notation, S_0^{fp} is the contribution up until the first free return and S_0^{bp} is an empty sum and equals 0. First we observe that S^{tail} has no more than $\sim -\log\delta$ terms, and is therefore $\leqslant C(\delta)$. We now estimate $\sum_{i=0}^{T-1} S_i^{\mathrm{fp}}$, using (a) and (b) of lemma 3:

$$\sum_{i=0}^{T-1} S_i^{\mathrm{fp}} = \sum_{i=0}^{T-1} \sum_{l=t_i+p_i}^{t_{i+1}-1} \Lambda_l \leqslant \sum_{i=0}^{T-1} \sum_{l=t_i+p_i}^{t_{i+1}-1} C\mathrm{e}^{-\lambda(t_{i+1}-l)} \Lambda_{t_{i+1}}$$
$$\leqslant C_1 \sum_{i=1}^{T} \Lambda_{t_i} \leqslant C_1 \sum_{i=1}^{T} 2^{i-T} \Lambda_{t_T} \leqslant C_2.$$

We now turn to $\sum_{i=0}^{T-1} S_i^{\mathrm{bp}}$. First, we estimate the individual terms in S_i^{bp}. Now $\Lambda_l = |\xi_l(a) - \xi_l(a_k)|$, and $(a; a_k) \subset \omega_{n_k} \in \mathcal{P}_{n_k}$. Since t_i is a free return, $t_i < t_T = n_k$, it follows that $\xi_{t_i}((a; a_k)) \subset$ some $I^+_{\mu_i,\nu_i} \subset I^*$. Using lemmas 2 and 4, and the binding condition (2.3), we see that for $1 \leqslant j < p_i$ and any $b \in (a; a_k)$,

$$\Lambda_{t_i+j} \lesssim \left|f_b^j\left(\xi_{t_i}((a; a_k))\right)\right| = \frac{\left|f_b^j\left(\xi_{t_i}((a; a_k))\right)\right|}{\left|f_b^j\left(\hat{I}_{\mu_i}\right)\right|} \left|f_b^j\left(\hat{I}_{\mu_i}\right)\right|$$
$$\lesssim \frac{\left|f_b\left(\xi_{t_i}((a; a_k))\right)\right|}{\left|f_b\left(\hat{I}_{\mu_i}\right)\right|} \mathrm{e}^{-2\alpha j} \sim \frac{\left|f_b\left(\xi_{t_i}((a; a_k))\right)\right|}{\left|f_b\left(I_{\mu_i}\right)\right|} \mathrm{e}^{-2\alpha j}$$
$$\sim \frac{\left|\left(\xi_{t_i}((a; a_k))\right)\right|}{\left|I_{\mu_i}\right|} \mathrm{e}^{-2\alpha j}.$$

In the last two steps we have also used the fact that $\left|f_b\left(\hat{I}_\mu\right)\right| \sim \left|f_b\left(I_\mu\right)\right|$, and that the distortion of f_b restricted to I_μ has a bound independent of μ.

Obviously $\Lambda_{t_i} < \left|\xi_{t_i}((a;a_k))\right| / \left|I_{\mu_i}\right|$, so we obtain

$$S_i^{\mathrm{bp}} \lesssim \sum_{j=0}^{\infty} \frac{\left|\left(\xi_{t_i}((a;a_k))\right)\right|}{\left|I_{\mu_i}\right|} \mathrm{e}^{-2\alpha j} \lesssim \frac{\left|\left(\xi_{t_i}((a;a_k))\right)\right|}{\left|I_{\mu_i}\right|} \leqslant 3 \frac{\left|I_{\mu_i,\nu_i}\right|}{\left|I_{\mu_i}\right|} \lesssim \frac{1}{\mu_i^2}.$$

Finally, we estimate $\sum S_i^{\mathrm{bp}}$:

$$\sum_{i=0}^{T-1} S_i^{\mathrm{bp}} \lesssim \sum_{\mu_i} \sum_{\substack{\text{returns} \\ \text{to } I_{\mu_i}}} \mu_i^{-2} \lesssim \sum_{\mu_i} \sum_{\substack{\text{last return} \\ \text{to } I_{\mu_i}}} \mu_i^{-2}$$

$$\lesssim \sum_{|\mu| \geqslant -\log\delta} \mu^{-2} < C.$$

The second '$\lesssim$' holds because of (b) of lemma 3. This concludes the proof of theorem A.

5. Proof of theorem D

If f_{a^*} is a quadratic Misiurewicz map with finite critical orbit, we claim that we can carry out the Benedicks–Carleson construction in a neighbourhood of a^*. This is so because the aforementioned sufficient conditions given in [MS93] (cf section 1.4) are satisfied. The crucial point is that our post-critically finite map f_{a^*} unfolds in the required generic way. The precise condition is that

$$D_a\left(\zeta(a) - \xi_1(a)\right)\big|_{a=a^*} \neq 0 \tag{5.1}$$

where $\zeta(a)$ is defined to be the continuation of $\zeta(a^*) = \xi_1(a^*)$, that is, the point which for a close to a^* has the same itinerary under f_a as $\xi_1(a^*)$ has under f_{a^*}.

