Control Synthesis Using LMIs

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What we will learn

- Elimination lemma
  - Eliminates parameters in design problem.
  - Results in Linear Matrix Inequalities (LMI).
  - Used in control synthesis, optimal filter design, model reduction.
- Developments in the 1990s
- We apply it to control synthesis

Good References for the Lecture

(1) S. Boyd, L. EL Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM Studies in Applied Mathematics, Philadelphia, 1994.

(2) G.E. Dullerud and F. Paganini. *A course in Robust Control Theory.* Springer, New York, 2000.

(3) L. El Ghaoui and S-I Niculescu, editors. *Advances in Matrix Inequality Methods in Control.* Advances in Design and Control. SIAM, 2000.

(4) R.E. Skelton, T. Iwasaki, and K. Grigoriadis. *A Unified Algebraic Approach to Linear Control Design.* Taylor and Francis, 1998.

Some contributers

- A. Packard (first gain scheduling paper e.t.c.)
- Apkarian/Gahinet ($\mathbf{H}_\infty$-paper, Matlab toolbox, e.t.c.)
- Iwasaki/Skelton ($\mathbf{H}_\infty$-paper, book, e.t.c.)
- A. Helmersson (gain scheduling, model reduction e.t.c.)
- C. Scherer (many nice synthesis results)
- Beck/Doyle/Glover (model reduction)

Highlights of the Lecture

- Output feedback synthesis
  - Elimination lemma (main theoretical result)
  - Stabilization theorem
- Design for performance
  - KYP Lemma
  - $\mathbf{H}_\infty$-synthesis (special case)

Quotation from book by Skelton, Iwasaki, and Grigoriadis:

The Elimination theorem is the most important result in this book. Almost all control problems in this book can be analytically solved by this theorem, i.e., approximately 20 different control problems all reduce to this problem of linear algebra. The point of the book is to show how to rearrange these problems so that they take the form of the elimination theorem

Notation

- $\mathcal{S}_+^{n\times n} = \{M \in \mathbf{R}^{n\times n} : M = M^T > 0\}$
- Finite dimensional transfer functions

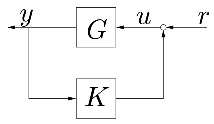
$$H(s) = C(sI-A)^{-1}B + D =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

- $\mathbf{H}_\infty$-norm of a finite dimensional transfer function

$$\begin{aligned}\|H\|_{\mathbf{H}_\infty} &= \sup_{\operatorname{Re} s\geq 0} \sigma_{\max}(H(s)) \\ &= \begin{cases} H \text{ is stable, i.e., } \operatorname{Re}\operatorname{eig}(A) < 0 \\ \sup_{\omega\in[0,\infty]} \sigma_{\max}(H(j\omega)) \end{cases}\end{aligned}$$

where $\sigma_{\max}(\cdot)$ is the largest singular value

Output Feedback Synthesis



$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = C(sI-A)^{-1}B$$

$$K(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] = C_k(sI-A_k)^{-1}B_k + D_k$$

$$G_{cl}(s) = (I - G(s)K(s))^{-1}G(s) = \left[\begin{array}{cc|c} A+BD_kC & BC_k & B \\ B_kC & A_k & 0 \\ \hline C & 0 & 0 \end{array}\right]$$

Closed Loop System Matrix

$$A_{cl} = \begin{bmatrix} A + BD_kC & BC_k \\ B_kC & A_k \end{bmatrix}$$

Define

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}$$

Then $A_{cl} = \mathcal{A} + \mathcal{B}K\mathcal{C}$, where

$$K = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$$

is the matrix of controller parameters

Design for Quadratic Stability

Simplest design objective is stability of A_{cl}

Theorem 1 (Lyapunov). *The following are equivalent*

(i) A_{cl} *is stable* ($\mathrm{Re}\,\mathrm{eig}(A_{cl}) < 0$)

(ii) $\exists P \in \mathcal{S}_+^{(n_G+n_K)\times(n_G+n_K)}$ *s.t.* $PA_{cl} + A_{cl}^TP < 0$

Design Problem: Find $P \in \mathcal{S}_+^{(n_G+n_K)\times(n_G+n_K)}$ and $K = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$ such that

$$P(\mathcal{A} + \mathcal{B}K\mathcal{C}) + (\mathcal{A} + \mathcal{B}K\mathcal{C})^TP < 0$$

- Not convex since P multiplies K

Elimination Lemma

Definition 1. An orthogonal complement $N_\perp$ is any matrix of maximal rank such that $N_\perp N = 0$ and $N_\perp N_\perp^T > 0$.

