Elementary Operator Theory

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References

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Outline

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Motivation

- Signals are represented as functions from normed vector spaces
  - The norm allow us to measure the size of the signals
  - We mostly consider Hilbert spaces which contains additional structure because of the inner product.
- The systems are represented as operators on the signal space
  - The induced norm allow us to measure the signal amplification

Signal Spaces

Definition 1. A vector space $\mathcal{V}$ over the field $\mathbf{F}$ (here $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$) is a nonempty set satisfying

1. $v_1 + v_2 = v_2 + v_1$, $\forall v_1, v_2 \in \mathcal{V}$
2. $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$, $\forall v_1, v_2, v_3 \in \mathcal{V}$
3. There exists $0 \in \mathcal{V}$ such that $v + 0 = v$, $\forall v \in \mathcal{V}$ (existence of zero)
4. $\forall v \in \mathcal{V}\ \exists - v \in \mathcal{V}$ such $v + (-v) = 0$ (existence of inverse)
5. $(\alpha_1 + \alpha_2)v = \alpha_1 v + \alpha_2 v$, $\forall \alpha_1, \alpha_2 \in \mathbf{F}$, $\forall v \in \mathcal{V}$
6. $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$, $\forall \alpha \in \mathbf{F}$, $\forall v_1, v_2 \in \mathcal{V}$
7. $(\alpha_1\alpha_2)v = \alpha_1(\alpha_2 v)$, $\forall \alpha_1, \alpha_2 \in \mathbf{F}$, $\forall v \in \mathcal{V}$
8. $1 \in \mathbf{F}$ satisfies $1v = v$, $\forall v \in \mathcal{V}$

Definition 2. A normed vector space $(\mathcal{V}, \|\cdot\|_\mathcal{V})$ is a vector space equiped with a norm, i.e. a function $\|\cdot\|_\mathcal{V} : \mathcal{V} \to \mathbf{R}^+$ satisfying

1. $\|v\|_\mathcal{V} = 0$ iff $v = 0$
2. $\|\alpha v\|_\mathcal{V} = |\alpha| \cdot \|v\|_\mathcal{V}$ $\forall \alpha \in \mathbf{F}$, $\forall v \in \mathcal{V}$
3. $\|v_1 + v_2\|_\mathcal{V} \le \|v_1\|_\mathcal{V} + \|v_2\|_\mathcal{V}$, $\forall v_1, v_2 \in \mathcal{V}$

- The norm allow us to define convergence in $\mathcal{V}$. A sequence $\{v_k\}_{k=0}^\infty \subset \mathcal{V}$ converges to $v \in \mathcal{V}$, which is denoted $\lim_{k\to\infty} v_k = v$, iff $\lim_{k\to\infty} \|v_k - v\|_\mathcal{V} = 0$.
- A normed vector space is complete if every Cauchy sequence converges. Here a sequence $\{v_k\}_{k=0}^\infty \subset \mathcal{V}$ is a Cauchy sequence if $\forall \epsilon > 0\ \exists N$ such that

$$\|v_k - v_l\|_\mathcal{V} \le \epsilon, \quad \forall k, l \ge N \tag{1}$$

If $\mathcal{V}$ is complete then (1) implies that there exists $v \in \mathcal{V}$ s.t.

$\lim_{k\to\infty} v_k = v$.

- A complete normed space is called a Banach space
- A subset $\mathcal{U} \subset \mathcal{V}$ which is closed under the operations in Definition 1 is called a subspace. A necessary and sufficient condition for closedness of the vector space operations is

  (*i*) $u_1 + u_2 \in \mathcal{U}$, $\forall u_1, u_2 \in \mathcal{U}$

  (*ii*) $\alpha u \in \mathcal{U}$, $\forall u \in \mathcal{U}$ and $\alpha \in \mathbf{F}$

  The norm on $\mathcal{U}$ is the same as on $\mathcal{V}$.
- A subspace $\mathcal{U} \subset \mathcal{V}$ is called closed if every sequence $\{u_k\}_{k=0}^\infty \subset \mathcal{U}$, which converges in $\mathcal{V}$ (i.e. $\lim_{k\to\infty} u_k = u \in \mathcal{V}$) has its limit in $\mathcal{U}$ (i.e. $u \in \mathcal{U}$)
- A closed subspace of a Banach space is a Banach space.

