An elementary approach to dynamics and bifurcations of skew tent maps

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Abstract

In this paper the dynamics of skew tent maps are classified in terms of two bifurcation parameters. In time series analysis such maps are usually referred to as continuous threshold autoregressive models (TAR(1) models) after Tong (1990). This study contains results simplifying the use of TAR(1) models considerably, e. g. if a periodic attractor exists it is unique. On the other hand we also claim that care must be exercised when threshold autoregressive (TAR) models are used. In fact, they possess a very special type of dynamical pattern with respect to the bifurcation parameters and their transition to chaos is far from standard.

Keywords Threshold autoregressive model; skew tent map; uniqueness of periodic attractors; difference equation; chaos; bifurcation diagram; unimodal map.

AMS2000 Subject classification codes 37E05, 37E15, 37G15, 37G35, 62M10, 92D25.

1 Introduction

In this paper we formulate and prove the dynamical properties of skew tent maps. These maps are often referred to as TAR(1)-models and are frequently used in several applied scientific branches for describing nonlinearities encountered in data, see e. g. Caner and Hansen (2001, 2004), Coakley and Fuertes (2006), Strikholm and Teräsvirta (2006), and Stenseth, Chan, Framstad, and Tong (1998), Tong (1990) and references therein.

Also from the mathematical point of view dynamic properties of skew tent maps have been studied previously. Misiurewisz and Visinescu (1991) and Marcuard and Visinescu (1992) completely described how kneading sequences and topological entropy depends on the parameters for skew tent maps like the ones that are to be considered here. Similair results are found in Ichimura (1998), Ito (1998), Ichimura and Ito (1998). Many results given below are already present in these previous studies. The objective of our study is to provide elementary proofs of the central properties of these maps. By elementary proofs we mean proofs that do not rely on abstract theory of one-dimensional dynamics and which should be accessible to all readers familiar with the most basic notions of dynamical systems and measure theory. Typical introductory courses based on for instance textbooks like Devaney (1989) and Royden (1988)) should be more than sufficient (but perhaps not necessary).

The mathematical results referred to above and our results are of remarkable interest for the applied scientific branches making use of TAR(1) models for describing nonlinearities in data. The reason is that all results can be collected in a plane two-parametric bifurcation plot giving rise to a quite complete classification of the dynamical properties of the deterministic skeleton of continuous TAR(1) models. In particular, we would like to emphasize that these families *do not* exhibit period doubling phenomena. There are regions in parameter space corresponding to periodic attractors of any given period, but these are isolated from each other and chaos is reached from these regions by one single bifurcation. Since our results also guarantee uniqueness of periodic attractors whenever they exist, they open a broad access for evaluating the power of statistical methods used in nonlinear data-analysis.

Our paper is organized as follows. We start by introducing a linear change of variables that reduces the number of parameters involved in TAR(1)-models (skew tent maps) with the result that two parameters remain. After introducing notation and terminology, these two parameters are then used as bifurcation parameters, and the bifurcation diagram that is used without proof in Stenseth, Chan, Framstad, and Tong (1998) is proved and completed in Sections 4-5. We distinguish between generic cases, that is, phenomena that appears with positive probability in parameter space, and non-generic cases appearing on bifurcation value curves. In Section 6 we proceed by proving uniqueness of periodic attractors. Problems attempting to provide estimates of number of attractors from above are considered as acknowledged mathematical problems for many systems. However, we still claim that our proofs of these facts for the system in question are kept on a reasonably elementary level. We end up in Section 7 with a discussion of results presented here.

2 TAR(1)-models reparametrized

A TAR(1) model (Tong (1990)) has four parameters and can be stated as

$$X' = \begin{cases} A - k(X - x^*), & X \geq x^*, \\ A + c(X - x^*), & X < x^*. \end{cases} \tag{1}$$

The parameters A and x^* can readily be eliminated by linear coordinate transformations, leaving the slope parameters c and k as the essential ones.

- If $A > x^*$, the new coordinate $\xi = \frac{X - x^*}{A - x^*}$ yields

$$\xi' = t(\xi) = \begin{cases} 1 - k\xi, & \xi \geq 0, \\ 1 + c\xi, & \xi < 0 \end{cases} \tag{2}$$

- If $A < x^*$, the new coordinate $\xi = \frac{X - x^*}{A - x^*}$ yields

$$\xi' = t(\xi) = \begin{cases} 1 - k\xi, & \xi \leq 0, \\ 1 + c\xi, & \xi > 0 \end{cases} \tag{3}$$

- If $A = x^*$, the case where the turning point is the fixed point, then $\xi = X - x^*$ yields

$$\xi' = t(\xi) = \begin{cases} -k\xi, & \xi \leq 0, \\ c\xi, & \xi > 0 \end{cases} \tag{4}$$

Each of the maps (3) is equivalent to one of the maps (2), by relabelling the constants. The case (4) is easily analyzed using for example so called graphic iteration (a k a "cobwebbing") and is not treated in this paper. Consequently, we will focus on equation (2).

3 Notation and terminology

From here on the maps $t = t_{c,k}$, $t : \mathbb{R} \to \mathbb{R}$ is defined by (2).

As usual t^n denotes the n:th iterate of t,

$$t^n(\xi) = t\left(t^{n-1}(\xi)\right), \quad n = 2, 3, \ldots.$$

For the first and second iterate we will sometimes write $\xi' := t(\xi)$ and $\xi'' := t^2(\xi)$ respectively.

