$\mathbf{H}_\infty$-Optimal Control

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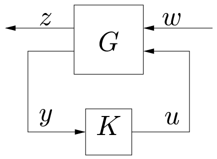
References

- Essentials of Robust Control, by Zhou and Doyle
- A course in Robust Control Theory, by Dullerud and Paganini
- $\mathbf{H}_\infty$-Optimal Control and Related Minimax Design Problems
- State-Space Solutions to Standard $\mathbf{H}_2$ and $\mathbf{H}_\infty$ Control Problems (DGKF), by Doyle, Glover, Khargonekar, and Francis.

Outline

- The $\mathbf{H}_\infty$-control problem
- More about the Riccati equation
- The operator theoreretic approach
- The dynamic game approach
- The DGKF approach (based on Riccati theory)
- The LMI approach (see lectures on LMI)

The $\mathbf{H}_\infty$ Control Problem



Given a system $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$, find a proper rational controller such that

(i) G is internally stabilized

(ii) $\|\mathcal{S}_l(G,K)\|_{\mathbf{H}_\infty} = \|T_{zw}\|_{\mathbf{H}_\infty}$ is minimized

Motivation

- It has been proven that LQ optimal control is robust (full state feedback). For example, in the SISO case the closed loop system has $6dB$ gain margin and $60°$ phase margin. For more details see Gain and Phase Margins for Multiloop LQG Regulators, by Safonov and Athans, or 5B1872 (Optimal Control)
- LQG regulators (the $\mathbf{H}_2$-optimal controller is a special case) does not have any robustness guarantees, see the Honeywell interoffice correspondence by Doyle.
- The $\mathbf{H}_\infty$-controller is robust since if $\|T_{zw}\| < \gamma$ then the system

  Delta-G-K connection

  is internally stable for all $\|\Delta\| \leq \gamma^{-1}$.

The Riccati Operator

From KYP we know that there is a close connection between the

$$A^T X + XA + Q - XRX = 0 \qquad \text{(ARE)}$$

(where Q and R are symmetric) and

$$H = \begin{bmatrix} A & -R \\ -Q & -A^T \end{bmatrix} \qquad \text{Hamiltonian matrix}$$

Some properties we proved in the KYP lecture

- H is symplectic, i.e. $J^{-1}HJ = -H^T$ where $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$
- This implies that the eigenvalues of H are symmetric w.r.t the imaginary axis. Hence, if H does not have any eigenvalues on the

imaginary axis then there are n stable and n unstable eigenvalues (where n is the dimension of A). Let $\mathcal{X}_-(H)$ be the invariant subspace corresponding to stable eigenvalues (evals in LHP) and let the basis vectors be partitioned as

$$\mathcal{X}_-(H) = \operatorname{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Definition 1. We say that $H \in \operatorname{dom}(\operatorname{Ric})$ if

(i) H has no eigenvalues on the imaginary axis

(ii) X_1 defined above is nonsingular.

and then $X = \operatorname{Ric}(H) := X_2 X_1^{-1}$ is uniquely defined.

Lemma 1. *If $H \in \operatorname{dom}(\operatorname{Ric})$ then $X = \operatorname{Ric}(H)$ is symmetric matrix such that*

(a) $A^T X + XA + Q - XRX = 0$

) *$A + RX$ is stable*

Proof: *We proved this during the KYP lecture.*

The next lemma shows that the condition that X_1 is nonsingular often can be removed in LQ problems

Lemma 2. *If $R \geq 0$ or $R \leq 0$ then the following are equivalent*

(i) $H \in \operatorname{dom}(\operatorname{Ric})$

(ii) *H has no eigenvalues on the imaginary axis and (A, R) is stabilizable.*

Proof: *See e.g. Zhou.*

Let us specialize to the LQ optimal control problem

$$\min \int_0^\infty (|Cx|^2 + |u|^2)dt \quad \text{subj. to} \quad \dot{x} = Ax + Bu \qquad (1)$$

