Kungl Tekniska Högskolan

Matematiska institutionen
Avdelningen för Optimeringslära och systemteori

Exam: 5B5765: Robust Control Theory Fall 2003
Deadline: January 30, 2004.

The final test in 5B5765 is an oral exam where you will be asked to solve a selection of the following problems. You are not allowed to bring any notes to the exam.

1. State four equivalent conditions for stabilizability (not including the statement).

2. Let

$$G_1(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right], \qquad G_2(s) = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right].$$

Derive a state space realization for the concatenation

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

3. Prove the following equivalences

(i) for every state feedback F and any invertible input transformation M we have

$$(A, B) \text{ stabilizable} \iff (A + BF, BM) \text{ is stabilizable}$$

(ii) for every output feedback L and any invertible output transformation M we have

$$(C, A) \text{ detectable} \iff (MC, A + LC) \text{ is detectable}$$

4. Let

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Show that $G(s)^{-1}$ is well defined and proper $\iff$ D is invertible.

5. Prove that the system

$$e_1 = w_1 + Ke_2$$
$$e_2 = w_2 + Pe_1$$

is internally stable if P and $K(I - PK)^{-1}$ are stable.

6. Let $G(s) = C(sI - A)^{-1}B \in \mathbf{RL}_\infty$. This transfer function defines a multiplication operator on $\mathbf{L}_2(j\mathbf{R})$. Derive the adjoint operator $G^* \in \mathcal{L}(\mathbf{L}_2(j\mathbf{R}))$. When is G self-adjoint?

7. Let $G(s) = C(sI - A)^{-1}B + D \in \mathbf{RH}_\infty$. Derive equivalent conditions for G to be strictly positive real in terms of

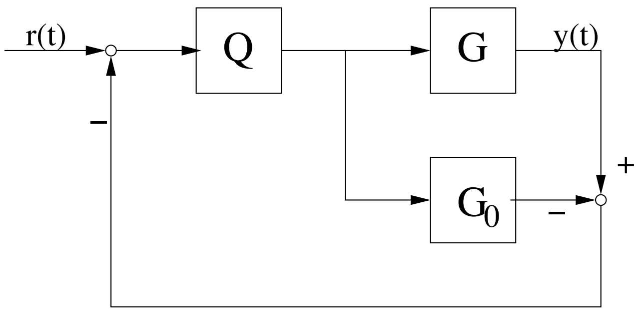
(*a*) Frequency domain inequality

(*b*) Existence of solution to an algebraic Riccati equation

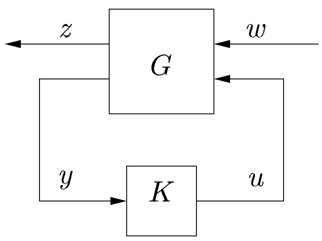
(*c*) LMI

(*d*) Conditions on an Hamiltonian matrix

8. Suppose we know that the plant $G(s)$ is stable but unknown. Let $G_0(s)$ be a model of the plant. In internal model control (IMC) the controller is implemented as in the figure below, where $Q(s) \in \mathbf{RH}_\infty$. I.e., the error model, $G(s) - G_0(s)$, is used to define the control. Show that this controller can be interpreted as the Youla parametrization of $G_0(s)$.



9. Consider the control system in the figure below



where

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ I & 0 & 0 \end{array}\right]$$

and where (A, B_1) stabilizable and (C_1, A) is detectable. Determine the $\mathbf{H}_2$ optimal controller.

10. Consider the same system as in Problem 9 for the case when

(*i*) (A, B_1) is stabilizable and (C_1, A) is detectable

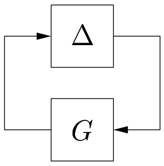
(*ii*) (A, B_2) is stabilizable

(*iii*) $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

Write down the lowest state degree $\mathbf{H}_\infty$ sub-optimal controller which gives $\|\mathcal{S}_l(G, K)\|_{\mathbf{H}_\infty} < \gamma$.

11. Derive the upper bound (based on D-scaling) for the structured singular value.

12. Use the result in problem 11 together with the zero exclusion principle to derive an IQC criterion for stability of the feedback interconnection



where $G(s) \in \mathbf{RH}_\infty^{m \times m}$ and $\Delta \in \mathbf{RH}_\infty^{m_f \times m_f}$ with

(*i*) $\|\Delta\|_{\mathbf{H}_\infty} < 1$

(*ii*)

$$\Delta(s) \in \left\{\Delta = \operatorname{diag}(\delta_1 I_1, \ldots, \delta_S I_S, \Delta_1, \ldots, \Delta_F),\ \delta_s \in \mathbf{C},\ \Delta_f \in \mathbf{C}^{m_f \times m_f}\right\}$$

for Re $s \geq 0$. In other words, you should find $\Pi = \Pi^*$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty]$$

13. For norm-bounded LDI systems, derive the LMIs for L_2-gain and passivity analysis stated in lecture note number 12, "Linear matrix inequalities in system analysis" (That is, the LMIs in pages 39 and 42. In the exam, you might be asked to derive one of them).