We also use the notation $\xi_n := t^n(\xi_0)$ when we think about the *orbit* $\{\xi_0, \xi_1, \xi_2, \ldots\}$ of a point ξ_0. Iteration schemes like (2) might be considered as difference equations; it is thus also natural to speak of orbits as *solutions*.

A point z such that $t(z) = z$ is called a fixed point. If $t^p(z_1) = z_1$ for some $p > 0$ and $t^j(z_1) \neq z_1$ for $0 < j < p$, then z_1 is a *periodic point* of period p with corresponding *periodic orbit* $\{z_1, z_2, \ldots, z_p\}$. A point that is not periodic in itself, but that is mapped onto a periodic point in a finite number of steps is called *preperiodic*.

A fixed point z is called *stable* or *attracting* if the absolute value of the slope t is less than one at z. Then there is an interval J containing z such that $\xi_n \to z$ as $n \to \infty$ for all $\xi_0 \in J$. If the slope equals $+1$ or -1 at z, z is called a *neutral* fixed point. Finally, if the slope has absolute value greater than one at a fixed point z, then z is called an *unstable* or *repelling* fixed point. In this case there is an interval $J \ni z$ such that points in J move away from z under iteration. A periodic point z_1 of period p is called *stable* (*attracting*), *neutral* or *unstable* (*repelling*) depending on the character of z_1 as a fixed point for t^p.

By the chain rule, the derivative of t^n at a point ξ_0 equals

$$(Dt^n)(\xi_0) = \prod_{i=0}^{n-1} Dt(\xi_i).$$

In particular, the derivative of t^p at a periodic point of period p equals the product of derivatives along the periodic orbit. Thus the value of (Dt^p) is constant along a p-periodic orbit. This common value is called the *multiplier* of the periodic orbit.

Orbits $\{\xi_0, \xi_1, \xi_2, \dots\}$ which remain bounded and have *positive Lyapunov exponent*, that is

$$|Dt^n(\xi_0)| \geq Ce^{\lambda n} \quad n = 1, 2, \dots$$

for some positive constants C and λ, are called *chaotic* orbits. On intervals of chaotic orbits small perturbations grow exponentially fast, we thus have a strong form of *sensitive dependence on initial conditions*. It is also known that this implies the existence of an invariant probability measure absolutely continuous with respect to Lebesgue measure.

Remark. For maps like ours that are locally linear around fixed points, neutral fixed points implies special kinds of local dynamics. If the slope at the fixed point equals $+1$, then the map is a local identity around the fixed point. If the slope equals -1, there is a symmetric interval around the fixed point where all points are on periodic orbits of length 2, cycling around the fixed point.

4 The bifurcation diagram — generic cases

In the next theorem we formulate the generic dynamical patterns of (2), and how these depend on the parameters c and k. With generic we mean patterns that appear for parameter sets with positive area (Lebesgue) measure; these sets will be open regions or open regions with some boundary curves added. In Subsection 5 we describe the dynamics on the remaining parts of parameter space (a countable collection of smooth curves corresponding to various types of bifurcations).

Theorem 4.1. *The parameter space of* (2) *contains the following regions of positive area measure corresponding to different types of dynamics.*

(a) *If* $c \leq 1$ *and* $k \leq -1$*, then* $\xi_n \to \infty$ *as* $n \to \infty$ *for all initial conditions* ξ_0.

(b) *If* $c > 1$ *and* $k \leq -1$*, then as* $n \to \infty$*,* $\xi_n \to +\infty$ *as for all* $\xi_0 \in]-1/(c-1), +\infty[$ *and* $\xi_n \to -\infty$ *if* $\xi_0 \in]-\infty, -1/(c-1)[$.

(c) *If* $c \leq 1$ *and* $-1 < k < 1$*, then the fixed point* $\xi = 1/(1+k)$ *attracts all initial conditions.*

(d) *If* $c > 1$ *and* $-1 < k < 1$ *the fixed point* $\hat{\xi} = 1/(k+1)$ *is attracting. For* $-1 < k \leq 0$*, initial conditions in* $]-1/(c-1), +\infty[$ *tend to* $\hat{\xi}$*, and initial conditions in* $]-\infty, -1/(c-1)[$ *tend to* $-\infty$*. For* $0 < k < 1$ *orbits starting in* $]-1/(c-1), c/(k(c-1))[$ *are attracted to* $\hat{\xi}$*, the point* $c/(k(c-1))$ *is the preimage of the negative unstable fixed point, and all other orbits tend to* $-\infty$*.*

(e) *If* $k > 1$ *and* $c \leq -1/k$*, then we have increasing oscillations around the fixed point* $\xi = 1/(1+k)$ *such that* $|\xi_n| \to \infty$ *for* $\xi_0 \neq 1/(1+k)$*.*

(f) *If* $k > 1$ *and* $-1/k < c < 1/k$*, then there exists a 2-periodic orbit at* $\hat{\xi}_1 = (1-k)/(1+kc) < 0$ *and* $\hat{\xi}_2 = (1+c)/(1+kc) > 1/k$ *attracting almost all initial conditions.*