Definition 3. An inner product space $(\mathcal{V}, \langle\cdot,\cdot\rangle_\mathcal{V})$ is a vector space equiped with an inner product, i.e. a bilinear function $\langle\cdot,\cdot\rangle_\mathcal{V} : \mathcal{V} \times \mathcal{V} \to \mathbf{F}$ satisfying

1. $\langle v, v\rangle_\mathcal{V} \ge 0$ $\forall v \in \mathcal{V}$ and $\langle v, v\rangle_\mathcal{V} = 0$ iff $v = 0$.
2. $\langle \alpha_1 v_1 + \alpha_2 v_2, v_3\rangle_\mathcal{V} = \alpha_1 \langle v_1, v_3\rangle_\mathcal{V} + \alpha_2 \langle v_2, v_3\rangle_\mathcal{V}$
3. $\langle v_1, v_2\rangle_\mathcal{V} = \overline{\langle v_2, v_1\rangle}_\mathcal{V}$

- If the underlying scalar field $\mathbf{F} = \mathbf{R}$ then $\langle v_1, v_2\rangle_\mathcal{V} = \langle v_2, v_1\rangle_\mathcal{V}$
- The inner product induces the norm $\|v\|_\mathcal{V} = \langle v, v\rangle_\mathcal{V}$
- The Cauchy-Schwartz inequality: $|\langle v_1, v_2\rangle_\mathcal{V}| \le \|v_1\|_\mathcal{V} \cdot \|v_2\|_\mathcal{V}$. If $\mathbf{F} = \mathbf{R}$ then this simplifies to $\langle v_1, v_2\rangle_\mathcal{V} \le \|v_1\|_\mathcal{V} \cdot \|v_2\|_\mathcal{V}$

- Two elements $v_1, v_2 \in \mathcal{V}$ are orthogonal if $\langle v_1, v_2 \rangle_{\mathcal{V}} = 0$, which is denoted $v_1 \perp v_2$.
- Two subsets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{V}$ are orthogonal if $\langle v_1, v_2 \rangle_{\mathcal{V}} = 0$ $\forall v_1 \in \mathcal{S}_1, v_2 \in \mathcal{S}_2$. This is denoted $\mathcal{S}_1 \perp \mathcal{S}_2$.
- If $\mathcal{S} \subset \mathcal{V}$ then the orthogonal complement $\mathcal{S}^\perp$ is the set of all vectors that are orthogonal to the set S.
- $\mathcal{S}^\perp$ is a closed subspace of $\mathcal{V}$ for every set $\mathcal{S}$ and if $\mathcal{S}$ in addition is a closed subspace of $\mathcal{V}$ then we have the direct sum decomposition

$$\mathcal{V} = \mathcal{S} \oplus \mathcal{S}^\perp,$$

i.e. for each $v \in \mathcal{V}$ there exists a unique $v_1 \in \mathcal{S}$ and $v_2 \in \mathcal{S}^\perp$ s.t. $v = v_1 + v_2$.

- If $v_1 \perp v_2$ then $\|v_1 + v_2\|_{\mathcal{V}} = \|v_1\|_{\mathcal{V}} + \|v_2\|_{\mathcal{V}}$ (Pythagaros theorem)
- A complete inner product space is called a Hilbert space
- A closed subspace of a Hilbert space is a Hilbert space

Time-Domain Signal Spaces

We consider the following time domain spaces

$\mathbf{L}_2(\mathbf{R})$ The space of square integrable functions on $\mathbf{F} = \mathbf{R}$ with inner product

$$\langle v_1, v_2 \rangle = \int_{-\infty}^{\infty} v_1(t)^T v_2(t) dt$$

The corresponding norm is $\|v\| = \sqrt{\int_{-\infty}^{\infty} |v(t)|^2 dt}$. It can be proven that $\mathbf{L}_2(\mathbf{R})$ is a Hilbert space.

$\mathbf{L}_2[0, \infty)$ The closed subspace of $\mathbf{L}_2(\mathbf{R})$ consisting of functions with support on $t \geq 0$, i.e., $v(t) = 0$ when $t < 0$.

$\mathbf{L}_2(-\infty, 0]$ The closed subspace of $\mathbf{L}_2(\mathbf{R})$ consisting of functions with support on $t \leq 0$, i.e., $v(t) = 0$ when $t > 0$.

- It is obvious that $\mathbf{L}_2[0, \infty) \perp \mathbf{L}_2(-\infty, 0]$, which implies

$$\mathbf{L}_2[0, \infty) = \mathbf{L}_2(-\infty, 0] \oplus \mathbf{L}_2[0, \infty)$$

- Let $P : \mathbf{L}_2(\mathbf{R}) \to \mathbf{L}_2[0, \infty)$ be the projection operator defined by

$$(Pv)(t) = \begin{cases} v(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Then $(I - P)\mathbf{L}_2(\mathbf{R}) = \mathbf{L}_2(-\infty, 0]$.