14. For norm-bounded LDI systems, derive LMI for finding a state-feedback controller which ensures that the L_2-gain of some input/output channel is less than a given number (That is, the LMI in pages 26).

15. State the matrix elimination lemma mentioned in lecture note number 14.

16. State and prove the Schur's lemma.

17. Consider the following system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \quad x(0) = 0 \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned}$$

Under which circumstances the problem of finding a LTI controller $u = K(s)y$ which minimizes the L_2-gain from f to z can be formulated as a convex optimization problem? And why?

18. Consider the following systems

$$\begin{bmatrix} z \\ v \end{bmatrix} = G(s) \begin{bmatrix} f \\ w \end{bmatrix} := \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} f \\ w \end{bmatrix}, \quad w = \Delta(v) \tag{1}$$

Assume that $G(s)$ is stable, Δ is causal and bounded, and system (1) is well-posed. Furthermore, assume that $\Delta \in \mathrm{IQC}(\Pi)$; i.e.,

$$\sigma(v, w) = \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \begin{bmatrix} \Pi_{11}(j\omega) & \Pi_{12}(j\omega) \\ \Pi_{12}(j\omega)^* & \Pi_{22}(j\omega) \end{bmatrix} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0$$

for all $(v, w) \in L_2$, $w = \Delta(v)$, where $\Pi_{11} \geq 0$ and $\Pi_{22} \leq 0$.

I. Is it true that the system is stable if

$$\begin{bmatrix} G_{22}(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \Pi_{11}(j\omega) & \Pi_{12}(j\omega) \\ \Pi_{12}(j\omega)^* & \Pi_{22}(j\omega) \end{bmatrix} \begin{bmatrix} G_{22}(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\, \omega \in [0, \infty]$$

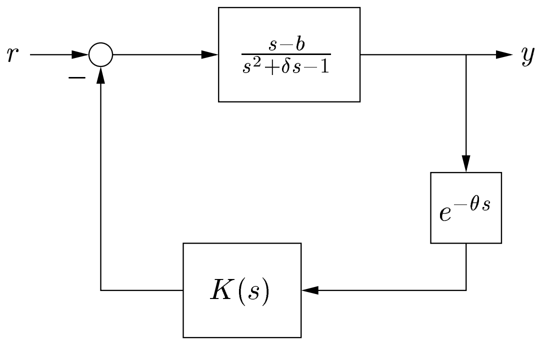
(One can view stability of a system this way: any finite-energy input results in a finite energy output.)

II. Show that the system is stable if

$$\begin{aligned} &\begin{bmatrix} G_{11}(j\omega) & G_{12}(j\omega) \\ I & 0 \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} G_{11}(j\omega) & G_{12}(j\omega) \\ I & 0 \end{bmatrix} + \\ &\begin{bmatrix} G_{21}(j\omega) & G_{22}(j\omega) \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \Pi_{11}(j\omega) & \Pi_{12}(j\omega) \\ \Pi_{12}(j\omega)^* & \Pi_{22}(j\omega) \end{bmatrix} \begin{bmatrix} G_{21}(j\omega) & G_{22}(j\omega) \\ 0 & I \end{bmatrix} < 0, \end{aligned} \tag{2}$$

for all $\omega \in [0, \infty]$

III. Show that (2) implies that the L_2-gain from f to z is less than or equal to γ.



19. Consider the following feedback control system where δ is an unknown constant in the interval $[-1, 1]$, and $e^{\theta s}$ represents an unknown constant delay. The delay parameter θ is known to be bounded by θ_0; i.e., $\theta \in [0, \theta_0]$. A stabilizing controller $K(s)$ is designed based on the nominal plant $P_0(s) = \frac{s-b}{s^2-1}$ and ignoring the delay effect. That means, a controller $K(s)$ is found so that

$$\frac{P_0(s)}{1+P_0(s)K(s)}, \quad \frac{P_0(s)K(s)}{1+P_0(s)K(s)}$$

are strictly proper and have no poles in the closed right half complex plane. Two colleagues from your consulting firm are asked to check whether the real feedback control system is stable based on the controller $K(s)$. Colleague K did the following: he expressed the system in the following feedback interconnection:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = G(s) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} := \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

$$w_1 = e^{-\theta s} v_1, \quad w_2 = \delta v_2$$

and computed the H_∞-norm of $G(s)$. Colleague J did some loop transformation and represented the system in the following form:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = H(s) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} := \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

$$f_1 = \frac{e^{-\theta s} - 1}{\theta_0 s} z_1, \quad f_2 = \delta z_2$$

and also computed the H_∞-norm of the linear subsystem $H(s)$.

I. Which colleague do you think is more likely to be successful in proving that the control system is stable? Why?

II. It turns out that neither of your colleagues was successful. However, after reviewing what colleagues K and J did, your boss sent colleague K back to school and asked you to help colleague J. Colleague J told you that $|\frac{e^{-j\theta\omega}-1}{j\theta_0\omega}| \leq 1$ for all ω. Is that right?

III. Colleague J told you that he computed the H_∞-norm of $H(s)$ and it is larger than 1! Use what you learn in this course to help your colleague. State and explain what you plan to do to prove that the control system is stable.