KTH

The corresponding Hamiltonian is

$$H = \begin{bmatrix} A & -BB^T \\ -C^TC & -A^T \end{bmatrix} \tag{2}$$

Lemma 3. *If (A, B) is stabilizable and (C, A) detectable then H in (2) belongs to* $\mathrm{dom(Ric)}$. *Moreover* $X = \mathrm{Ric}(H) \geq 0$ *and* $\mathrm{Ker}(X) \subset \mathrm{Ker}(\mathcal{O})$, *where $\mathcal{O}$ is the observability matrix.*

Proof: *One need to show that (A, B) is stabilizable and (C, A) detectable implies that H has no eigenvalues on the imaginary axis, see Zhou. Note that the LQ problem in (1) is convex. The detactability and stabilizability conditions ensure that a stabilizing solution is obtained.*

We will next see that the situation is more difficult in $\mathbf{H}_\infty$-problems.

KTH

Consider the LQ problem

$$\min \int_0^\infty (-|Cx|^2 + |u|^2)dt \quad \text{subj. to} \quad \begin{cases} \dot{x} = Ax + Bu, \ x(0) = 0 \\ (x, u) \in \mathbf{L}_2[0, \infty) \end{cases} \tag{3}$$

The corresponding Hamiltonian is

$$H = \begin{bmatrix} A & -BB^T \\ C^TC & -A^T \end{bmatrix} \tag{4}$$

The optimization problem (3) is not necessarily convex and $H \in \mathrm{dom(Ric)}$ is no longer guaranteed by stabilizability and detectability of the system matrices. Instead it is closely related to the $\mathbf{H}_\infty$-norm of the system (when A is stable). The next result is a special case of the KYP lemma.

KTH

Lemma 4. *Let* $G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \in \boldsymbol{RH}_\infty$. *The following are equivalent*

(i) $\|G\|_{\boldsymbol{H}_\infty} < \gamma$

(ii) $H = \begin{bmatrix} A & \frac{1}{\gamma^2}BB^T \\ -C^TC & -A^T \end{bmatrix}$ *has no eigenvalues on the imaginary axis.*

(iii) *H in (4) belongs to* $\mathrm{dom(Ric)}$ *and* $\mathrm{Ric}(H) \geq 0$.

It is important to note the distinction between $\mathbf{H}_2$-norm computation and $\mathbf{H}_\infty$-norm computation.

KTH

- $\|G\|_{\mathbf{H}_2}^2 = \mathrm{tr}(CPC^T) = \mathrm{tr}(B^TQB)$ where

$$AP + PA^T + BB^T = 0$$
$$A^TQ + QA + C^TC = 0$$

- $\|G\|_{\mathbf{H}_\infty}^2 < \gamma^2$ iff $H \in \mathrm{dom(Ric)}$, where H is defined in Lemma 4.

To obtain the exact $\mathbf{H}_\infty$-norm we need to iterate (bijection algorithm) over γ. This is in stark contrast to the $\mathbf{H}_2$-norm compuation where only a linear equation system (the Lyapunov equation) has to be solved.

Operator Theoretic Approach

By using the Youla parametrization we get

$$\min_K \|\mathcal{S}_l(G,K)\|_{\mathbf{H}_\infty} = \min_{Q\in\mathbf{RH}_\infty} \|T_{11} + T_{12}QT_{21}\|_{\mathbf{H}_\infty}$$

where $T_{11}, T_{12}, T_{21} \in \mathbf{RH}_\infty$. The following steps are taken

- Verify that $T_{12}(j\omega)^*T_{12}(j\omega) \geq \epsilon I$ and $T_{21}(j\omega)T_{21}(j\omega)^* \geq \epsilon I$, $\forall\omega$ (the problem is nonsingular).
- Introduce inner outer transformations of T_{12}, T_{21} and transform the problem to the form

$$\min_{Q\in\mathbf{RH}_\infty} \left\| \begin{bmatrix} Q + R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty$$

here R belongs to $\mathbf{RL}_\infty$ but not in $\mathbf{RH}_\infty$.