Frequency Domain Signal Spaces

$\mathbf{L}_2(j\mathbf{R})$ The space of square integrable functions in the frequency domain

$$\langle \hat{v}_1, \hat{v}_2 \rangle = \int_{-\infty}^{\infty} \hat{v}_1(i\omega)^* \hat{v}_2(i\omega) d\omega$$

The corresponding norm is $\|\hat{v}\| = \sqrt{\int_{-\infty}^{\infty} |\hat{v}(i\omega)|^2 d\omega}$. This is a Hilbert space that can be obtained as the Fourier transform of functions in $\mathbf{L}_2(\mathbf{R})$

$$\hat{u}(i\omega) = (\mathcal{F}u)(i\omega) := \int_{-\infty}^{\infty} u(t) e^{i\omega t} dt, \qquad a.e.$$

The inverse Fourier transform is defined as

$$u(t) = (\mathcal{F}^{-1}\hat{u})(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(i\omega) e^{i\omega t} d\omega, \qquad a.e.$$

The following result from function theory is important

Theorem 1 (Plancherel-Parceval). *The Fourier transform and its inverse satisfies*

(i) *If $\hat{u} = \mathcal{F}u$ and $\hat{v} = \mathcal{F}v$ for some $u, v \in \boldsymbol{L}_2(\boldsymbol{R})$ then*

$$\langle u, v \rangle = \langle \mathcal{F}u, \mathcal{F}v \rangle$$

(ii) *If $u = \mathcal{F}^{-1}\hat{u}$ and $v = \mathcal{F}^{-1}\hat{v}$ for some $\hat{u}, \hat{v} \in \boldsymbol{L}_2(j\boldsymbol{R})$ then*

$$\langle \hat{u}, \hat{v} \rangle = \left\langle \mathcal{F}^{-1}\hat{u}, \mathcal{F}^{-1}\hat{v} \right\rangle$$

This in particular means that $\|\mathcal{F}u\| = \|u\|$.

$\mathbf{H}_2$ The closed subspace of of $\mathbf{L}_2(i\mathbf{R})$ consisting of functions that are analytic in the open right half plane and satisfy the uniformly square integrable in the open right half plane. The norm is

$$\|\hat{u}\|_{\mathbf{H}_2} = \sup_{\sigma>0} \|\hat{u}(\sigma + i\omega)\| = \sup_{\sigma>0} \sqrt{\int_{-\infty}^{\infty} |\hat{u}(\sigma + i\omega)|^2 d\omega}$$

Any function in $\mathbf{H}_2$ can be obtained as the right-sided Laplace transform of a function in $u \in \mathbf{L}_2[0,\infty)$, i.e.for $s = \sigma + i\omega$, $\sigma > 0$, we have

$$\hat{u}(s) = (\mathcal{L}_+ u)(s) = \int_0^{\infty} u(t) e^{-st} dt$$

and vice versa

$$u(t) = (\mathcal{L}_+^{-1} u)(t) = \int_0^{\infty} \hat{u}(\sigma + i\omega) e^{(\sigma + i\omega)t} d\omega$$

$\mathbf{H}_2^{\perp}$ This is the orthogonal complement of $\mathbf{H}_2$ in $\mathbf{L}_2(i\mathbf{R})$. It consists of functions that are analytic in the open left half plane. The norm on $\mathbf{H}_2^{\perp}$ is defined as

$$\|\hat{u}\|_{\mathbf{H}_2^{\perp}} = \sup_{\sigma<0} \|\hat{u}(\sigma + i\omega)\| = \sup_{\sigma<0} \sqrt{\int_{-\infty}^{\infty} |\hat{u}(\sigma + i\omega)|^2 d\omega}$$

Any function in $\mathbf{H}_2^{\perp}$ can be obtained as the left sided Laplace

transform of a function in $u \in \mathbf{L}_2(-\infty, 0)$, i.e. for $s = \sigma + i\omega$, $\sigma < 0$, we have

$$\hat{u}(s) = (\mathcal{L}_- u)(s) = \int_0^{\infty} u(t) e^{-st} dt$$

and vice versa

$$u(t) = (\mathcal{L}_-^{-1} \hat{u})(t) = \int_0^{\infty} \hat{u}(\sigma + i\omega) e^{(\sigma + i\omega)t} d\omega$$

- Note that the $\mathbf{H}_2$-norm can be interpreted as

$$\|\hat{u}\|_{\mathbf{H}_2} = \begin{cases} \|\hat{u}\|, & \hat{u} \in \mathbf{H}_2 \\ \infty, & \text{otherwise} \end{cases}$$

where $\|\cdot\|$ is the norm on $\mathbf{L}_2(i\mathbf{R})$, i.e.

$$\|\hat{u}\| = \sqrt{\int_{-\infty}^{\infty} |\hat{v}(i\omega)|^2 d\omega}$$

Hence, the $\mathbf{H}_2$-norm is finite only if $\hat{u}$ is analytic in the RHP and then the norm can be evaluated on the imaginary axis (this is a maximum modulus theorem, i.e. the supremum is obtained on the imaginary axis $\sup_{\sigma>0} \|\hat{u}(\sigma + i\omega)\| = \|\hat{u}(i\omega)\|$).

