12.3.3.

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}\ ,\ \ 0<x<L\ ,\ \ t>0\ .$$

$$\text{Randvillkor}:\begin{cases}\dfrac{\partial u}{\partial x}(0,t)=0\\ \dfrac{\partial u}{\partial x}(L,t)=0\end{cases}\qquad t>0\ .$$

$$\text{Begynnelsevillkor}: u(x,0)=f(x)\ ,\ \ 0<x<L\ .$$

Separera variablerna: $u(x,t) = X(x)T(t)$.

$$kX''(x)T(t) = X(x)T'(t)$$

Multiplicera med $\dfrac{1}{kX(x)T(t)}$.

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{konstant} = \lambda \; .$$

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ T'(t) - \lambda k T(t) = 0 \end{cases}$$

$$\lambda > 0,\ \lambda = \mu^2,\ \ \mu \quad R.$$

$$X''(x) - \mu^2 X(x) = 0$$

Lösningarna ges av $X(x) = A_1 e^{\mu x} + B_1 e^{-\mu x}$.

$$\lambda = 0$$

$$X''(x) = 0$$

$$X(x) = A_2 x + B_2$$

$$\lambda < 0,\ \lambda = -\mu^2,\ \ \mu \quad R.$$

$$X''(x) + \mu^2 X(x) = 0$$

$$X(x) = A_3 \cos \mu x + B_3 \sin \mu x$$

Substitutionen ger att randvillkoren kan skrivas

$$0 = \frac{\partial u}{\partial x}(0,t) = \mathrm{X}'(0)\mathrm{T}(\mathrm{t})$$

$$0 = \frac{\partial u}{\partial x}(L,t) = \mathrm{X}'(\mathrm{L})\mathrm{T}(\mathrm{t})$$

Dessa samband skall gälla för alla t .

Detta innebär att: $0 = \mathrm{X}'(0)$, $0 = \mathrm{X}'(\mathrm{L})$.

$$\lambda > 0$$

$$X'(x) = \mu(A_1 e^{\mu x} - B_1 e^{-\mu x})$$

$$\begin{cases} 0 = X'(0) = \mu(A_1 - B_1) \\ 0 = X'(L) = \mu(A_1 e^{L\mu} - B_1 e^{-L\mu}) \end{cases}$$

$$A_1 = B_1 = 0$$

Endast den triviala lösningen: $u = 0$.

$$\lambda = 0$$

$$X'(x) = A_2$$

$$\begin{cases} 0 = X'(0) = A_2 \\ 0 = X'(L) = A_2 \end{cases}$$

$$X(x) = B_2$$

$$T(t) = C_2$$

$$\lambda < 0$$

$$X'(x) = \mu(-A_3 \sin \mu x + B_3 \cos \mu x)$$

$$\begin{cases} 0 = X'(0) = \mu(B_3) \\ 0 = X'(L) = \mu(-A_3 \sin \mu L + B_3 \cos \mu L) \end{cases}$$

$$B_3 = 0 \qquad A_3 \sin \mu L = 0$$

Icke triviala lösningar erhålles då: $\mu L = n\pi$.

$$X(x) = A_3 \cos \frac{n\pi x}{L} \qquad T(t) = C_3 e^{-(\frac{n\pi}{L})^2 kt}$$

$$u(x,t) = B_2C_2 + \sum_{n=1}^{\infty} (B_3C_3)_n \cos\frac{n\pi x}{L} e^{-(\frac{n\pi}{L})^2 kt}$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} e^{-(\frac{n\pi}{L})^2 kt}$$

Begynnelsevillkoret ger:

$$f(x) = u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} e^{-(\frac{n\pi}{L})^2 kt}$$