Matematiska Institutionen
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Examen i Algebra g.k., 5B1309
17. augusti 2004, klockan 14.00-19.00

Preliminary limits. Grade 3 requires at least 28 points, grade 4 at least 38 points and grade 5 at least 48 points. Good luck!

I. Groups. In a group G we let e denote the identity.

a. Describe, up to isomorphism, all abelian groups of order 24. (4p)

b. Let g be any element in a finite group G. Show that $g^{|G|} = e$. (4p)

c. When is a subgroup $N \subseteq G$ normal? (4p)

d. Give an example of a subgroup which is not normal. (6p)

II. Rings. Let R be a commutative ring.

a. What is the characteristic of a ring R? (4p)

b. When is R an integral domain? (4p)

c. Given an example of a prime ideal not being a maximal ideal (6p)

III. Polynomial rings. We let $\mathbf{Z}[X]$ denote polynomial ring over the integers. We fix the polynomial $F = 2X^2 - 4$ and let $R = \mathbf{Z}[X]/(F)$ denote the quotient ring.

a. Show that R has zero-divisors. (4p)

b. Show that the quotient ring $R/(2)$ is an integral domain. (4p)

IV. Group action. Let G be a finite group acting by conjugation on $X = G$. For any element $x \in G$ we let $G.x$ denote its orbit and G_x its isotropy group. Let $B_1, \dots, B_n$ be the orbits of length > 1, and let p be a prime number.

a. For any $x \in G$, show that we have a bijection (of sets) $G/G_x = G.x$. (4p)

b. Show that $|G| = |\mathrm{Cent}(G)| + \sum_{j=1}^{n} |B_j|$. (4p)

c. Show that center of a p-group G contains at least two elements. (4p)

d. Show that any group of order p^2 is abelian. (8p)

ANSWERS

I. Groups.

a. The abelian groups of order $24 = 2 \cdot 2 \cdot 2 \cdot 3$ are

$$\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_6, \mathbf{Z}_4 \times \mathbf{Z}_6, \mathbf{Z}_2 \times \mathbf{Z}_{12} \quad \text{and} \quad \mathbf{Z}_{24}.$$

b. As the order of any subgroup H divides the order of the group G, we have that the order m the cyclic subgroup $< g >$ divides H. That is $g^m = e$ and $|G| = tm$ for some integer t. So in particular we have $e = e^t = g^{mt} = g^{|G|}$.

c. A subgroup N of G is normal if $gNg^{-1} \subseteq N$ for all elements $g \in G$.

d. Consider the group S_3 of permutations of $\{1, 2, 3\}$ and the subgroup $H = \{e, \tau\}$ where τ is the permutation of $1, 2$. Consider $g \in S_2$ that permutes $\{2, 3\}$. We then look at the permutation $g\tau g^{-1}$. It is clear that $g = g^{-1}$ and we have that the effect of the permuation $g\tau g^{-1}$ is

$$1 \mapsto 3, \quad 2 \mapsto 3 \quad \text{and } 3 \mapsto 1.$$

Thus $g\tau g^{-1}$ is not in H, and consequently H is not normal.

II. Rings.

a. If there is an integer n such that $n \cdot 1 = 0$ in a ring R then the characteristic of R is the smallest positive such n. If $n \cdot 1 \neq 0$ for all integers n, the characteristic of R is zero.

b. A commutative ring is an integral domain if the zero ideal is a prime ideal.

c. In the ring of integers $R = \mathbf{Z}$ we have the zero ideal which is prime. However, as $\mathbf{Z}$ is not a field the zero ideal is not maximal.

III. Polynomial rings.

a. We have that neither 2 or $X^2 - 2$ is in the ideal generated by $2X^2 - 4$, but clearly their product is. Hence $(2X^2 - 4)$ is not prime and equivalently the quotient ring is not an integral domain.

b. We clearly have a proper inclusion of ideals $(F = 2X^2 - 4) \subset (2)$, and consequently the ideal (2) is a non-trivial ideal in the quotient ring $R = \mathbf{Z}[X]/(F)$. Consequently we have that $R/(2) = \mathbf{Z}[X]/(2) = F_2[X]$, the polynomial ring over the field of 2 elements. The polynomial ring over a field (or a domain) is an integral domain.

IV. Group action.

a. We have a surjective map $G \to G.x$ sending $g \mapsto g.x$, and we will first show that the map is constant on the conjugacy classes of G_x. Let a and b be two elements in G representing the same conjugacy class, i.e. $aG_x = bG_x$. It follows that $a = bh$ for some $h \in G_x$, and consequently that

$$a.x = (bh).x = (b(h.x)) = b.x \tag{*}$$

Hence we have an induced surjective map of sets $G/G_x \to G.x$. The map is also injective; Let a and b be elements in G such that their conjugacy class determine the same element in the orbit. That is, the elements a and b satisfy the equation (*). And consequently we have $1.x = x = a^{-1}b.x$, that is $a^{-1}b$ is in the isotropy group G_x. We then have that the conjugacy classes determined by a and b are equal and the map is injective.

b. For any G-set X we have that the orbits form a partition of X. As X is finite each orbit is of finite length. Thus we have $|X| = \sum |\text{Orbits}|$. Let $A_1, \ldots, A_m$ be the orbits of lenght 1, i.e. fix points, and we have $|X| = \sum_i |A_i| + \sum_j |B_j|$. However as the fixed points in X under the conjugate action is precicely the elements in the center, we are done.

c. Let $x \in B_j$ be an element in the orbit B_j. From a) we have that $|B_j| = |G : G_x|$ as the index is precisely the number of conjugacy classes. Furthermore, by definition of B_j the orbit contains at least two elements, and by the index Theorem of Langrange it follows that $|B_j| \cong 0 \mod (p)$. Consequently the equality in b) gives $0 \cong |G| \cong |\text{Cent}(G)|$. In particular we have that the center can not consist of one element, hence the center is not trivial.

d. Let $x \in G$ be an element different from the identity element, but in the center of G. By c) we can find such an element. Let $H =< x >$ be the subgroup generated by x. The order of H is not 1 by assumption on x, hence the order is either p or p^2. If the order is p^2 the group G is cyclic and we are done. We need therefore only consider the case when $|H| = p$. Let y be an element in G, but not in H. As x is in the center we have $y^m x^n = x^n y^m$ for any m, n. We have furthermore that the set $G' = \{y^m x^n \mid m, n \in \mathbf{Z}\}$ form a subgroup of G. The group G' contains H as a proper subgroup, and it follows that $G' = G$. However G' is abelian, hence so is G.