Matematiska Institutionen
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Examen i Algebra g.k., 5B1309
20 april, klockan 08.00-13.00

Preliminary limits. Grade 3 requires at least 28 points, grade 4 at least 38 points and grade 5 at least 48 points. Good luck!

I. Homomorphism. Let $\mathbf{Q}$ denote the rational numbers.

a. Define what a homomorphism of groups is. (4p)

b. Show that the compositon $a * b = a + b - ab$ makes $\mathbf{Q} \setminus 1$ into a group. (4p)

c. Show that $(\mathbf{Q} \setminus 1, *)$ is isomorphic to the multiplicative group $\mathbf{Q}^*$. (4p)

d. Is the group $(\mathbf{Z}, +)$ isomorphic to $(\mathbf{Q}, +)$? (4p)

II. Definitions.

a. What are the prime ideals in $K[X]$, with K a field. (4p)

b. Let R an integral domain, define what an irreducible element is. (4p)

c. Is the element $2x^2 - 4$ irreducible in $\mathbf{Z}[X]$? (4p)

III. Polynomial rings. We let ξ denote the residue class of the variable X in $\mathbf{Z}[X]/(F)$, where $F = X^2 - 1$.

a. Show that F is irreducible in $\mathbf{Z}[X]$. (4p)

b. Is F irreducible considered as an element in $\mathbf{R}[X]$? (4p)

c. Compute the norm and the discriminant of the element $5 + 2\xi$. (4p)

d. Is $5 + 2\xi$ a divisor in $7 - \xi$? (4p)

IV. Signature. We let σ be a permutaion of n (unordered) numbers X, and we represent σ with the matrix $M_\sigma = \binom{x_1, \cdots, x_n}{y_1, \cdots, y_n}$. Given such a matrix representation we define the fraction

$$F(\sigma) = \prod_{i<j} \frac{x_j - x_i}{y_j - y_i}.$$

a. Show that the value of the fraction $F(\sigma)$ is ± 1. (4p)

b. Show that the fraction $F(\sigma)$ is independent of the matrix representation. (4p)

c. Show that $F(\sigma\nu) = F(\sigma)F(\nu)$, for any two permutations σ and ν of X. (4p)

d. Show that the fraction $F(\sigma)$ equals the signature of σ. (4p)

Answers

I. Homomorphism.

a. A homomorphism of two groups G and G' is a map of sets $f : G \to G'$ such that $f(ab) = f(a)f(b)$ for all a, b in G.

b. Let $G = \mathbf{Q} \setminus 1$. First we note that the composition $a * b := a + b - ab$ is well-defined and associative on $\mathbf{Q}$. We first show that when restricting to G we get a map $G \times G \to G$. However the identity $a * b = a + b - ab = 1$ is equivalent with

$$0 = 1 - (a + b) + ab = (1 - a)(1 - b).$$

And consequently we have that $*$ is a well-defined map $G \times G \to G$. As the composition is associative and commutative, we need only to check left identity and left inverse. Cleary $0 \in G$ is an identity. Let $x \in G$ be given, then an inverse to x is obtained by solving

$$0 = x * y = x + y - xy \leftrightarrow x = y(x - 1).$$

As $x \neq 1$ we obtain $y = x/(x - 1) \in G$.

c. We define the map $f : G \to \mathbf{Q}^*$ by $f(x) = 1 - x$, which is clearly well-defined. It is furthermore a homomorphism as

$$f(x * y) = f(x + y - xy) = 1 - (x + y) + xy = (1 - x)(1 - y) = f(x)f(y).$$

The morphism f is clearly injective and surjective, hence an isomorphism.

d. No, the two groups $\mathbf{Z}$ and $\mathbf{Q}$ can not be isomorphic as the first one is cyclic, whereas the second is not.

II. Definitions.

a. The primeideals are ideals generated by irreducible polynomials and the zero ideal.

b. An element x in a integral domain R is irreducible if it non-zero and not a unit and if it contains only trivial divisors.

c. No since $2x^2 - 4 = 2(x^2 - 2)$ and 2 is not a trivial divisor.

III. Polynomial rings.

a, b. The polynomial in question is $F(x) = X^2 + 1$. As the roots of $F = X^2 + 1$ are the complex it follows that F is irreducible in $\mathbf{Z}[X]$ and $\mathbf{R}[X]$.

c. The norm $N(5 + 2\xi) = 25 + 4 = 21$, while the discriminant is $D(5 + 2\xi) = 4D(\xi) = -16$.

d. The norm of $5 + 2\xi$ is 29, whereas the norm of $7 - \xi$ is 49+1=50. The norm is multiplicative and consquently as 29 does not divide 50 we can not have $5 + 2\xi$ as a divisor of $7 - \xi$.

IV. Signature.

a. Each factor $(x_i - x_j)$ in the denominator appears once either as $(x_i - x_j)$ or once as $(x_j - x_i) = -(x_i - x_j)$ in the numerator. Hence, clearly the fraction has value ± 1.

b. Note that the value of the fraction $F(\sigma)$ remains unchanged if two coloumns in the matrix representation is interchanged. As any two matrix representations of the permutation differ by a rearrangement of the rows it follows that the fraction $F(\sigma)$ is independent of the matrix representation.

c. Let us fix a matrix representation $M_\nu = \binom{x_1,\cdots,x_n}{y_1,\cdots,y_n}$ and chose a matrix representation of σ in such a way that the upper row corresponds to the lower row of M_ν. That is we write $M_\sigma = \binom{y_1,\cdots,y_n}{z_1,\cdots,z_n}$. We then have

$$F(\sigma)F(\nu) = \prod_{i<j} \frac{y_j - y_i}{z_j - z_i} \prod_{k<l} \frac{x_l - x_k}{y_l - y_k} = \prod_{i<j} \frac{x_j - x_i}{z_j - z_i}.$$

The latter product equals the fraction of $F(\sigma\nu)$ as the composition has the matrix representation $M_{\sigma\nu} = \binom{x_1,\cdots,x_n}{z_1,\cdots,z_n}$.

d. As any permutation can be written as a product of transpositions and we have the result of c, we need only check that the fraction $F(\tau)$ of a transposition τ is -1. This is however clear.