Matematiska Institutionen
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Examen i Algebra g.k., 5B1309
9. mars 2004, klockan 14.00-19.00

Preliminary limits. Grade 3 requires at least 28 points, grade 4 at least 38 points and grade 5 at least 48 points. Good luck!

I. Groups.

a. Describe, up to isomorphism, all abelian groups of order 18. (4p)

b. Let G be a group of prime order. Show that G is cyclic. (4p)

c. Give an example of an abelian group of order 4, which is not cyclic. (4p)

II. Definitions. Let R be a commutative ring.

a. Let $I \subseteq R$ be an ideal. What is the definition of I being a prime ideal. (4p)

b. Describe the prime ideals in $R = \mathbf{Z}$. (4p)

c. Let R an integral domain. Define what a prime element is. (4p)

III. Polynomial rings. We let $\mathbf{Z}$ denote the integers, we let $\mathbf{Q}$ denote the field of rational numbers, and we fix the polynomial $F = X^2 - 5$ in the variable X.

a. Show that F is irreducible in $\mathbf{Z}[X]$. (4p)

b. Is F irreducible considered as an element in $\mathbf{Q}[X]$? (4p)

c. Is the quotient ring $\mathbf{Z}[X]/(F)$ a field? (4p)

d. Compute the norm $N(\xi)$ and the discriminant $D(\xi)$ of a root ξ of F. (4p)

IV. Group action. Let Σ be a fixed square, and let l be a vertical line through the center of Σ and fix also one of the diagonals d of Σ. We let λ denote the reflection through the line l and we let δ denote the reflection through d.

a. Describe the operations λ^2, $\lambda\delta$. (4p)

b. The group D of symmetries of the square has order 8. Show that any symmetry of the square can be describe by successive use of λ and δ. (4p)

c. We name the four vertices of the square as A, B, C and D. Each symmetry σ of the square gives a permutation of the vertices. Can any permutation of the four elements A, B, C, D be realized as a symmetry operation of the square? (4p)

d. We want to paint the edges of the square, having 5 different colors at our disposal. How many different ways can that be done - two paintings are considered as equal if a symmetry operation moves one painted square over to another. (8p)

Answers

I. Groups.

a. The abelian groups of order $18 = 2 \cdot 3 \cdot 3$ are $\mathbf{Z}_{18}$ and $\mathbf{Z}_6 \times \mathbf{Z}_3$.

b. As G is of prime order there exists at least one element x different from the identity element in G. Consequently the subgroup generated by x contain at least two elements. The order of $< x >$ must divide the order of G, which is a prime. Hence the order of $< x >$ equals the order of G, and consequently $< x >= G$.

c. The Klein Vier group $\mathbf{Z}_2 \times \mathbf{Z}_2$ is not cyclic, but abelian and with four elements.

II. Definitions.

a. A proper ideal I of a commutative ring R is prime if for any a, b in R such that $ab \in I$ it follows that either a or b is in I.

b. Let I be an ideal of $\mathbf{Z}$. Then we can find a non-negative number n that generates I. The ideal I is prime if and only if $n = 0$ or if n is a prime number.

c. An element f in an integral domain R is prime if f is non-zero and not a unit and is such that if f is a divisor in a product ab then f is a divisor in of the factors a or b.

III. Polynomial rings.

a. The roots of the polynomial $F = X^2 - 5$, as an element of $\mathbf{C}[X]$ are $\pm\sqrt{5}$. As $\sqrt{5}$ is not an integer, we have that F is irreducible.

b. As F is irreducible in $\mathbf{Z}[X]$ we have that the roots of F are not rational, hence F is also irreducible in $\mathbf{Q}[X]$.

c. We have the norm map $\mathbf{Z}[X]/(F) \to \mathbf{Z}$, and those elements in $\mathbf{Z}[X]/(F)$ that have norm ± 1 are invertible. As the norm of any integer n is n^2 and the norm of 0 is zero, it follows that there are plenty of non-zero elements that are not invertible. Hence the quotient ring is not a field.

d. The norm of a root ξ of F is $\xi\xi' = -5$, where ξ is the conjugate root; $F = (x - \xi)(x - \xi')$. The discriminant is $(\xi - \xi')^2 = (2\sqrt{5})^2 = 20$.

IV. Group action.

a. We clearly have that $\lambda^2 = 1$, and that $\lambda\delta$ is a quarter of a rotation. The rotation is either counter clockwise or clockwise depending on which diagonal d you have fixed.

b. Let us denote $\rho = \lambda\delta$ which is our rotation. We the have the four elements $1, \rho, \rho^2$ and ρ^3 in addition to the two reflections λ and δ, which gives us a total of 6 elements. The two other elements in the group D are the following. The reflection λ' along the horizontal line, and the reflection δ' along the other diagonal. As we have that $\rho^2\lambda = \lambda'$ and $\rho^2\delta = \delta'$ we have that the group D is generated by δ and λ.

c. No, the group S_4 has order $4! = 24$ which is strictly bigger than $\mid D \mid= 8$.

d. Let X denote the set of all $5 \cdot 5 \cdot 5 \cdot 5$ squares we obtain by using 5 colors on the edges. The group D acts as described on the set X and we want to compute the number of orbits. By Burnsides Theorem we have the following formula

$$\mid X/D \mid = \frac{1}{\mid D \mid} \sum_{g \in D} \mid X^g \mid .$$

We use the notation of exercise b). We have $X^e = X$, and $\mid X^\rho \mid = \mid X^{\rho^3} \mid = 5$, and $\mid X^{\rho^2} \mid = 5 \cdot 5$. Furthermore we have that $\mid X^\lambda \mid = \mid X^{\lambda'} \mid = 5 \cdot 5 \cdot 5$, and that $\mid X^\delta \mid = \mid X^{\delta'} \mid = 5 \cdot 5$. When we plug these numbers in the Burnside formula we end up with 120 orbits.