(g) *If* $k > 1$ *and* $1/k < c < 1$ *and if, for some integer* $n \geq 3$*,*

$$\sum_{i=0}^{n-2} \frac{1}{c^i} < k < \frac{1}{c^{n-1}}, \tag{5}$$

then an n-periodic orbit attracting almost all initial conditions exist at

$$\left\{ \frac{\sum_{i=0}^{n-1} c^i}{1 + c^{n-1}k}, -k\frac{\sum_{i=0}^{n-1} c^i}{1 + c^{n-1}k} + 1, \ldots, -kc^{n-2}\frac{\sum_{i=0}^{n-1} c^i}{1 + c^{n-1}k} + \sum_{i=0}^{n-2} c^i \right\}.$$

(h) *If* $k > 1$ *and* $1/k < c \leq 1$ *and* k *is outside the regions specified by* (5)*, almost all orbits enter and remain in the interval* $[1-k, 1]$ *and have positive Lyapunov exponent.*

(i) *If* $k > 1$ *and* $1 < c < k/(k-1)$ *then almost all orbits starting in* $[-1/(c-1), c/(kc-k)]$ *are bounded and have positive Lyapunov exponent. Solutions starting outside the interval above are attracted to* $-\infty$*.*

(j) *If* $c > k/(k-1)$*,* $k > 1$*,* $c > 1$*, then almost every solution tends to* $-\infty$*.*

Remark. Since in cases (a) - (g) almost all points tends to either a stable periodic orbit or to infinity under iteration, finite multiple attractors are ruled out in these cases. Our numerical work did not provide any evidence for multiple attractors in the remaining cases either, but there are still some work to be done before we know under what conditions uniqueness of attractors might be true in the non-periodic case. Ichimura (1998) and Ito (1998) provide some more details concerning the chaotic regions in parameter space; different subregions corresponding to dynamics with different kneading sequences and chaotic attractors with different topologies are identified.

Proof. (a) If $\xi < 0$, then $\xi' = 1 + c\xi \geq 1 + \xi$. Similarly, if $\xi > 0$, then $\xi' = 1 - k\xi \geq 1 + \xi$. Hence, $\xi_n \to \infty$ for all initial conditions.

(b) A repelling fixed point exists at $\xi = -1/(c-1)$. If $\xi_0 < -1/(c-1)$, then $\xi_n \to -\infty$. If $\xi_0 > -1/(c-1)$, then ξ_n grows first until it becomes greater than zero. Then by a similar argument as in (a), it grows to infinity.

(c) Since $|k| < 1$, the fixed point $\xi = 1/(1+k)$ is attracting. Initial conditions $\xi_0 < 0$ will in a finite number of step be mapped to the positive half-axis. So it suffices to prove that positive initial conditions are attracted to the fixed point. We distinguish two cases: (i) If $-1 < k \leq 0$ it is easily seen from the graph that all positive initial conditions tend to the fixed point. (ii) If $0 < k < 1$, $t(0) = 1$ is the maximum value and since $t(1) = 1 - k > 0$, the interval $[0, 1]$, which contains the fixed point, is mapped into itself. Points $\xi_0 \in [0, 1/k]$ remain on the positive side and oscillate towards the fixed point. If $\xi_0 < 0$ the orbit will sooner or later enter $[0, 1]$. If $\xi_0 > 1/k$ the orbit will make an initial detour do the negative side before being absorbed by $[0, 1]$.

(d) The situation is much the same as in the previous case, only that we now also have an unstable fixed point on the negative side at $\bar{\xi} = 1/(1-c)$. Initial conditions to the left of $\bar{\xi}$ will move towards $-\infty$. For $0 < k < 1$ there is a positive preimage of $\bar{\xi}$ at $\bar{\xi}_{-1} = c/(k(c-1))$. Initial conditions $\xi_0 > \bar{\xi}_{-1}$ will thus also tend to $-\infty$ when $0 < k < 1$. In remaining cases, orbits tend to the stable fixed point by the same arguments as in (c).

(e) Since $c \leq 0$ we note that,

$$\xi'' = \begin{cases} 1 - k - kc\xi, & \xi < 0 \\ 1 - k + k^2\xi, & 0 < \xi < \frac{1}{k} \\ 1 + c - kc\xi, & \xi > \frac{1}{k} \end{cases} . \tag{6}$$

By the assumptions, the derivative of the map (6) is greater than or equal to one everywhere. In particular, it is greater than one at the unique fixed point at $\xi = 1/(1+k)$. This repelling fixed point splits the real axis, so that for the map (6), initial conditions with $\xi < 1/(1+k)$ tends to $-\infty$ and initial conditions with $\xi > 1/(1+k)$ tends to $+\infty$. For the map (2) this corresponds to orbits with increasing unbounded oscillations.

(f) We take the case $c < 0$ first. The map (6) is everywhere increasing and has three fixed points. One of these fixed points correspond to the unstable fixed point of the map (2), the other ones are located at $\hat{\xi}_1 = (1-k)/(1+kc) < 0$ and $\hat{\xi}_2 = (1+c)/(1+kc) > 1/k$, and correspond to a 2-cycle of (2). We have

$$\xi'' > \xi \quad \text{if} \quad \xi \in]-\infty, (1-k)/(1+kc)[\cup]1/(1+k), (1+c)/(1+kc)[$$

and

$$\xi'' < \xi \quad \text{if} \quad \xi \in](1-k)/(1+kc), 0[\cup](1+c)/(1+kc), +\infty[.$$

Hence the two-cycle of (2) is globally attracting.