Lemma 1. *Let* $M \in \boldsymbol{R}^{n\times k}$, $N \in \boldsymbol{R}^{n\times m}$, *and* $H = H^T \in \boldsymbol{R}^{n\times n}$. *The following statements are equivalent*

(i) *There exists* $X \in \boldsymbol{R}^{m\times k}$ *such that*

$$NXM^T + MX^TN^T + H < 0$$

(ii) *The following two conditions hold*

$$N_\perp HN_\perp^T < 0 \quad \text{or} \quad NN^T > 0$$
$$M_\perp HM_\perp^T < 0 \quad \text{or} \quad MM^T > 0$$

Applied to our problem

$$P(\mathcal{A} + \mathcal{B}K\mathcal{C}) + (\mathcal{A} + \mathcal{B}K\mathcal{C})^TP < 0$$
$$\Leftrightarrow$$
$$\underbrace{(P\mathcal{B})}_{N}\underbrace{K}_{X}\underbrace{\mathcal{C}}_{M^T} + \mathcal{C}^TK^T(\mathcal{B}^TP) + \underbrace{P\mathcal{A} + \mathcal{A}^TP}_{H} < 0$$

By elimination lemma this is equivalent to

$$N_\perp(P\mathcal{A} + \mathcal{A}^TP)(N_\perp)^T < 0$$
$$M_\perp(P\mathcal{A} + \mathcal{A}^TP)(M_\perp)^T < 0$$

- $N_\perp = \mathcal{B}_\perp P^{-1} = \begin{bmatrix} B_\perp & 0 \end{bmatrix} P^{-1}$
- $M_\perp = (\mathcal{C}^T)_\perp = \begin{bmatrix} (C^T)_\perp & 0 \end{bmatrix}$

Equivalent stabilization problem:
Find $P \in \mathcal{S}_+^{(n_G+n_K)\times(n_G+n_K)}$ such that

$$\begin{bmatrix} B_\perp & 0 \end{bmatrix} \left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} P^{-1} + P^{-1} \begin{bmatrix} A^T & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} B_\perp^T \\ 0 \end{bmatrix} < 0$$

$$\begin{bmatrix} (C^T)_\perp & 0 \end{bmatrix} \left(P \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A^T & 0 \\ 0 & 0 \end{bmatrix} P \right) \begin{bmatrix} ((C^T)_\perp)^T \\ 0 \end{bmatrix} < 0$$

- Still not convex.
- We fix this by introducing block structure for P and P^{-1}
- $P = \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}$
- The next lemma shows that it is enough to find $X, Y \in \mathcal{S}_+^{n_G \times n_G}$

Lemma 2. *Suppose $X, Y \in \mathcal{S}_+^{n_G \times n_G}$. The following are equivalent*

(*i*) *There exists $X_2, Y_2 \in \boldsymbol{R}^{n_G \times n_K}$ and $X_3, Y_3 \in \mathcal{S}_+^{n_K \times n_K}$ s.t.*

$$\begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \tag{1}$$

(*ii*) *We have*

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \quad \text{and} \quad \operatorname{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n_G + n_K \tag{2}$$

Stabilization Theorem

Theorem 2. *The following are equivalent*

- *$G(s) = C(sI - A)^{-1}B$ can be stabilized by n_K-dim controller.*
- *There exists $X, Y \in S_+^{n_G \times n_G}$ s.t.*

$$B_\perp(AY + YA^T)B_\perp^T < 0$$
$$(C^T)_\perp(XA + A^TX)((C^T)_\perp)^T < 0$$
$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \text{ and rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n_G + n_K$$

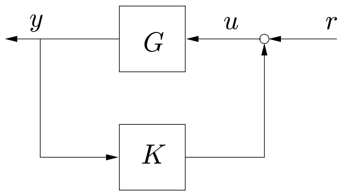
- Convex if $n_K \geq n_G$ since the rank constraint can be removed.
- How do we get the controller parameters?
- How to solve rank constrained problem?