- A consequence of the above is that there exists $u(t) = (\mathcal{F}^{-1}\hat{u})(t) \in \mathbf{L}_2[0,\infty)$ for any $\hat{u} \in \mathbf{H}_2$.
- $\mathbf{H}_2$ inherits the inner product from $\mathbf{L}_2(i\mathbf{R})$, i.e. for any $\hat{u}, \hat{v} \in \mathbf{H}_2$ we have the inner product

$$\langle \hat{u}, \hat{v} \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(i\omega)^* \hat{v}(i\omega) d\omega = \int_0^{\infty} u(t)^T v(t) dt$$

since $u = \mathcal{F}^{-1}\hat{u}, v = \mathcal{F}^{-1}\hat{v} \in \mathbf{L}_2[0,\infty)$

- Similar properties hold for $\mathbf{H}_2^{\perp}$, in particular $u(t) = (\mathcal{F}^{-1}\hat{u})(t) \in \mathbf{L}_2(-\infty, 0]$ for any $\hat{u} \in \mathbf{H}_2^{\perp}$
- From the Plancherel-Parseval Theorem we immediately see that

$\mathbf{H}_2 \perp \mathbf{H}_2^{\perp}$ because if $\hat{u} \in \mathbf{H}_2$ and $\hat{v} \in \mathbf{H}_2^{\perp}$ then

$$\langle \hat{u}, \hat{v} \rangle = \left\langle \mathcal{F}^{-1}\hat{u}, \mathcal{F}^{-1}\hat{v} \right\rangle = 0$$

since $\mathcal{F}^{-1}\hat{u} \in \mathbf{L}_2[0,\infty)$ and $\mathcal{F}^{-1}\hat{v} \in \mathbf{L}_2(-\infty, 0]$, which implies that

$$\mathbf{L}_2(j\mathbf{R}) = \mathbf{H}_2 \oplus \mathbf{H}_2^{\perp}$$

Summary of relationships

$$\begin{array}{ccccc}
\mathbf{H}_2 & \subset & \mathbf{L}_2(i\mathbf{R}) & \supset & \mathbf{H}_2^{\perp} \\
\mathcal{L}_+ \uparrow \quad \downarrow \mathcal{L}_+^{-1} & & \mathcal{F} \uparrow \quad \downarrow \mathcal{F}^{-1} & & \mathcal{L}_- \uparrow \downarrow \mathcal{L}_-^{-1} \\
\mathbf{L}_2[0,\infty) & \subset & \mathbf{L}_2(\mathbf{R}) & \supset & \mathbf{L}_2(-\infty, 0]
\end{array}$$

Important subspaces

- $\mathbf{RL}_2$ Strictly proper real rational transfer functions with no poles on the imaginary axis. $\mathbf{RL}_2 \subset \mathbf{L}_2(j\mathbf{R})$ is a dense subspace.
- $\mathbf{RH}_2$ Strictly proper real rational transfer functions with all poles in the open left half plane. $\mathbf{RH}_2 \subset \mathbf{H}_2$ is a dense subspace.
- $\mathbf{RH}_2^{\perp}$ Strictly proper real rational transfer functions with all poles in the open right half plane. $\mathbf{RH}_2^{\perp} \subset \mathbf{H}_2^{\perp}$ is a dense subspace.

Operators

An operator $G : \mathcal{U} \to \mathcal{V}$ is a mapping with domain $\mathcal{U}$ and co-domain $\mathcal{V}$. For each $u \in \mathcal{U}$ there exists a unique $v \in \mathcal{V}$ s.t. $v = Gu$.

Definition 4. We let $\mathcal{L}(\mathcal{U}, \mathcal{V})$ denote the set of all bounded linear operators maping the normed vector space $\mathcal{U}$ into the normed vector space $\mathcal{V}$. Any $G \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ satisfies

(*i*) $G(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 G u_1 + \alpha_2 G u_2$, for all $\alpha_1, \alpha_2 \in \mathbf{F}$ and $u_1, u_2 \in \mathcal{U}$.

(*ii*) There exists $c > 0$ such that $\|Gu\|_{\mathcal{V}} \le c\|u\|_{\mathcal{U}}$, for all $u \in \mathcal{U}$.

The norm on $\mathcal{L}(\mathcal{U}, \mathcal{V})$ is defined to be the smallest c such that (*ii*) holds, i.e.

$$\|G\|_{\mathcal{U}\to\mathcal{V}} = \sup_{u \neq 0} \frac{\|Gu\|_{\mathcal{V}}}{\|u\|_{\mathcal{U}}}$$

Remark 1.