In the case $c > 0$ we prove local stability and postpone the proof of the global stability properties to the proof of Proposition 6.1. We note that

$$\xi'' = \begin{cases} 1+c+c^2\xi, & \xi < -\frac{1}{c} \\ 1-k-kc\xi, & -\frac{1}{c} < \xi < 0 \\ 1-k+k^2\xi, & 0 < \xi < \frac{1}{k} \\ 1+c-kc\xi, & \xi > \frac{1}{k} \end{cases} \tag{7}$$

If $k > 1$, $0 < c < 1/k$ map (7) has three fixed points. One corresponds to the unstable fixed point of (2) at $\xi = 1/(1+k)$. The other fixed points of (7) are stable and correspond to a 2-cycle of (2). This two-cycle is given by $\xi = (1-k)/(1+kc)$ and $\xi = (1+c)/(1+kc)$, so (f) is proved.

Before proving statements (g) and (h) we make the following observation. For each point (c, k) in the region where $k > 1$ and $1/k < c \leq 1$ there is a unique integer $n = n(c,k) \geq 3$ such that

$$\frac{1}{c^{n-2}} \leq k < \frac{1}{c^{n-1}}.$$

For a fixed n, the region defined by the above inequality is split in to two sub-regions: If $k > \sum_{i=0}^{n-2} 1/c^i$ we are in the periodic case (g); if on the other hand $k < \sum_{i=0}^{n-2} 1/c^i$ we have the chaotic case (h). Please note that condition (5) and $k > 1$ implies that $1/k < c < 1$.

(g) We postpone the global stability properties of these periodic orbits to the proof of Proposition 6.2. Assume that there is an orbit $\{\xi_1, \xi_2, \ldots, \xi_n\}$ where $\xi_1 > 0$ and $\xi_2 < \xi_3 < \cdots < \xi_n < 0$. Then $\xi_2 = t(\xi_1) = 1 - k\xi_1 < 0$, and $\xi_p = 1 + c\xi_{p-1} > \xi_{p-1}$ for $3 \leq p \leq n+1$. Unwinding this recursive definition we find that points on such an orbit are given by

$$\xi_p = -kc^{p-2}\xi_1 + \sum_{i=0}^{p-2} c^i, \quad 2 \leq p \leq n+1. \tag{8}$$

If this is to be an n-periodic orbit we also need $\xi_{n+1} = \xi_1$, that is,

$$-kc^{n-1}\xi_1 + \sum_{i=0}^{n-1} c^i = \xi_1.$$

Solving for ξ_1 we find

$$\xi_1 = \frac{\sum_{i=0}^{n-1} c^i}{1 + c^{n-1}k} \tag{9}$$

as the only possible choice.

To prove (g) it now suffices to verify that for c and k as specified in (5) and ξ_1 as in (9) the following holds:

(I) $\xi_1 = \frac{\sum_{i=0}^{n-1} c^i}{1+c^{n-1}k} > 1/k$;

(II) $\xi_p = -kc^{p-2}\xi_1 + \sum_{i=0}^{p-2} c^i < 0$, for $p = 2, \ldots, n$;

(III) The multiplier of the orbit is < 1.

(I) is equivalent to

$$k \sum_{i=0}^{n-2} c^i > 1.$$

We use the assumption that $k > 1$ to deduce that this last inequality is true.

We now verify (II). First

$$kc^{p-2}\xi_1 > \sum_{i=0}^{p-2} c^i \quad \Longleftrightarrow \quad kc^{p-2}\frac{\sum_{i=0}^{n-1} c^i}{1+c^{n-1}k} > \sum_{i=0}^{p-2} c^i$$
$$\Longleftrightarrow \quad k\frac{\sum_{i=0}^{n-1} c^i}{1+c^{n-1}k} > \sum_{i=0}^{p-2} \frac{1}{c^i}.$$

We want this to hold for all $p = 2, \dots, n$. From the last inequality it is obvious that this is true if and only if it is true for $p = n$

Putting $p = n$ and multiplying both sides with $1 + c^{n-1}k$ in the last inequality, after simplifying, we obtain the equivalent statement

$$k > \sum_{i=0}^{n-2} \frac{1}{c^i}$$

which holds true by assumption.

Finally (III) reduces to the condition

$$c^{n-1}k < 1$$

which also is true by assumption.

(h) Every periodic orbit must visit the region $\xi > 0$ at least once. Since for $\xi < 0$ we have $t(\xi) = 1 + c\xi$ with $0 < c < 1$, it is clear that once orbits are on the left half-axes, they move steadily to the right until they pass the origin. Since $t(0) = 1$ is the maximal value, it also clear that no periodic orbit can have longer runs on the negative side than the orbit of $\xi_0 = 1$.

Now define the decreasing sequence $z^{(r)}$, $r = 1, 2, \dots$ of negative preimages of 0 defined by $t(z^{(1)}) = 0$ and $t(z^{(r)}) = z^{(r-1)}$, $r \geq 2$. It is easily seen that

$$z^{(r)} = -\sum_{i=1}^{r} \frac{1}{c^i}.$$

The condition $k < \sum_{i=0}^{n-2} 1/c^i$ is equivalent to $1 - k > -\sum_{i=1}^{n-2} 1/c^i$, which in turn is equivalent to the dynamical statement

$$t(1) > z^{(n-2)}.$$

We conclude that no periodic orbit spends more than $(n-2)$ iterates in the region $\xi < 0$ in a row. Thus, the multiplier M of any periodic orbit satisfies

$$|M| \geq \left|kc^{n-2}\right| > 1,$$

hence all periodic orbits are unstable. The number of periodic orbits is countable since there are only a finite number of periodic orbits of any given period (consider the intersection of the graph of t^p and the diagonal). Finally, it is easy to see that the interval $[1-k, 1]$ is invariant and absorbing, and the same type of reasoning as above shows that all points must have positive Lyapunov exponent (except for the singular turning point and its preimages).

