PROJECT DESCRIPTION: NON-LINEAR DYNAMICS

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1. Background

1.1. **Notation.** We denote the n^{th} fold iteration of a map f by f^n. Most maps considered will be interval maps with a unique critical point, which then is denoted c. Also let $c_1 = f(c)$ be the critical value.

An interval map is *unimodal* if it does have a unique critical point c and is strictly increasing on one side of c, and strictly decreasing on the other.

A C^3 interval map has *negative Schwarzian derivative* if

$$Sf(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 < 0, \quad \forall x \in I \setminus \{c \mid f'(c) = 0\}.$$

If μ is a measure invariant under f, that is if $\mu(E) = \mu(f^{-1}(E))$ for every measurable set E, we say that a point x_0 is *generic* for μ if

$$\mu \overset{\text{weak}^*}{=} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x_0)}.$$

1.2. **Chaotic interval maps.** The following theorem forms a starting point for several parts of this project.

Theorem 1 (Benedicks and Carleson, [BC85], [BC91]). *Let $f_a(x) = 1 - ax^2$. There exists positive constants C and λ and a set $\mathcal{A} \subset (0, 2]$ of positive Lebesgue measure, with 2 a Lebesgue density point of $\mathcal{A}$, such that if $a \in \mathcal{A}$, then*

(1) *f_a has positive Lyapunov exponent at the critical value: $|Df^n(c_1)| \geq Ce^{\lambda n}$, for all $n \geq 1$;*
(2) *f_a admits an acip (absolutely continuous invariant probability measure) μ_a.*

The first proof of the existence of a positive measure set of quadratic maps supporting an acip is due to Jakobson, [Jak81] . Many versions of this theorem has appeared. The theorem generalizes to generic families f_a passing through a so-called Misiurewicz point $a = a^*$. Suitable generic conditions are given in [TTY94], in [MS93] and in [Tsu93]. A generalization to families with flat critical point is given in [Thu99], see Section 2.1.

For S-unimodal maps (unimodal, C^3 interval maps f with negative Schwarzian derivative and non-flat critical point), it is known that a a certain sub-exponential growth of $|Df^n(c_1)|$ is enough to imply the existence of an acip, [NvS91]. This in turn is equivalent, for this class of maps, to positive Lyapunov exponents everywhere, [Kel90]. Such systems are considered chaotic.

1.3. **Natural measures.** An invariant measure is called a *natural* or *observable* measure if the *basin of* μ

$$B(\mu) := \{x \mid x \text{ is generic for } \mu\}$$

has positive Lebesgue measure. These are the measures that one observes in numerical experiments and in real-life processes. Of course, point masses on a periodic attractor is a natural measure. For S-unimodal maps it also holds that every acip is natural, with a basin of full measure. In the special case of quadratic maps it is also known that for almost every $a \in [0,2]$, $f_a(x) = 1 - ax^2$ has either a periodic attractor or an acip, [Lyu97]. Recently Ávila, Lyubich and de Melo has generalized this to unimodal real-analytic families with quadratic critical point. One also knows from the generalizations of Theorem 1 that maps with acips generically occupy a positive measure set in parameter-space, and due to [Koz97] it is known that maps with periodic attractors form a open and dense set in real-analytic families. (This was first proved for the quadratic case in [GSa97].) Therefore natural measures (as a function of the parameter) is a very natural object study.

1.4. **Applications.** Single-species populations are sometimes modeled with discrete one-dimensional dynamical systems. See for example [Ric54] and [HLM74]. The famous article [May76] promoted the interest in these kind of models, by pointing out the possibility of complicated, chaotic dynamics in standard models with relatively simple one-step dynamics. Lately there has been an interest in models with some stochastic component, [GHK94], [HV97] and [Hög97], and discrete-continuous hybrids, [GHL97].

2. Earlier results

2.1. **Flat-topped families.** We consider the following families of flat-topped unimodal systems

$$F_a(x) = F_{a,q}(x) = Q\left(1 - ae^{\frac{q+1}{q} - \frac{1}{|x|^q}}\right) \tag{2.1}$$

where $Q = Q(q)$ is a normalizing constant. Flatness is increasing in q, and keeping q fixed we get one-parameter families $\{F_{a,q}\}_a$.

In [Thu94] I did a first attempt to numerically investigate the Feigenbaum scenario for these families. This was followed up in [Blo95]. The numeric results are not completely conclusive, and a good theoretical understanding of the period-doubling scenario in this degenerate case is still missing.

In [Thu99] I proved that for $q < 1/8$, Theorem 1 generalizes to these families. There are reasons to believe that the theorem should be true for any $q < 1$, and that this is sharp. The basic idea of the proof is the same as in the work by Benedicks and Carleson, but some crucial estimates on the size of excluded parameter sets had to be done in a different way. In particular this involved invoking results in [Nag79] on large deviations of sums of random variables with only finitely many moments.

Lately I have returned to these maps, and I can, using the general framework put fourth in [You98] and [You99], prove that the maps considered in the flat-top version of Theorem 1 exhibits polynomial decay of correlations. In the non-flat case it is known that for maps f_a, $a \in \mathcal{A}$ (c.f. Theorem 1), one has exponential decay of correlations. This is the first examples of so-called Collet-Eckmann maps (positive Lyapunov exponent at the critical value) with polynomial rather that exponential decay. This is still work in progress.