- It easy easy to see that $\mathcal{L}(\mathcal{U},\mathcal{V})$ is a normed vector space. Furthermore, if $\mathcal{V}$ is Banach space then $\mathcal{L}(\mathcal{U},\mathcal{V})$ is a Banach space
- The norm is submultiplicative, i.e. if $G \in \mathcal{L}(\mathcal{U},\mathcal{V})$ and $H \in \mathcal{L}(\mathcal{V},\mathcal{W})$ then $\|HG\|_{\mathcal{U}\to\mathcal{W}} \le \|H\|_{\mathcal{V}\to\mathcal{W}}\|G\|_{\mathcal{U}\to\mathcal{V}}$

If $\mathcal{V} = \mathcal{U}$ then we use the simplified notation $\mathcal{L}(\mathcal{U})$ instead of $\mathcal{L}(\mathcal{U},\mathcal{U})$. This is an important case because

- The identity operator $I : \mathcal{U} \to \mathcal{U}$ can be defined as $Iu = u$ $\forall u \in \mathcal{U}$.
- If $H, G \in \mathcal{L}(\mathcal{U})$ then $HG \in \mathcal{L}(\mathcal{U})$

This makes $\mathcal{L}(\mathcal{U})$ a so-called Banach algebra (when U is a Banach space) which satisfies (subset of several easily proven properties)

- $IG = GI = G$ for all $G \in \mathcal{L}(\mathcal{U})$
- $H(\alpha_1 G_1 + \alpha_2 G_2) = \alpha_1 HG_1 + \alpha_2 HG_2$

Note however that $HG \neq GH$ except in special cases.

An element $G^{-1} \in \mathcal{L}(\mathcal{U})$ satisfying $G^{-1}G = GG^{-1} = I$ is called the inverse of G.

Robustness analysis is essentially the problem of finding invertibility conditions and norm bounds for the closed loop system. The following results is a small gain theorem

Theorem 2. *Suppose $G \in \mathcal{L}(\mathcal{U})$ where $\mathcal{U}$ is a Banach space. If $\|G\|_{\mathcal{U}\to\mathcal{U}} < 1$ then $(I-G)^{-1} \in \mathcal{L}(\mathcal{U})$ and*

$$\|(I-G)^{-1}\|_{\mathcal{U}\to\mathcal{U}} \le \frac{1}{1-\|G\|_{\mathcal{U}\to\mathcal{U}}}$$

Proof: *The Taylor series*

$$(I-G)^{-1} = \sum_{k=0}^{\infty} G^k \tag{2}$$

is well defined because $G_n = \sum_{k=0}^{n} G^k$ can easily be shown to be a Cauchy sequence since $\|G\| < 1$. Hence $\lim_{n\to\infty} G_n \in \mathcal{L}(\mathcal{U})$ since

$\mathcal{L}(\mathcal{U})$ is complete. The infinite sum is equal to the inverse since

$$(I-G)\sum_{k=0}^{\infty} G^k = I - \lim_{n\to\infty} G^n = I$$

We have now shown that $(I-G)^{-1} = \sum_{k=0}^{\infty} G^k \in \mathcal{L}(\mathcal{U})$

Finally, from (2) we have

$$\|(I-G)^{-1}\| = \|\sum_{k=0}^{\infty} G^k\| \le \sum_{k=0}^{\infty} \|G\|^k = \frac{1}{1-\|G\|}$$

Adjoint Operator

Definition 5. Let $G \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ where $\mathcal{U}, \mathcal{V}$ are Hilbert spaces. The adjoint of G is the operator $G^* \in \mathcal{L}(\mathcal{V}, \mathcal{U})$ defined by the relation

$$\langle Gu, v\rangle_{\mathcal{V}} = \langle u, G^* v\rangle_{\mathcal{U}}, \quad \forall u \in \mathcal{U}, v \in \mathcal{V}$$

Remark 2.

- The adjoint always exists and has norm $\|G^*\|_{\mathcal{V}\to\mathcal{U}} = \|G\|_{\mathcal{U}\to\mathcal{V}}$.
- The easiest example is a matrix $M \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m)$ in which case $M^* = M^T$.

Definition 6 (Self-adjoint Operator). An operator $G \in \mathcal{L}(\mathcal{U})$ is called self-adjoint if $G = G^*$.

Definition 7 (Positive Definite Operator). A self-adjoint operator $G \in \mathcal{L}(\mathcal{U})$ is called positive definite if there exists $\epsilon > 0$ such that

$$\langle Gu, u\rangle_{\mathcal{U}} \geq \epsilon \|u\|_{\mathcal{U}}^2, \quad \forall u \in \mathcal{U}$$

Definition 8 (Isometric and Unitary Operators). An operator $U \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ is called isometric if

$$\langle Uu_1, Uu_2\rangle_{\mathcal{V}} = \langle u_1, u_1\rangle_{\mathcal{U}}, \quad \forall u_1, u_2 \in \mathcal{U}$$

The operator is called unitary if it in addition is invertible. It follows that a unitary operator satisfies $U^*U = I$, which implies $U^{-1} = U^*$.