In [Tsu00] it is shown that for any Collet–Eckmann map f_{a^*} in the quadratic family

$$\lim_{n\to\infty} \frac{\xi_n'(a^*)}{D_x f_{a^*}^{n-1}(\xi_1(a^*))} > 0$$

from which (5.1) can be proved with elementary calculations if f_{a^*} is post-critically finite.

We now consider a fixed post-critically finite Misiurewicz map in the quadratic family, which we denote by f_a (rather than f_{a^*}) for the rest of this proof. Let $\Gamma_a = \{x_1, x_2, \ldots, x_p\}$ be the hyperbolic repelling periodic orbit for f_a which absorbs the critical orbit. Let N be minimal such that $f_a^N(0) \in \Gamma_a$, and assume that $f_a^N(0) = x_1$. Fix a small $\gamma > 0$ and define $J_\gamma = \bigcup_{i=1}^p [x_i - \gamma, x_i + \gamma]$.

Using continuity, uniform expansion away from the critical point and the fact that parameter and space derivatives are comparable, we can now for sufficiently small γ find a sequence $\{\omega_k\}_{k=1}^\infty$ of parameter intervals, and an increasing sequence of positive integers $\{n_k\}_{k=1}^\infty$, $n_1 = N$, such that

- $\omega_{k+1} \subset \omega_k$ and $a = \bigcap_{k=1}^\infty \omega_k$;
- $\xi_{n_j}(\omega_k) \subset J_\gamma$ for $j = 1, \ldots, k$;
- $\xi_{n_k}(\omega_k) = [x_{i_k} - \gamma, x_{i_k} + \gamma]$ for some $i_k \in \{1, 2, \ldots, p\}$;
- there is a natural number $M = M(\gamma)$ such that for each k there is an $m_k \leqslant M$ such that $\xi_{n_k+m_k}(\omega_k) \ni 0$.

From this we conclude that for all $k > 0$, there is an $a_k \in \omega_k$ such that f_{a_k} has a super-attractor of length $n_k + m_k$ such that

$$\#\{j \leqslant n_k + m_k \mid f_{a_k}^j(0) \notin J_\gamma\} = N - 1 + m_k \leqslant N + M(\gamma).$$

Now pick a $\varphi \in C^0(I)$ and an $\epsilon > 0$. Since φ is uniformly continuous, we can choose γ such that $|x - y| < \gamma$ implies $|\varphi(x) - \varphi(y)| < \epsilon/2$. If μ_{a_k} is the natural measure for f_{a_k}, where a_k is as above, we have that

$$\begin{aligned}\int \varphi \, d\mu_{a_k} &= \frac{1}{n_k + m_k} \sum_{i=1}^{n_k+m_k} \varphi\left(f_{a_k}^i(0)\right) \\ &\leqslant \frac{1}{n_k + m_k} \sum_{i=N}^{n_k} \varphi\left(f_{a_k}^i(0)\right) + \frac{N - 1 + m_k}{n_k + m_k} \sup_I \varphi \\ &\leqslant \frac{1}{n_k + m_k} \sum_{i=N}^{n_k} \varphi\left(f_{a_k}^i(0)\right) + \frac{N + M}{n_k + M} \sup_I \varphi.\end{aligned}$$

It is clear that the last term vanishes when $n_k \to \infty$. We have to show that $\frac{1}{n_k+m_k} \sum_{i=N}^{n_k} \varphi\left(f_{a_k}^i(0)\right)$ tends to $\int \varphi \, d\mu_a^{\mathrm{sing}} = \frac{1}{p} \sum_{i=1}^{p} \varphi(x_i)$, when n_k tends to infinity. Without loss of generality we may assume that $n_k - N + 1 = R_k p$ for some integer R_k. Let $y_i = f_{a_k}^{N+i-1}(0)$ for $1 \leqslant i \leqslant n_k + 1 - N$, and for $i > p$ define $x_i = f_a^{i-1}(x_1)$. Then

$$\begin{aligned}\frac{1}{n_k + m_k} \sum_{i=N}^{n_k} \varphi\left(f_{a_k}^i(0)\right) &= \frac{1}{n_k + m_k} \sum_{i=1}^{n_k+1-N} \varphi(y_i) \\ &\leqslant \frac{pR_k}{n_k + m_k} \frac{1}{pR_k} \sum_{i=1}^{n_k+1-N} (\varphi(x_i) + \epsilon/2) \\ &= \frac{n_k - N + 1}{n_k + m_k} \left(\int \varphi \, d\mu_a^{\mathrm{sing}} + \epsilon/2 \right) \\ &\leqslant \int \varphi \, d\mu_a^{\mathrm{sing}} + \epsilon\end{aligned}$$

where the last inequality holds for n_k sufficiently large. In the same way

$$\int \varphi \, d\mu_{a_k} \geqslant \int \varphi \, d\mu_a^{\mathrm{sing}} - \epsilon$$

for n_k sufficiently large.