Controller Reconstruction

- Suppose $X, Y \in \mathcal{S}_+^{n_G \times n_G}$ satisfies (*ii*) in Theorem 2.
- By Schur complement $X - Y^{-1} = X_2 X_2^T$ for some $X_2 \in \mathbf{R}^{n_G \times n_K}$
- Then $P = \begin{bmatrix} X & X_2 \\ X_2^T & I \end{bmatrix} > 0$ (by Schur complements formula).
- $\begin{bmatrix} X & X_2 \\ X_2^T & I \end{bmatrix}^{-1} = \begin{bmatrix} Y & -YX_2 \\ -X_2^TY & I + X_2^TYX_2 \end{bmatrix}$
- We have found our Lyapunov matrix!
- Find controller parameters from LMI

$$P(\mathcal{A} + \mathcal{B}K\mathcal{C}) + (\mathcal{A} + \mathcal{B}K\mathcal{C})^T P < 0$$

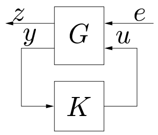
Static Output Feedback



Find conditions that ensure existence of a stabilizing static output feedback controller $K \in \mathbf{R}^{m\times p}$

The previous result gives the next corollary

Design for Performance



- e disturbance signal
- z measured output

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

$$K(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right]$$

Heuristic for Rank Constrained Problems

One heuristic: Solve

$$\begin{aligned} &\min \operatorname{trace}(X+Y) \quad \text{subj. to} \\ &B_\perp(AY + YA^T)B_\perp^T < 0 \\ &(C^T)_\perp(XA + A^TX)((C^T)_\perp)^T < 0 \\ &\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \end{aligned}$$

The hope is that we get rank $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n_G + n_K$

orollary 1. *The following are equivalent*

(i) $\exists\ K \in \boldsymbol{R}^{m\times p}$ *s.t.* $A + BKC$ *is stable.*

(ii) $\exists\ P, Q \in \mathcal{S}_+^{n_G\times n_G}$ *s.t.*

$$\begin{aligned} &B_\perp(AQ + QA^T)B_\perp^T < 0 \\ &(C^T)_\perp(PA + A^TP)((C^T)_\perp)^T < 0 \\ &\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0 \text{ and rank} \begin{bmatrix} P & I \\ I & Q \end{bmatrix} \leq n_G \end{aligned}$$

Proof. Use $P = X$ and $Q = Y = P^{-1}$ in previous result. □

- This is an NP hard problem
- It is an open problem whether there exists a test with polynomial complexity.

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Closed Loop System

- $G_{cl} = G_{11} + G_{12}(I - KG_{22})^{-1}G_{21}$

$$G_{cl}(s) = \left[\begin{array}{cc|c} A + B_2 D_k C_2 & B_2 C_k & B_1 + B_2 D_k D_{21} \\ B_k C_2 & A_k & B_k D_{21} \\ \hline C_1 + D_{12} D_k C_2 & D_{12} C_k & D_{11} + D_{12} D_k D_{21} \end{array}\right]$$

$$=: \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right]$$

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Let $K = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$ and define

$$\mathcal{A}_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_0 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix},$$

$$\mathcal{C}_0 = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}, \quad \mathcal{D}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix},$$

$$\mathcal{D}_{12} = \begin{bmatrix} 0 & D_{12} \end{bmatrix}, \quad \mathcal{D}_0 = D_{11}$$

Then

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ \mathcal{C}_0 & \mathcal{D}_0 \end{bmatrix} + \begin{bmatrix} \mathcal{B} \\ \mathcal{D}_{12} \end{bmatrix} K \begin{bmatrix} \mathcal{C} & \mathcal{D}_{21} \end{bmatrix}$$