Remark 3. If there exists a unitary operator $U \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ then $\mathcal{U}$ and $\mathcal{V}$ are called isomorphic because they have exactly the Hilbert space structure ($U : \mathcal{U} \to \mathcal{V}$ defines a bijection that preserves the inner product and thus also the norm).

Remark 4. From the Plancherel-Parceval theorem it follows that the Fourier transform is a unitary operator, i.e. the frequency domain space $\mathbf{L}_2(j\mathbf{R})$ and the time domain space $\mathbf{L}_2(\mathbf{R})$ are isomorphic.

In the course we consider several applications of adjoint operators

- Self-adjoint and positive definite operators
  - Integral quadratic constraints are defined in terms of self-adjoint operators and the corresponding stability criterium is to verify that a particular operator is positive definite.
  - The Kalman-Yakubovich-Popov lemma shows that positive definiteness of particular self-adjoint operators is equivalent to a large number of important properties related to LQ optimal control.
- Unitary operators are used in the proof of the $\mathbf{H}_2$-optimal control theorem.

Frequency Domain Operator Spaces

In the course we consider operators from the following Banach spaces

$\mathbf{L}_\infty(i\mathbf{R})$ The Banach space of matrix valued functions that are bounded on the imaginary axis with norm ($G : i\mathbf{R} \to \mathbf{C}^{n\times m}$)

$$\|G\|_\infty = \sup_{\omega\in\mathbf{R}} \bar{\sigma}(G(j\omega))$$

$\mathbf{H}_\infty$ The closed subspace of $\mathbf{L}_\infty(i\mathbf{R})$ with functions that are analytic and bounded in the open right half plane.The norm is defined as

$$\|G\|_{\mathbf{H}_\infty} = \sup_{\text{Re } s>0} \bar{\sigma}(G(s))$$

$\mathbf{H}_\infty^-$ The closed subspace of $\mathbf{L}_\infty(i\mathbf{R})$ with functions that are analytic and bounded in the open left half plane.The norm is defined as

$$\|G\|_{\mathbf{H}_\infty^-} = \sup_{\text{Re } s<0} \bar{\sigma}(G(s))$$

- If $G \in \mathbf{L}_\infty(i\mathbf{R})$ then

$$\|G\|_{\mathbf{H}_\infty} = \begin{cases} \|G\|_\infty, & G \in \mathbf{H}_\infty \\ \infty, & otherwise \end{cases}$$

which means

– The $\mathbf{H}_\infty$-norm is finite only if a transfer function is *stable*, i.e. $G \in \mathbf{H}_\infty$

– For stable systems, i.e. $G \in \mathbf{H}_\infty$, we have a maximum modulus result $\|G\|_{\mathbf{H}_\infty} = \|G\|_\infty = \sup_{\omega\in\mathbf{R}} \bar\sigma(G(j\omega))$.

Proposition 1. *An element $G \in \boldsymbol{L}_\infty(j\boldsymbol{R})$ defines an operator $M_G : \boldsymbol{L}_2(j\boldsymbol{R}) \to \boldsymbol{L}_2(j\boldsymbol{R})$ via the multiplication $(M_G\hat u)(j\omega) = G(j\omega)\hat u(j\omega)$. Moreover the induced norm*

$$\|M_G\|_{\boldsymbol{L}_2(j\boldsymbol{R})\to\boldsymbol{L}_2(j\boldsymbol{R})} = \sup_{0\neq\hat u\in\boldsymbol{L}_2(j\boldsymbol{R})} \frac{\|G\hat u\|_{\boldsymbol{L}_2(j\boldsymbol{R})}}{\|\hat u\|_{\boldsymbol{L}_2(j\boldsymbol{R})}} = \sup_{\omega\in\boldsymbol{R}} \bar\sigma(G(j\omega)) = \|G\|_\infty$$

Proof: *The main condition to prove is the norm relation. We have*

$$\begin{aligned}\|G\hat u\|^2_{\boldsymbol{L}_2(j\boldsymbol{R})} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat u(j\omega)^* G(j\omega)^* G(j\omega)\hat u(j\omega)d\omega \\ &= \|G\|_\infty^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat u|^2 d\omega = \|G\|^2_\infty \|\hat u\|_2^2\end{aligned}$$

For the other direction we only outline the idea. Let ω_0 be such that $\sigma_1 := \bar\sigma(G(j\omega_0)) = \|G\|_\infty$. The singular value decomposition $G(j\omega_0) = \sum_{k=1}^m \sigma_k u_k v_k^*$ (where $\sigma_1 \geq \sigma_2 \geq \ldots$) leads us to the

following approximate optimal candidate

$$\hat u(j\omega) = \begin{cases} v_1, & |\omega - \omega_0| \leq \epsilon \\ \bar v_1, & |\omega + \omega_0| \leq \epsilon \\ 0, & otherwise \end{cases}$$

