KUNGL TEKNISKA HÖGSKOLAN

Matematiska institutionen
Avdelningen för Optimeringslära och systemteori

Homework 5: 5B5765: Robust Control Theory Fall 2003
Deadline: December 5, 2003.

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1. (Exactness of the upper bound of the structure singular value) Recall that for $M \in C^{m\times m}$, the structure singular value is defined as

$$\mu_{\mathbf{\Delta}}(M) = \frac{1}{\inf\{\|\Delta\| \mid \det(I - M\Delta) = 0,\ \Delta \in \mathbf{\Delta}\}}$$

Consider the case where

$$\mathbf{\Delta} = \begin{bmatrix} \Delta_1 & & \\ & \Delta_2 & \\ & & \Delta_3 \end{bmatrix}$$

where $\Delta_i \in C^{m_i \times m_i}$. Let

$$\mathcal{D} = \{\mathbf{diag}(d_1 I_{m_1}, d_2 I_{m_2}, d_3 I_{m_3}) \mid d_i \in R,\ d_i > 0,\ i = 1, \cdots, 3\}$$

where I_{m_i} is the $m_i \times m_i$ identity matrix. Let $\bar{\sigma}(M)$ denotes the largest singular value of M. The goal of this problem is to show that for the uncertainty structure $\mathbf{\Delta}$ considered, the upper bound

$$\inf_{D \in \mathcal{D}} \bar{\sigma}(D^{-1}MD)$$

for $\mu_{\mathbf{\Delta}}(M)$ is tight; i.e., $\mu_{\mathbf{\Delta}}(M) = \inf_{D\in\mathcal{D}} \bar{\sigma}(D^{-1}MD)$. We saw in the lecture that $\mu_{\mathbf{\Delta}}(M) \leq \inf_{D\in\mathcal{D}} \bar{\sigma}(D^{-1}MD)$. In this problem, you will see that one can show, by using S-procedure lossless theorem, the opposite inequality also holds.

Consider the following system of equations

$$\begin{aligned} v &= Mw, \quad v, w \in C^{m_1+m_2+m_3} \\ w_i &= \Delta_i v_i, \quad i = 1, 2, 3 \end{aligned}$$

and quadratic forms $\sigma_i(v, w) = \|v_i\|^2 - \gamma^2 \|w_i\|^2$, $i = 1, 2, 3$.

(1.1) (Optional) Show that if $\mu_{\boldsymbol{\Delta}}(M) < \gamma$, then

$$\{w \mid \sigma_i(Mw, w) \geq 0,\ i = 1, 2, 3\} = \{0\}$$

which means that if $w \neq 0$, and $\sigma_2(Mw, w) \geq 0$, $\sigma_3(Mw, w) \geq 0$, then $\sigma_1(Mw, w) < 0$. In other words,

$$\sigma_1(Mw, w) < 0, \quad \forall\, w \in \{w \mid w \neq 0,\ \sigma_2(Mw, w) \geq 0,\ \sigma_3(Mw, w) \geq 0\} \tag{1}$$

(1.2) Show that (1) implies that

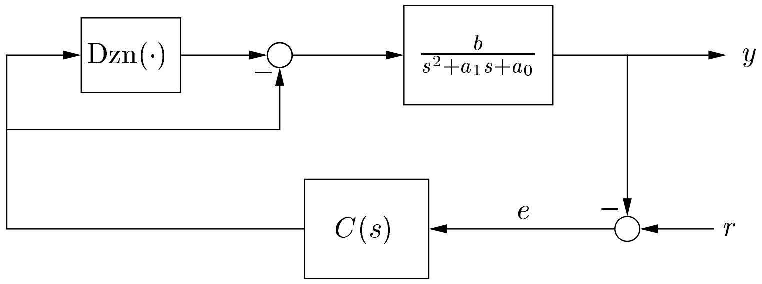
$$\bar{\sigma}(D^{-1}MD) < \gamma$$

for some $D \in \mathcal{D}$. This gives the following statement: for any γ, if $\mu_{\boldsymbol{\Delta}}(M) < \gamma$, then $\bar{\sigma}(D^{-1}MD) < \gamma$ for some $D \in \mathcal{D}$. Let $\gamma = \mu_{\boldsymbol{\Delta}}(M) + \epsilon$ for some $\epsilon > 0$. We have

$$\inf_{D \in \mathcal{D}} \bar{\sigma}(D^{-1}MD) < \mu_{\boldsymbol{\Delta}}(M) + \epsilon$$

Since ϵ is arbitrary, the above inequality implies $\inf_{D \in \mathcal{D}} \bar{\sigma}(D^{-1}MD) \leq \mu_{\boldsymbol{\Delta}}(M)$. (*Hint: use S-procedure*).