Since much of the general theory for interval dynamics has non-flat critical point as a standing assumption, much remain unknown about these families.

2.2. **Parameter dependence of natural measures.** In [Thu96] I proved that the acip μ_a of a quadratic map f_a with $a \in \mathcal{A}$ of Theorem 1, can be approximated with invariant measures supported on periodic attractors for nearby quadratic maps. This result was generalized to a two dimensional setting in [Ure95] and [Ure96]. In [Thu01b] I showed , in spite of the results in [Thu96], that the map

$$a \mapsto \mu_a \quad \text{(the natural measure of } f_a = 1 - ax^2),$$

which by Lyubich's result is well defined a.e. in parameter space, is severely discontinuous in the weak* topology at every point of the set $\mathcal{A}$. To recover continuity, a positive measure set of parameters must be deleted. The proofs show that these discontinuity results generalizes to both the flat-topped families (2.1) and to those generic families considered in the generalizations of Theorem 1 in [TTY94] and [MS93]. It also follows that in all these cases the set $\mathcal{A}$ is in the closure of the set of parameters corresponding to maps with periodic attractors.

Using quite different techniques (kneading theory) one also proves that these results holds at every strictly finitely renormalizable map in the quadratic family. The proof in [Thu01b] uses certain rigidtity properties which presently only are known for the quadratic family.

2.3. **Applications to population models.** In [Thu01a] (see also [Thu00b] and [Thu00a])I investigated the *Ricker* model [Ric54]

$$R_{\lambda,\beta}(x) = \lambda x e^{-\beta x} \qquad \lambda > 1, \quad \beta > 0 \tag{2.2}$$

and the *Hassell* model [Has74]

$$H_{\lambda,\beta}(x) = \frac{\lambda x}{(1+x)^{\beta}}, \qquad \lambda > 1, \quad \beta > 1. \tag{2.3}$$

These are two standard models in population dynamics. Using state-of-the-art theorems, one readily sees that each of these maps has a unique, metric attractor attracting a.e. initial condition. For an open and dense set of parameters this attractor is a periodic cycle, and for a completely disconnected, positive measure set of parameters the attractor consists of a finite union of intervals, supporting a natural absolutely continuous invariant measure. In the latter case it also holds that we have positive Lyapunov exponents a.e. in phase space. Also the discontinuity results of natural measures mentioned in the previous section and the stochastic stability of acips according to [BV96] do hold.

3. Plan for future work

3.1. **Flat-topped families.** I intend to continue the investigations of the families (2.1). First I want to finish proving the above mentioned theorem about polynomial decay of correlations for certain types of chaotic flat-topped Collet-Eckmann maps.

There are also many other questions of future interest in the flat setting: Properties of the invariant acips previously constructed. To prove the sharp bound $q < 1$ (c.f. Section 2.1). Renormalization theory. Uniqness of attractors ...

3.2. **Natural measures.** I would like to continue the study of the parameter dependence of natural measures. One objective is to get estimates on the size of parameter sets around points in $\mathcal{A}$ where $a \mapsto \mu_a$ is (dis)continues (c.f. Theorem 1 and Section 2.2). Some results in this direction should be available by a careful analysis of the proof of (the generalized) Theorem 1 and the proofs in [Thu01b]. There is also some hope that one could use the tower constructions in [You98] and [You99] to this end.

3.3. **Persistence of weak hyberbolicity.** Generalizing the methods introduced by [BC85] and [BC91] (see also for example [Luz00] and [Thu99]), we hope to prove that within generic one-parameter families, even maps with weak non-uniformly hyperbolic behavior (e.g. polyomial growth of derivatives) are Lebesgue density points of, say, Collet-Eckmann maps. One source of inspiration here is the work in [BLS00], which indeed uses some generalizations of Benedick's and Carleson's methods. This is an ongoing joint project with Dr. Stefano Luzzatto of Imperial College in London.

3.4. **Skew tent-maps.** Suprisingly little seems to been known about skew tent maps, that is, continuous affine interval maps with two branches with different slopes. In particular when the left branch lies above the diagonal and has slope less than one, and the right branch intersects the diagonal and has slope of modulus larger than one, there is an interesting mixture of contracting and expanding dynamics. This was introduced as a possible joint project by Dr. Torsten Lindström from University of Kalmar.

3.5. **Population models.** I plan to continue investigations of standard models in population dynamics, using up-to-date theory of dynamical systems, with a focus on robust, generic behavior seen with positive probability among parameters and initial conditions. One interesting question is the relation between stochastic and deterministic models. One would like to relate the invariant measures of a deterministic model to the quasi-stationary distributions of a suitable stochastic version. This is done in [Hög97], where the deterministic models are Ricker maps, (2.2), and the stochastic versions are certain branching process. The results in this paper is for Ricker maps with period attractors of length 2^k, below the Feigenbaum map. An idea is to generalize this to maps with periodic attractors above the Feigenbaum point, and also to maps with natural acips.

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