This gives

$$\begin{aligned}\|G\hat u\|^2_{\boldsymbol{L}_2(j\boldsymbol{R})} =& \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat u(j\omega)^* G(j\omega)^* G(j\omega)\hat u(j\omega)d\omega \\ &\approx \frac{1}{2\pi}(\bar\sigma(G(j\omega_0))^2 + \bar\sigma(G(-j\omega_0))^2)\epsilon \\ &= \|G\|^2_\infty \|\hat u\|_2^2\end{aligned}$$

Remark 5. In this way, each $G \in \mathbf{L}_\infty(j\mathbf{R})$ defines an operator $M_G \in \mathcal{L}(\mathbf{L}_2(j\mathbf{R}))$. We will mostly ignore the notation M_G and simply write $G \in \mathcal{L}(\mathbf{L}_2(j\mathbf{R}))$. Next we will see that each $G \in \mathbf{L}_\infty(j\mathbf{R})$ via the Fourier transform defines a time domain convolution operator. We use the same notation for both the time domain $G \in \mathcal{L}(\mathbf{L}_2(\mathbf{R}))$ and the frequency domain interpretations $G \in \mathcal{L}(\mathbf{L}_2(j\mathbf{R}))$ of the operator.

We can also consider the time domain representation $g = \mathcal{F}^{-1}G\mathcal{F}$, where $\mathcal{F}$ denotes the Fourier transform. Formally, an "impulse response" function defined in this way defines a convolution operator on $\mathbf{L}_2(\mathbf{R})$ as follows

$$(Gu)(t)) = (g * u)(t) = \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau \qquad (3)$$

here the impulse response function is

$$g(t) = \int_{-\infty}^{\infty} e^{j\omega t} G(j\omega) d\omega$$

This follows from the following formal derivation

$$\begin{aligned}(Gu)(t) &= \int_{-\infty}^{\infty} e^{j\omega t} G(j\omega)\hat{u}(j\omega) d\omega \\ &= \int_{-\infty}^{\infty} e^{j\omega t} G(j\omega) \int_{-\infty}^{\infty} e^{-j\omega\tau} u(\tau) d\tau \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{j\omega(t-\tau)} G(j\omega) d\omega \right) u(\tau) d\tau\end{aligned}$$

The impulse response function is often a fairly "wild" function and the integral (3) most be interpreted appropriately. In the course we mainly consider real rational transfer functions for which the interpretation is easy.

he following properties follows from the Plancherel-Parceval theorem.

Proposition 2. *Let $g = \mathcal{F}^{-1} G \mathcal{F}$ where $G \in \boldsymbol{L}_\infty(j\boldsymbol{R})$. Then*

(i) *$G \in \mathcal{L}(\boldsymbol{L}_2(\boldsymbol{R}))$ (the time domain $\boldsymbol{L}_2$) is defined by the convolution (3) (interpreted with care)*

(ii) $\|G\|_{\boldsymbol{L}_2(\boldsymbol{R}) \to \boldsymbol{L}_2(\boldsymbol{R})} = \sup_{0 \neq u \in \boldsymbol{L}_2(\boldsymbol{R})} \dfrac{\|g * u\|_{\boldsymbol{L}_2(\boldsymbol{R})}}{\|u\|_{\boldsymbol{L}_2(\boldsymbol{R})}} = \|G\|_\infty$

(iii) *G is a time invariant operator, i.e. it commutes with the time shift operator, $GS_T = S_T G$, where*

$$(S_T u)(t) = u(t-T)$$

Remark 6. The result can be streghtened to (see e.g. Dullerud/Paganini)

$$G \in \mathbf{L}_\infty(j\mathbf{R}) \quad \Leftrightarrow \quad \mathcal{F}^{-1} G \mathcal{F} \in \mathcal{L}(\mathbf{L}_2(R)) \text{ is time invariant}$$

Proof of Proposition: (i) and (ii) follows since the Fourier transform is a unitary operator, i.e. we have

$$\begin{aligned}\sup_{u \in \mathbf{L}_2(\mathbf{R})} \|g * u\|^2_{\mathbf{L}_2(\mathbf{R})} = \sup_{\hat{u} \in \mathbf{L}_2(\mathbf{R})} \|\mathcal{F}^{-1} G \mathcal{F} u\|^2_{\mathbf{L}_2(\mathbf{R})} &= \sup_{\hat{u} \in \mathbf{L}_2(j\mathbf{R})} \|G\hat{u}\|^2 \\ &= \|G\|_\infty \|\hat{u}\|_{\mathbf{L}_2(j\mathbf{R})}\end{aligned}$$