- The above is a so called 2×2 block problem. Operator theory results such as Parrots lemma and results on interpolation theory are used to solve this approximation problem.
- Elementary introductions to these topics can be found in "A course in $\mathbf{H}_\infty$ control theory", by Francis and "An Introduction to Hilbert Space", by N. Young.

t us consider the standard simplifying assumptions

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

where

(i) (A, B_1) is stabilizable and (C_1, A) is detectable

(ii) (A, B_2) is stabilizable and (C_2, A) is detectable

(iii) $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$

ıe inner outer factorization can now be simplified if we let the nominal controller be the $\mathbf{H}_2$-optimal controller then the Youla parametrization becomes $T_{zw} = T_{\mathbf{H}_2} + UQV$, where U is inner and V is co-inner, i.e. $U^*U = I$ and $VV^* = I$. If we let $U_\perp$ be a complementary inner factor such that $\mathbf{U} = \begin{bmatrix} U & U_\perp \end{bmatrix}$ is unitary (i.e. inner and invertible) and similarly let $V_\perp$ be such that $\mathbf{V} = \begin{bmatrix} V^* & V_\perp^* \end{bmatrix}$ is unitary. This gives

$$\begin{aligned} \min_K \|\mathcal{S}(G,K)\|_{\mathbf{H}_\infty} &= \min_{Q\in\mathbf{RH}_\infty} \|T_{\mathbf{H}_2} + UQV\|_{\mathbf{H}_\infty} \\ &= \min_{Q\in\mathbf{RH}_\infty} \|\mathbf{U}^*(T_{\mathbf{H}_2} + UQV)\mathbf{V}^*\|_\infty \\ &= \min_{Q\in\mathbf{RH}_\infty} \left\| \begin{bmatrix} R_{11} + Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty \end{aligned}$$

where $R = \mathbf{U}^* T_{\mathbf{H}_2} \mathbf{V}$.

A simpler problem is the suboptimal $\mathbf{H}_\infty$-control problem

$$\text{Find } Q \in \mathbf{RH}_\infty \text{ s.t. } \quad \left\| \begin{bmatrix} R_{11}+Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty < \gamma$$

Solutions are discussed in

- The book "A course in $\mathbf{H}_\infty$ control theory", by Francis discusses how this problem can be solved using state space methods.
- The DGKF paper ends with a brief discussion on how this problem can be solved using alternative state space methods.

The Dynamic Game Approach

Let $T_{zw}(K)$ be the closed loop system with controller K. Since

$$\|T_{zw}\|_{\mathbf{H}_\infty} = \sup_{\hat{w}\in\mathbf{H}_2} \frac{\|T_{zw}\hat{w}\|_{\mathbf{H}_2}}{\|\hat{w}\|_{\mathbf{H}_2}}$$

it follows that the optimal $\mathbf{H}_\infty$ design problem can be formulated as the min max problem

$$\inf_{K\in\mathcal{K}} \sup_{\hat{w}\in\mathbf{H}_2} \frac{\|T_{zw}(K)\hat{w}\|_{\mathbf{H}_2}}{\|\hat{w}\|_{\mathbf{H}_2}}$$

where $\mathcal{K} = \{K \in \mathbf{RL}_\infty : K \text{ internally stabilizing}\}$. A much simpler problem is the suboptimal $\mathbf{H}_\infty$ control problem

$$\text{Find } K \in \mathcal{K} \text{ s.t. } \quad \|T_{zw}(K)\hat{w}\|_{\mathbf{H}_2} - \gamma\|\hat{w}\|_{\mathbf{H}_2} < 0, \ \forall \hat{w} \in \mathbf{H}_2 \qquad (5)$$

We will next derive a dynamic game (in the time domain) that

rresponds to this suboptimal $\mathbf{H}_\infty$ problem. We consider the case when $\mathcal{K}$ is the set of stabilizing state feedback controllers. Let

$$G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ I & 0 & 0 \end{array}\right]$$

where

(i) (A, B_1) is stabilizable and (C_1, A) is detectable

(ii) (A, B_2) is stabilizable

(iii) $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$

Let

$$J_\gamma(u,w) = \int_0^\infty (|C_1x + D_{12}u|^2 - \gamma^2|w|^2)dt = \int_0^\infty (|C_1x|^2 + |u|^2 - \gamma^2|w|^2)dt$$

d let $\mathcal{U} = \{u = Fx : F \text{ is a stabilizing state feedback}\}$. If the value of the game