6. Proof of theorem C

Since $\{x_1, \ldots, x_p\}$ is a hyperbolic repeller, it persists for nearby parameter values. Let $\Gamma_a = \{x_1(a), \ldots, x_p(a)\}$ by its continuation. By theorem B, we know that there is a sequence $\{b_n\}$ converging to a, such that f_{b_n} has critical point pre-periodic to $x_1(b_n)$. By theorem D, for each b_n there is a sequence $\{a_{n,m}\}_{m=1}^{\infty}$ corresponding to maps with super-stable attractors, such that $\lim_{m\to\infty} a_{n,m} = b_n$ and $\lim_{m\to\infty} \mu_{a_{n,m}} = \frac{1}{p} \sum_{i=1}^{p} \delta_{x_i(b_n)}$. Since Γ_a moves continuously with a, we obtain a sequence $\{a_{n,m(n)}\}_{n=1}^{\infty}$, corresponding to super-attractors and converging to a such that

$$\lim_{n\to\infty} \mu_{a_{n,m(n)}} \overset{\text{weak}^*}{=} \frac{1}{p} \sum_{i=1}^{p} \delta_{x_i(a)}.$$

7. Proof of theorem F

Step I. First, one proves that the statement in theorem B holds for every strictly finitely renormalizable quadratic map $f_a(x) = 1 - ax^2$. Let $z = z(a)$ be a point in the hyperbolic set $\Gamma = \Gamma(a)$ invariant under f_a, and let $z(b)$ and $\Gamma(b)$ be their continuations.

Since pre-images of any point accumulate on the critical point $c = 0$, we can find a sequence $\{y_n\}_{n=1}^{\infty}$ of points and an increasing sequence of positive integers $\{m(n)\}_{n=1}^{\infty}$ such that

- $y_n \to 0, n \to \infty$;
- $f_a^{m(n)}(y_n) = z(a)$;
- $f^k(y_n) \notin [-y_n, y_n], k = 1, 2, \ldots, m(n)$.

Let $I_b(x)$ and K_b denote the itinerary of x under f_b and the kneading sequence of f_b, respectively. From $f^k(y_n) \notin [-y_n, y_n]$, $k = 1, 2, \ldots, m(n)$, it follows that $I_a(f_a(y_n))$ is maximal, with respect to the usual ordering of itineries. Here we have assumed that $[-y_n, y_n] \bigcap \Gamma(a) = \emptyset$; this is true for n sufficiently large, since a hyperbolic set cannot accumulate on the critical point. From kneading theory it follows that there is a parameter a_n such that $K_{a_n} = I_a(f_a(y_n))$. Then

$$K_{a_n} = I_a(f_a(y_n)) \to K_a \qquad n \to \infty$$

since $\lim_{n\to\infty} y_n = 0$. From rigidity it follows that

$$a_n \to a \qquad n \to \infty.$$

We now show that the critical orbit of f_{a_n} eventually lands on $z(a_n)$. Note that $z(a_n)$ and $\Gamma(a_n)$ exist if a_n is sufficiently close to a.

Since the dynamics of f_{a_n} on $\Gamma(a_n)$ is conjugated to the dynamics of f_a on $\Gamma(a)$, it follows that $I_{a_n}(z(a_n)) = I_a(z(a))$ and so

$$K_{a_n} = I_a(f_a(y_n)) = BI_a(z(a)) = BI_{a_n}(z(a_n))$$

where B is a block of length $m(n) - 1$. Temporarily denote $\zeta_n := f_{a_n}^{m(n)}(0)$, and let σ be the shift. Then

$$I_{a_n}(\zeta_n) = \sigma^{m(n)-1}\left(K_{a_n}\right) = I_{a_n}(z(a_n))$$

so $f_{a_n}^{m(n)}(0) = \zeta_n = z(a_n)$, otherwise $\left(f_{a_n}^{m(n)}(0), z(a_n)\right)$ would be a homterval.

Step II. Using step I and theorem D, we can now prove the statement of theorem C exactly as before for any non-renormalizable quadratic map.

Step III. We now prove the third statement of theorem A for a finitely renormalizable quadratic map f_a (the first two are of course clear because of the density of axiom A in the quadratic family, cf [Sa92, GSa97]).

Now suppose that μ_a^c exists. Then μ_a^c will be an invariant probability measure for f_a, with its support contained in the finite union of intervals which constitutes the topological attractor of f_a. Since the attractor is topologically transitive and since the topological entropy of f_a is positive, we can apply a theorem due to Hofbauer, cf [Hof87, Hof88], which says that invariant measures supported on periodic orbits form a dense subset in the space of invariant probability measures endowed with the weak*-topology. So μ_a^c can be approximated by measures supported on repelling periodic orbits of f_a. In step II above we just proved that any measure on a periodic repeller for f_a can be approximated with measures supported on the periodic attractors of nearby quadratic maps. The third statement of theorem A follows.

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