(i) Similar arguments as in (b) proves the statements about the solutions attracted to $-\infty$, with the small modification that the interval $]c/(kc-k), +\infty[$ is mapped to $]-\infty, -1/(c-1)[$. It is easy to see that the points $-1/(c-1)$, $1/(1+k)$ and $c/(kc-k)$ are either mapped to or remain at the fixed points at $-1/(c-1)$ or $1/(1+k)$. Similar considerations for other possible unstable periodic points, gives a countable number of initial conditions in $[-1/(c-1), 1/(k+1)]$, which gives rise to periodic behavior. This set has Lebesgue-measure zero, hence something else happens almost everywhere in the set above. In the interval $[-1/(c-1), c/(kc-k)]$ the map is everywhere expanding, since $c > 1$ and $k > 1$, so a positive Lyapunov exponent is guaranteed. Since this interval is invariant all solutions starting here are bounded, and hence chaotic.

(j) The construction technique of the Cantor middle third set, cf. Devaney (1989), shows that a certain amount of points remain in the interval $[-1/(c-1), c/(kc-k)]$. These points are not countable, but have Lebesgue-measure zero. Therefore they provide what happens in exceptional cases only. The rest of the points are attracted to $-\infty$.

□

5 The bifurcation diagram — non-generic cases

The description in the previous section is not complete, in this section we treat the remaining cases. These are curves in parameter space corresponding to borderline cases between different types of dynamics. The significance of these cases are discussed in the concluding Section 7.

Theorem 5.1. *The complement of regions described in Theorem* 4.1 *in the parameter space of* (2) *consists of smooth curves, corresponding to different types of dynamics as described below.*

(A) *If* $k = 1$ *there is a neutral fixed point at* $\xi = 1/2$. *The interval* $[0, 1]$ *is invariant and every point* $1/2 \neq \xi \in [0, 1]$ *is on a neutral orbit of length two. Points* $\xi > 1$ *are mapped to* $\xi' = t(\xi) < 0$. *The structure of the orbits of points starting* $\xi < 0$ *depends on* c. *We have the following subcases.*

- (A_1) *If* $c > 1$ *there is an unstable fixed point* $\xi = 1/(1 - c)$. *Points* $\xi \in]-1/(c-1), 0[$ *are preperiodic to either the fixed point* $\xi = 1/2$ *or a neutral two-cycle in* $[0, 1]$. *Orbits in* $] - \infty, -1/(c - 1)[$ *goes to* $-\infty$.
- (A_2) *If* $0 < c \leq 1$ *all points outside* $[0, 1]$ *are preperiodic to the fixed point or a two-cycle in* $[0, 1]$.
- (A_3) *If* $c = 0$, *all points outside* $[0, 1]$ *are preperiodic to the neutral two-cycle* $\{0, 1\}$.
- (A_4) *If* $-1 < c < 0$ *orbits outside* $[0, 1]$ *stay outside* $[0, 1]$, *but converges to the neutral two-cycle* $\{0, 1\}$.
- (A_5) *If* $c = -1$, *all points* $\xi \neq 1/2$ *are on a neutral two-cycle.*
- (A_6) *If* $c < -1$ *and* $\xi \notin [0, 1]$, *the orbit of* ξ *oscillates to infinity.*

(B) *If* $k > 1$ *and* $c = 1/k$, *the intervals* $[1 - k, 0]$ *and* $[1/k, 1]$ *are mapped onto each other. There is one neutral two-cycle* $\left\{\frac{(1-k)}{2}, \frac{k+1}{2k}\right\}$, *all other points in* $[1 - k, 0] \cup [1/k, 1]$ *are on neutral four-cycles. In particular the endpoints forms the cycle*

$$0 \mapsto 1 \mapsto 1 - k \mapsto \frac{1}{k} \mapsto 0.$$

The fixed point $\xi = 1/k$ *is unstable. All points* $\xi \notin [1 - k, 0] \cup \{1/(k + 1)\} \cup [1/k, 1]$ *are preperiodic to the unstable fixed point or one of the cycles in* $[1 - k, 0] \cup [1/k, 1]$.

(C) *If* $k > 1$ *and* $1/k < c < 1$ *and if, for some integer* $n \geq 3$,

$$k = \sum_{i=0}^{n-2} \frac{1}{c^i} \leq \frac{1}{c^{n-1}}, \tag{10}$$

then the turning point $\xi_0 = 0$ *is on a cycle of length* n*, where* $\xi_1 = 1$ *and* $\xi_p < 0$ *for* $p = 2, \ldots, n-1$*. All orbits enters the interval* $[\xi_2, \xi_1]$ *which is mapped into itself with chaotic dynamics almost everywhere.*

(D) *If* $k > 1$ *and* $1/k < c < 1$ *and if, for some integer* $n \geq 3$*,*

$$\sum_{i=0}^{n-2} \frac{1}{c^i} < k = \frac{1}{c^{n-1}}, \tag{11}$$

then there are closed intervals J_i*, and points* $\xi_i \in J_i$*,* $i = 1, 2, \ldots, n$ *such that*