The time invariance follows since

$$\mathcal{F}^{-1} G \mathcal{F} S_T u = \mathcal{F}^{-1} G e^{j\omega T} \mathcal{F} u = \mathcal{F}^{-1} e^{j\omega T} G \mathcal{F} u = S_T \mathcal{F}^{-1} G \mathcal{F} u$$

The relationships are summarized by

$$\begin{array}{ccc} & g & \\ \mathbf{L}_2(\mathbf{R}) & \longrightarrow & \mathbf{L}_2(\mathbf{R}) \\ \mathcal{F} \downarrow & & \downarrow \mathcal{F} \\ & G & \\ \mathbf{L}_2(j\mathbf{R}) & \longrightarrow & \mathbf{L}_2(j\mathbf{R}) \end{array}$$

e next show that $\mathbf{H}_\infty$ provides us with operators with similar properties as $\mathbf{L}_\infty(j\mathbf{R})$ but with the additional property that the operators are causal. We first notice that Proposition 1 must hold also for $\mathbf{H}_\infty$ since $\mathbf{H}_\infty \subset \mathbf{L}_2(j\mathbf{R})$. However, it can strenghtened because we have the additional property that $\mathbf{H}_2$ is invariant under multiplication with transfer functions from $\mathbf{H}_\infty$:

Proposition 3. *A function $G \in \boldsymbol{H}_\infty$ defines an operator $M_G : \boldsymbol{H}_2 \to \boldsymbol{H}_2$ via the multiplication $(M_G\hat{u})(j\omega) = G(j\omega)\hat{u}(j\omega)$, i.e. $G \in \boldsymbol{H}_\infty \Rightarrow G \in \mathcal{L}(\boldsymbol{H}_2)$. Moreover the induced norm is*

$$\|M_G\|_{\boldsymbol{H}_2\to\boldsymbol{H}_2} = \sup_{0\neq\hat{u}\in\boldsymbol{H}_2} \frac{\|G\hat{u}\|_{\boldsymbol{H}_2}}{\|\hat{u}\|_{\boldsymbol{H}_2}} = \sup_{\sigma>0} \bar{\sigma}(G(\sigma + j\omega)) =: \|G\|_{\boldsymbol{H}_\infty}$$

Proof idea: *We only have to notice that $G(s)\hat{u}(s)$ is analytic in the open right half plane when $G \in \boldsymbol{H}_\infty$ and $\hat{u} \in \boldsymbol{H}_2$.*

e next consider the time domain representation $g = \mathcal{L}_+^{-1}G\mathcal{L}_+$, of $G \in \mathbf{H}_\infty$. A formal derivation gives the convolution interpretation

$$(Gu)(t) = \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau \tag{4}$$

where

$$g(t) = \int_{-\infty}^{\infty} e^{(\sigma+j\omega)t}G(\sigma + j\omega)d\omega, \quad \mathrm{Re}\ \sigma > 0$$

The Payley-Wiener theorem shows that $g(t) = 0$ for $t < 0$. If $u \in \mathbf{L}_2[0,\infty)$ then this means that (4) reduces to

$$(Gu)(t) = \int_0^t g(t-\tau)u(\tau)d\tau \tag{5}$$

which clearly shows that g is causal, i.e. the ouput $(Gu)(t)$ does not depend on future values of the input u. An equivalent way of stating this is $P_TGP_T = P_TG$.

e have

Proposition 4. *Let $g = \mathcal{L}_+^{-1}G\mathcal{L}_+$ where $G \in \boldsymbol{H}_\infty$. Then*

(i) $G \in \mathcal{L}(\boldsymbol{L}_2[0,\infty))$ (the time domain $\boldsymbol{L}_2$) is defined vi the convolution (5) (interpreted with care)

(ii)

$$\|G\|_{\boldsymbol{L}_2[0,\infty)\to\boldsymbol{L}_2[0,\infty)} = \sup_{0\neq u\in\boldsymbol{L}_2[0,\infty)} \frac{\|g * u\|_{\boldsymbol{L}_2[0,\infty)}}{\|u\|_{\boldsymbol{L}_2[0,\infty)}} = \|G\|_{\boldsymbol{H}_\infty}$$

(iii) G is a causal operator, i.e $P_TGP_T = P_TG$, where

$$(P_Tu)(t) = \begin{cases} u(t), & t \leq T \\ 0, & t > T \end{cases}$$

Proof: *The first two properties follows as before. For the third, see Dullerud/Paganini.*

The relationships are summarized by

$$\begin{array}{ccc} & g & \\ \mathbf{L}_2[0,\infty) & \longrightarrow & \mathbf{L}_2[0,\infty) \\ \mathcal{L}_+ \downarrow & & \downarrow \mathcal{L}_+ \\ & G & \\ \mathbf{H}_2 & \longrightarrow & \mathbf{H}_2 \end{array}$$