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Design Specifications

1. Closed loop system stable
2. Performance on chanel $e \to z$

$$\int_0^\infty \begin{bmatrix} z(t) \\ e(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} z(t) \\ e(t) \end{bmatrix} dt < 0 \quad \Leftrightarrow \quad \int_{-\infty}^\infty \begin{bmatrix} \hat{z}(j\omega) \\ \hat{e}(j\omega) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \hat{z}(j\omega) \\ \hat{e}(j\omega) \end{bmatrix} d\omega < 0$$

$$\Leftrightarrow \quad \begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix} < 0, \ \forall \omega \in [0, \infty]$$

Follows "since"

- $\hat{z}(j\omega) = G_{cl}(j\omega)\hat{e}(j\omega)$
- e can be chosen close to worst case sinusoidal

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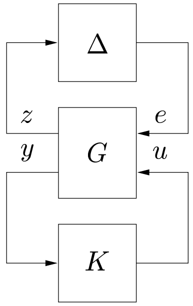
Special Case $\mathbf{H}_\infty$-synthesis

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \Rightarrow \mathbf{H}_\infty\text{-synthesis}$$



Interpretation: Closed loop stable for all $\|\Delta\|_{\mathbf{H}_\infty} \leq 1$

KYP Lemma

Lemma 3. *Assume $Q \geq 0$. Then the following are equivalent*

(*i*) G_{cl} *is stable and*

$$\begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty]$$

(*ii*) *There exists* $P \in \mathcal{S}_+^{(n_G+n_K)\times(n_G+n_K)}$ *s.t.*

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} + C_{cl}^T Q C_{cl} & P B_{cl} + C_{cl}^T Q D_{cl} + C_{cl}^T S \\ B_{cl}^T P + D_{cl}^T Q C_{cl} + S^T C_{cl} & D_{cl}^T Q D_{cl} + D_{cl}^T S + S^T D_{cl} + R \end{bmatrix} < 0$$

We assume $Q > 0$. Then by the Schur complement formula (*ii*) is equivalent to

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_{cl} + C_{cl}^T S & C_{cl}^T \\ B_{cl}^T P + S^T C_{cl} & D_{cl}^T S + S^T D_{cl} + R & D_{cl}^T \\ C_{cl} & D_{cl} & -Q^{-1} \end{bmatrix} < 0$$

The problem is to find P and K such that this inequality holds.

Synthesis Problem

Find $P \in \mathcal{S}_+^{(n_G+n_K)\times(n_G+n_K)}$ and $K = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$ such that

$$NKM^T + MKN^T + H < 0$$

where

$$H = \begin{bmatrix} \mathcal{A}_0^T P + P\mathcal{A}_0 & P\mathcal{B}_0 + \mathcal{C}_0^T S & \mathcal{C}_0^T \\ \mathcal{B}_0^T P + S^T \mathcal{C}_0 & \mathcal{D}_0^T S + S^T \mathcal{D}_0 + R & \mathcal{D}_0^T \\ \mathcal{C}_0 & \mathcal{D}_0 & -Q^{-1} \end{bmatrix}$$

$$N = \begin{bmatrix} P\mathcal{B} \\ S^T \mathcal{D}_{12} \\ \mathcal{D}_{12} \end{bmatrix}, \qquad M = \begin{bmatrix} \mathcal{C}^T \\ \mathcal{D}_{21}^T \\ 0 \end{bmatrix}$$

The elimination lemma gives the following necessary and sufficient conditions

$$N_\perp H N_\perp^T < 0$$
$$M_\perp H M_\perp^T < 0$$

where

$$N_\perp = \left[\begin{array}{cc|cc} V_1 & 0 & 0 & V_2 \\ 0 & 0 & I & -S^T \end{array}\right] \left[\begin{array}{ccc} P^{-1} & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right], \; \tilde{N}^T := \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ D_{12} \end{bmatrix}_\perp$$

$$M_\perp = \begin{bmatrix} W_1 & 0 & W_2 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad \text{where} \quad \tilde{M}^T := \begin{bmatrix} W_1 & W_2 \end{bmatrix} = \begin{bmatrix} C_2^T \\ D_{21}^T \end{bmatrix}_\perp$$

The orthogonal complements are as indicated since

$$N = \left[\begin{array}{c|cc} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \left[\begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hline 0 & S^T D_{12} \\ 0 & D_{12} \end{array}\right], \qquad M = \begin{bmatrix} 0 & S_2^T \\ I & 0 \\ 0 & D_{21}^T \\ 0 & 0 \end{bmatrix}$$

Next let $P = \begin{bmatrix} X & X_2 \\ X_2 & X_3 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2 & Y_3 \end{bmatrix}$. Lemma 2 and some manipulation of

$$N_\perp H N_\perp^T < 0$$
$$M_\perp H M_\perp^T < 0$$

gives the next theorem (it is a good exercise to fill in the details).