2. Consider the following feedback control system where $\mathrm{Dzn}(\cdot)$ denotes the dead-zone



operator; i.e., $\mathrm{Dzn}(v) = 0$ for $|v| < 1$, $\mathrm{Dzn}(v) = v - 1$ for $v \geq 1$, and $\mathrm{Dzn}(v) = v + 1$ for $v \leq -1$. Parameters a_0 and a_1 are unknown constants in the following ranges, respectively: $a_0 \in [1, 3]$ and $a_1 \in [0, 1]$. Controller $C(s)$ is either a PID controller (i.e., $k_p + \frac{k_1}{s} + k_d s$) or a PD controller (i.e.,$k_p + k_d s$).

(2.1) Let $a_0 = 2 + \delta_1$, $a_1 = 0.5 + 0.5\delta_2$, and model the dead-zone operator $y(t) = \mathrm{Dzn}(z(t))$ as $y(t) = \delta_3(t) \cdot z(t)$. Express the control system in the following state-space form

$$\begin{aligned} \dot{x} &= Ax + B_1 r + B_2 w \\ e &= C_1 x + D_{11} r + D_{12} w \\ v &= C_2 x + D_{21} r + D_{22} w \end{aligned}$$

where v and w are vector signals such that $w_1(t) = \delta_1 v_1(t)$, $w_2(t) = \delta_2 v_2(t)$, and $w_3(t) = \delta_3(t) v_3(t)$. What are matrices $A, B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}$, and D_{22}.

(2.2) For a given PID controller (i.e., fix the values of k_p, k_d and k_i), give a set of Linear Matrix Inequalities, such that if the LMIs are feasible then the energy of e is bounded by 1 for the following input

$$r(t) = \begin{cases} 1 & t \in [0, 2] \\ 0 & \text{otherwise} \end{cases}$$

(2.3) For a given PID controller, give a set of Linear Matrix Inequalities, such that if the LMIs are feasible then the amplitude of the unit step-input response is bounded by 1.1.

(2.4) Now consider only PD controller. Give a set of Linear Matrix Inequalities to find the values of k_p and k_d which guarantee that the amplitude of the unit step-input response is bounded by 1.1. (Note that we have one state less in this case – the state for the integrator is to be removed.)

3. (IQC analysis and LMI) Consider the following system

$$\begin{aligned} \dot{x} &= Ax + Bw, \quad x(0) = 0 \\ v &= Cx + Dw \\ w &= \Delta(v) \end{aligned} \tag{1}$$

where Δ is causal, bounded in $\mathbf{L}_2[0,\infty)$, and satisfies IQCs defined by

$$\sigma(v,w) = \int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \begin{bmatrix} \Pi_{11}(j\omega) & \Pi_{12}(j\omega) \\ \Pi_{12}(j\omega)^* & \Pi_{22}(j\omega) \end{bmatrix} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \tag{2}$$

where $\hat{v}$ denotes the Fourier transform of signal $v(t)$. That is $\sigma(v,w) \geq 0$ for all $(v,w) \in \mathbf{L}_2[0,\infty)$ and $w = \Delta(v)$.

Let $G(s) = C(sI - A)^{-1}B + D$. Assume that the $G(s)$ is stable, system (1) is well posed, $\Pi_{11}(j\omega) \geq 0$ and $\Pi_{22}(j\omega) \leq 0$. The IQC theory states that the system is stable if

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \Pi_{11}(j\omega) & \Pi_{12}(j\omega) \\ \Pi_{12}(j\omega)^* & \Pi_{22}(j\omega) \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\, \omega \in [0,\infty] \tag{3}$$

Now let $\Pi(j\omega)$ has the structure such that (2) can be expressed in time domain as

$$\sigma(v,w) := \int_0^{\infty} \begin{bmatrix} x_\pi \\ v \\ w \end{bmatrix}' \begin{bmatrix} Q & S_1 & S_2 \\ S_1' & R_{11} & R_{12} \\ S_2' & R_{12}' & R_{22} \end{bmatrix} \begin{bmatrix} x_\pi \\ v \\ w \end{bmatrix} dt$$

where $\dot{x}_\pi = A_\pi x_\pi + B_{\pi,v} v + B_{\pi,w} w$, $x_\pi(0) = 0$. Express condition (3) in terms of Linear Matrix Inequalities (LMIs).

Hint: Use KYP lemma