- $J_1 \subset]0, 1[$ *and* $J_i \subset]1-k, 0[$*;*
- $J_1 \mapsto J_2 \mapsto \ldots \mapsto J_n \mapsto J_1$*;*
- *the points* ξ_i *form a neutral* n*-cycle;*
- *all other points in* $\bigcup_{i=1}^{n} J_i$ *are on neutral* $2n$*-cycles.*

(E) *If* $c = k/(k-1)$*,* $k > 1$*,* $c > 1$*, solutions starting almost everywhere in the interval* $[-1/(c-1), c/(kc-k)]$ *possesses a chaotic solution and the itinerary generates the uniform distribution on that interval. All solutions starting outside the interval above tend to* $-\infty$*.*

Proof. Cases (A) and (B) are easily analyzed with graphic iteration and elementary computations. This is left to reader. For case (C), the existence of the periodic orbit is proved by using the formulas for orbit in the proof of case (g) in Theorem 4.1, and chaotic dynamics is deduced in the same way as in case (h) in Theorem 4.1.

We now turn to case (D). The existence of an n-cycle, with $n-1$ points to the left of the origin and one to right, follows by the same argument as in the proof of case (g) in Theorem 4.1. The condition $k = c^{1-n}$ then immediately implies that the multiplier of this orbit equals -1. By continuity, and since we are dealing with a piecewise linear map, there are intervals J_i, $i = 1, 2, \ldots, n$, containing the n-periodic points ξ_i in their interior, such that $t^n_{|J_i}$ is a linear map with slope -1. Consequently $t^{2n}_{|J_i}$ is linear map with a fixed point ξ_i and slope $+1$, so $t^{2n}_{|J_i}$ is the identity, and so all points $\xi_i \neq \xi \in J_i$ are neutral periodic points with period $2n$.

Finally, case (E) correspond to the asymmetric tent-map studied by Lagarias, Porta and Stolarsky (1993,1994). As in (i) in Theorem 4.1, we prove that all solutions starting in the domain $]-\infty, -1/(c-1)[\cup]c/(kc-k), +\infty[$ tend to $-\infty$. Likewise, at most a countable set of points in $[-1/(c-1), c/(kc-k)]$ are eventually mapped to unstable periodic points. It is easy to see that the points in the interval $[-c/(c-1), c/(kc-k)]$ are invariant. To see that the orbits in this interval, and which are not periodic or preperiodic, generate the uniform distribution, we solve the Perron-Frobenius equation, cf. Lauwerier (1986)

$$kP(\xi) = (k-1)P\left(\frac{(k-1)(\xi-1)}{k}\right) + P\left(\frac{1-\xi}{k}\right),$$

which has the Lebesque measure $P(\xi) = 1$, as its solution. □

6 The periodic attractors of TAR(1)-models are unique

In this section we prove that the periodic attractors of period ≥ 2 found in Section 4 are unique by showing that whenever they exist, they attract almost all orbits. First we introduce some notation.

We consider maps $t = t_{c,k}$ as defined in (2). Denote $\tau_0 = 0$ (the turning point), and $\tau_n = t(\tau_{n-1})$ its forward images. We note that $\tau_1 = 1$, that $\tau_2 = 1-k < 0$ for $k > 1$, and that t has a unique invariant interval $I = [\tau_2, \tau_1] \ni 0$ which absorbs all orbits whenever $c > 0$ and $k > 1$.

The map $t = t_{c,k}$ has a unique fixed point $p = \frac{1}{k+1}$. We denote by z^+ the positive preimage of $\tau_0 = 0$, that is, $z^+ = \frac{1}{k}$. Just like in the proof of (h) in Theorem 4.1, $z^{(n)}$, $n = 1, 2, \ldots$, denotes the negative preimages of the turning point defined by $t(z^{(1)}) = 0$ and $t(z^{(n)}) = z^{(n-1)}$, $n = 2, 3, \ldots$. Remember that

$$z^{(n)} = -\sum_{i=1}^{n} \frac{1}{c^i}.$$

We proved in Section 4 that t admits a stable two cycle whenever $k > 1$ and $c < \left|\frac{1}{k}\right|$. For negative c we know already this to be the global attractor, now we extend this results to the positive values of c. We shall then proceed to prove that also attractive periodic orbits of higher periods are in fact global attractors.

Proposition 6.1. *For $k > 1$, $0 < c < \frac{1}{k}$, the map $t = t_{c,k}$ has a stable 2-cycle attracting all initial conditions $x_0 \notin \cup_{i=0}^{\infty} t^{-i}(p)$*

Proof. Consider the intervals $J = [z^+, \tau_1]$ and $K = [\tau_2, \tau_0]$. Then t maps J bijectively onto K. Also t maps K bijectively, such that rightmost endpoint τ_0 of K maps to the rightmost endpoint of J. For the leftmost endpoint τ_2 we have that

$$t(\tau_2) = 1 + c(1 - k),$$

so $t(K) \subset J$ iff $1 + c(1-k) > \frac{1}{k}$ which is equivalent to $c < \frac{1}{k}$. So for these parameters t^2 maps J strictly into itself, that is, J is a sink for t^2 (but not for t). From this it follows that there is a stable 2-cycle inside $J \cup K$ with multiplier $ck < 1$ absorbing all points of J.