ıeorem 3. *The following are equivalent*

(*i*) *G_{cl} is stable and* $\begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix} < 0, \ \forall \omega \in [0, \infty]$

(*ii*) *There exists* $X, Y \in \mathcal{S}_+^{n_G \times n_G}$ *s.t.*

$$\begin{bmatrix} \widetilde{N} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY + YA^T & YC_1^T & B_1 \\ C_1 Y & -Q^{-1} & D_{11} - Q^{-1} S^T \\ B_1^T & D_{11}^T - SQ^{-1} & R - SQ^{-1} S^T \end{bmatrix} \begin{bmatrix} \widetilde{N} & 0 \\ 0 & I \end{bmatrix} < 0$$

$$\begin{bmatrix} \widetilde{M} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB_1 + C_1^T S & C_1^T \\ B_1^T X + S^T C_1 & D_{11}^T S + S^T D_{11} + R & D_{11}^T \\ C_1 & D_{11} & -Q^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{M} & 0 \\ 0 & I \end{bmatrix} < 0$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0, \quad \operatorname{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \le n_G + n_K$$

- Convex problem if $n_K \ge n_G$ since rank constraint dissapears
- An important special case is $Q = I$, $R = -I$, and $S = 0$. This is the $\mathbf{H}_\infty$-synthesis problem. It gives

$$\|G_{cl}\|_{\mathbf{H}_\infty} < 1$$

Controller Reconstruction

- Suppose $X, Y \in \mathcal{S}_+^{n_G \times n_G}$ satisfies (*ii*) in Theorem 3.
- By Schur complement $X - Y^{-1} = X_2 X_2^T$ for some $X_2 \in \mathbf{R}^{n_G \times n_K}$
- Use $P = \begin{bmatrix} X & X_2 \\ X_2^T & I \end{bmatrix} > 0$ (by Schur).
- Controller parameters from LMI

$$NKM^T + MKN^T + H < 0$$

where H, N, and M was defined on a previous slide.

References

[1] S. Boyd, L. EL Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM Studies in Applied Mathematics, Philadelphia, 1994.

[2] G.E. Dullerud and F. Paganini. *A course in Robust Control Theory.* Springer, New York, 2000.

[3] L. El Ghaoui and S-I Niculescu, editors. *Advances in Matrix Inequality Methods in Control.* Advances in Design and Control. SIAM, 2000.

[4] R.E. Skelton, T. Iwasaki, and K. Grigoriadis. *A Unified Algebraic Approach to Linear Control Design.* Taylor and Francis, 1998.

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$\mathbf{H}_\infty$-synthesis $(Q = I, S = 0, R = -I)$

Theorem 4. *The following are equivalent*

(i) $\|G\|_{\mathbf{H}_\infty} = \sup_{\mathrm{Re}\, s \geq 0} \sigma_{\max}(G(s)) < 1$

(ii) *There exists* $X, Y \in \mathcal{S}_+^{n_G \times n_G}$ *s.t.*

$$\begin{bmatrix} \widetilde{N} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY + YA^T & YC_1^T & B_1 \\ C_1 Y & -I & D_{11} \\ B_1^T & D_{11}^T & -I \end{bmatrix} \begin{bmatrix} \widetilde{N} & 0 \\ 0 & I \end{bmatrix} < 0$$

$$\begin{bmatrix} \widetilde{M} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB_1 & C_1^T \\ B_1^T X & -I & D_{11}^T \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} \widetilde{M} & 0 \\ 0 & I \end{bmatrix} < 0$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0, \quad \mathrm{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n_G + n_K$$