Now consider an arbitrary initial condition ξ_0. If $\xi_0 = \tau_0$, then $\xi_1 = \tau_1 \in J$. If $\xi_0 \in (\tau_0, z^+)$, $\xi_0 \neq p$, it is clear the orbit ξ_n will oscillate around p, increasing it's distance to p with a factor k for each iterate, so in a finite number of iterates the orbit will hit J, and so we are done. If $\xi_0 < 0$, then the sequence ξ_n will increase until it finally hits $[\tau_0, \tau_1] = [0, 1]$, and then we are back to case above. If finally $\xi_0 > \tau_1$, then $\xi_1 < \tau_0$, so this reduces to the negative case. □

Proposition 6.2. *For any integer $n \geq 3$ the following holds: If $k > 1$, $0 < c < 1$ and*

$$\sum_{i=0}^{n-2} \frac{1}{c^i} < k < \frac{1}{c^{n-1}}, \tag{12}$$

then $t = t_{c,k}$ admits a globally attracting n-cycle with itinerary $(RL^{n-1})^\infty$.

Proof. The existence of the attracting cycle with this itinerary has already been proved. The condition (12) is equivalent to

$$1 - \frac{1}{c^{n-1}} < 1 - k < -\sum_{i=1}^{n-2} \frac{1}{c^i}$$

and since, trivially, $1 - \frac{1}{c^{n-1}} > -\sum_{i=1}^{n-1} \frac{1}{c^i}$, (12) implies that

$$-\sum_{i=1}^{n-1} \frac{1}{c^i} < 1 - k < -\sum_{i=1}^{n-2} \frac{1}{c^i}.$$

This translates into the following dynamical property implied by (12) :

$$z^{(n-1)} < \tau_2 < z^{(n-2)}.$$

Let $K_i = [\tau_i, z^{(n-i)}]$ for $i = 2, 3, \ldots, n$, where $z^{(0)} := \tau_0 = 0$, and let $\hat{K} = [\tau_{n+1}, \tau_1]$. Then

$$K_2 \mapsto K_3 \mapsto \ldots \mapsto K_n \mapsto \hat{K},$$

and these maps are all bijective.

Lemma 6.3. *If $k > 1$ and $0 < c < 1$ are such that* (12) *holds, then*

$$t_{c,k}\left(\hat{K}\right) \subset K_2.$$

Proof of Lemma 6.3. We have that $t_{c,k}\left(\hat{K}\right) \subset K_2$ if and only if

$$\tau_{n+2} = t(\tau_{n+1}) = 1 - k\tau_{n+1} \leq z^{(n-2)}.$$

From equation (8) in the previous section it follows that

$$\tau_{n+1} = -kc^{n-1} + \sum_{i=0}^{n-1} c^i.$$

So to prove Lemma 6.3 we need show that

$$1 - k\left(-kc^{n-1} + \sum_{i=0}^{n-1} c^i\right) \leq -\sum_{i=1}^{n-2} \frac{1}{c^i}$$

which is the same as showing that

$$\phi(k) := Ak - Bk^2 \geq \sum_{i=0}^{n-2} \frac{1}{c^i}, \quad A = \sum_{i=0}^{n-1} c^i, B = c^{n-1}.$$

We need to verify this for $k \in \Gamma := \left(\sum_{i=0}^{n-2} 1/c^i, 1/c^{n-1}\right)$. It is clear that on the closure of Γ, $\phi(k)$ takes its minimal value on the boundary. A simple calculation shows that

$$\phi\left(\sum_{i=0}^{n-2} 1/c^i\right) = \phi\left(1/c^{n-1}\right) = \sum_{i=0}^{n-2} \frac{1}{c^i},$$

and the lemma is proved. It follows that $\hat{K}$ is sink for t^n, containing an attracting n-cycle with multiplier $c^{n-1}k < 1$.

Let $L_i =]z^{(n-i)}, \tau_{i+1}[$ for $i = 2, 3, \ldots, n$. Then for $i < n$, these are precisely the gaps between the intervals K_i on the negative side, and $L_n =]\tau_0, \tau_{n+1}[$. Then

$$L_2 \mapsto L_3 \mapsto \ldots \mapsto L_{n-1} \mapsto L_n.$$

Once again these are all bijections. The image of L_n will be large and cover all of the invariant interval $[\tau_2, \tau_1]$ except for a subinterval of K_2. We note that if an orbit never enters $\hat{K} \cup \bigcup_{i=2}^n K_i$, then this orbit must have positive exponent, so these exclude the possibility of a second periodic attractor.

We now prove that almost all initial conditions will be absorbed into the periodic attractor. We do this by proving that the subset $\Lambda \subset L_n$ of points whose orbit never hits $\hat{K} \cup \bigcup_{i=2}^n K_i$ has zero Lebesgue measure.

Let Λ_1 be those point in L_n that does not map into $\hat{K} \cup \bigcup_{i=2}^n K_i$ in one iterate. It is clear Λ_1 is a finite union of intervals, each of which is mapped bijectively onto L_n in a finite number of steps. Λ_2 is now defined as those points in Λ_1 that after returning to L_n does not hit $\hat{K} \cup \bigcup_{i=2}^n K_i$ in the subsequent iterate, and so on. It is clear that iterating this construction leads to a nested sequence of sets Λ_n such that $\Lambda = \bigcap_{i=1}^\infty \Lambda_i$, which is contained in a generalized Cantor set of bounded type, and so Λ has Lebesgue measure zero □

7 Summary

In this paper elementary proofs of the basic properties of skew tent maps are given. These maps have received a considerable attention in several branches of science during the last decades, e. g. time-series analysis, ecology, econometrics, etc, in which they are frequently referred to as TAR-models after Tong (1990). Simultaneously, mathematical results on the quite special dynamics possessed by these maps (Misiurewisz and Visinescu (1991) and Marcuard and Visinescu (1992), Ichimura (1998), Ito (1998), Ichimura and Ito (1998)) have been largely ignored.

We remark that the dynamical pattern possessed by these models are significantly different from the period doubling route commonly encountered, for example in one parameter families of unimodal maps with negative Schwartzian derivative. In particular, the bifurcation diagram of skew tent maps contain no one parameter curve along which standard period doublings can be observed. Instead, our results demonstrate that skew tent maps typically become chaotic after a sudden emergence of an interval of periodic solutions of period $2p$ around a single periodic solution of period p. Up to our knowledge, this transition has only received limited attention until now.

Our results greatly simplifies the use of continuous variants of TAR-models in one dimension, since they completely characterize their behavior in terms of these two parameters. In addition, we show that each parameter value allows for at most one type of attracting periodic behavior.

Our analysis contains several bifurcation cases that add valuable information concerning the transition to chaos in skew tent maps. Sharp analysis of bifurcation cases are usually needed in control and optimization theory of dynamical systems. As a system is optimized towards a specific in some sense, optimal behavior, difficult special cases are usually selected for. For an example of particularly difficult and demanding studies in this direction, see e. g. Söderbacka (1988) and references therein.

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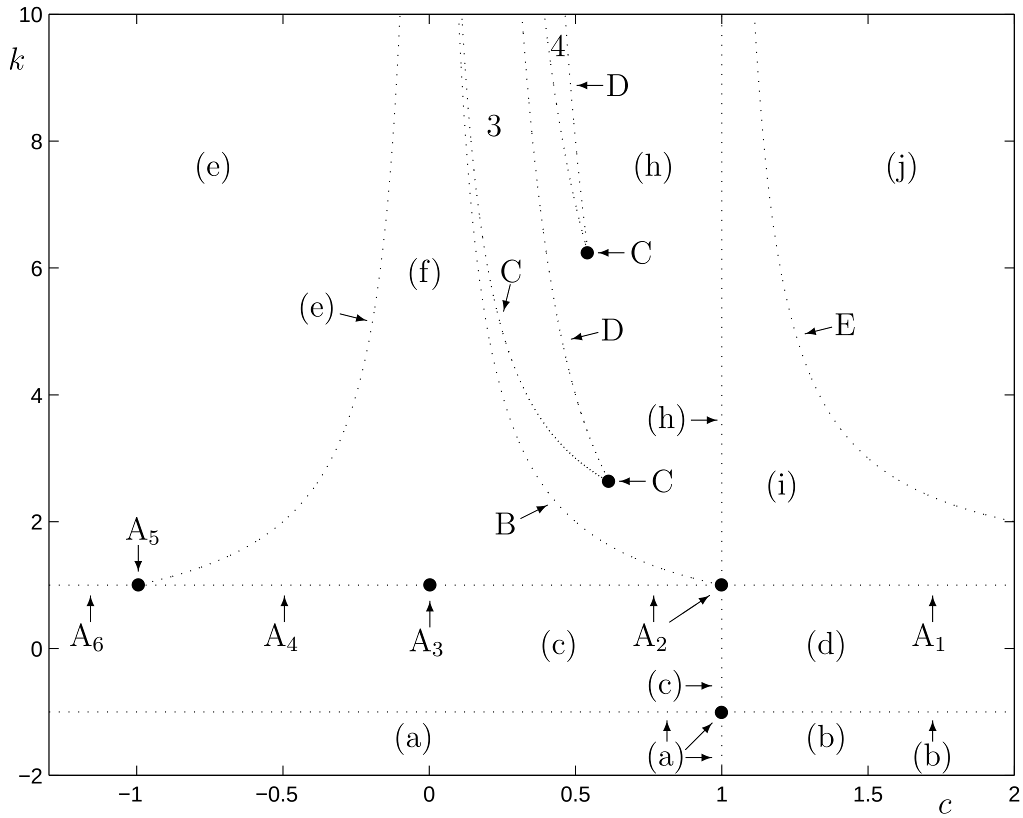

Figure 1: Bifurcation diagram of skew tent maps. The regions corresponding to different dynamical behavior are plotted with dotted lines. (a) All solutions explode to infinity. (b) Some solutions explode to minus infinity, the rest to infinity. (c) Fixed point globally stable. (d) Some solutions tend to the fixed point, the rest to minus infinity. (e) Unbounded oscillations. (f) A two-periodic solution that is globally stable almost everywhere exists, corresponding regions for 3- and 4-periodic almost everywhere globally stable solutions are denoted with 3 and 4, respectively. (h) Almost all solutions are attracted to a bounded chaotic solution. (i) Almost all solutions starting within a special interval are attracted to a bounded and chaotic solution, solutions outside the interval tend to minus infinity. (j) Almost all solutions tend minus infinity. Bifurcating cases are marked with arrows and symbols corresponding to the notation introduced in Sections 4-5.