$$\inf_{u\in\mathcal{U}} \sup_{w\in\mathbf{L}_2[0,\infty)} J_\gamma(u,w) \quad \text{s.t.} \quad \begin{cases} \dot{x} = Ax + B_1w + B_2u \\ x(0) = 0 \end{cases}$$

is zero then it can be shown that (5) holds. We have the following

Theorem 1. *Let* $H_\infty = \begin{bmatrix} A & \gamma^{-2}B_1B_1^T - B_2B_2^T \\ -C_1C_1^T & -A^T \end{bmatrix}$. *If* $H_\infty \in \mathrm{dom}(\mathrm{Ric})$ *and* $X_\infty = \mathrm{Ric}(H_\infty) > 0$ *then the optimal state feedback* $F_\infty = -B_2^T X_\infty$ *gives* $\|\mathcal{S}_l(G, F_\infty)\|_{\boldsymbol{H}_\infty} < \gamma$.

Remark 1. Note that we obtain the LQ ($\mathbf{H}_2$ solution when $\gamma \to \infty$)

Proof: $X_\infty = \mathrm{Ric}(H_\infty)$ *satisfies*

$$A^TX_\infty + X_\infty A + C_1^TC_1 - X_\infty(B_2B_2^T - \gamma^{-2}B_1B_1^T)X_\infty = 0$$

Let $V(x) = x^T X_\infty x$. Using the dynamics and the ARE we can verify that

$$\frac{d}{dt}V(x) + |C_1 x| + |u|^2 - \gamma^2 |w|^2 = |u - B_2^T X_\infty x|^2 - \gamma^2 |w + \gamma^{-2} B_1^T X_\infty x|^2$$

Integration gives

$$J_\gamma(u,w) = \int_0^\infty (|C_1 x|^2 + |u|^2 - \gamma^{-2}|w|^2)dt$$

$$= \lim_{T\to\infty}\left(x(T)^T X_\infty x(T) + \int_0^T (|u + B_2^T X_\infty x|^2 - \gamma^{-2}|w - \gamma^{-2} B_1^T X_\infty x|^2)dt \right)$$

and hence

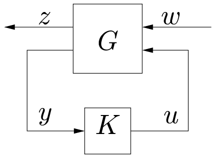
$$\inf_{u\in\mathcal{U}} \sup_{w\in \boldsymbol{L}_2[0,\infty)} J_\gamma(u,w) = \sup_{w\in \boldsymbol{L}_2[0,\infty)} \inf_{u\in\mathcal{U}} J_\gamma(u,w) = 0$$

where the optimum (saddle point solution) is obtained when

$$u_{opt} = F_\infty x, \qquad F_\infty = -B_2^T X_\infty$$
$$w_{opt} = \gamma^{-2} B_1^T X_\infty x$$

Note that $\dot{x} = Ax + B_1 w_{opt} + B_2 u_{opt} = (A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T)X_\infty)x$ *is a stable system since by the Riccati theory* $A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T)X_\infty$ *is Hurwitz.*

DGKF



- State space solution using two Riccati equations
- A famous milestone

Simplifying assumptions:

- $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$
- (A, B_1) is stabilizable and (C_1, A) detectable
- (A, B_2) is stabilizable and (C_2, A) detectable
- $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$
- $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$

Let

$$H_\infty = \begin{bmatrix} A & \gamma^{-2}B_1B_1^T - B_2B_2^T \\ -C_1^TC_1 & -A^T \end{bmatrix}$$

$$J_\infty = \begin{bmatrix} A^T & \gamma^{-2}C_1C_1^T - C_2^TC_2 \\ -B_1B_1^T & -A \end{bmatrix}$$

Theorem 2. *There exists an proper internally stabilizing controller such that* $T_{zw} = \mathcal{S}_l(G,K)\|_{\boldsymbol{H}_\infty} < \gamma$ *iff*

(i) $H_\infty \in \mathrm{dom(Ric)}$ *and* $X_\infty = \mathrm{Ric}(H_\infty) \geq 0$

(ii) $J_\infty \in \mathrm{dom(Ric)}$ *and* $Y_\infty = \mathrm{Ric}(J_\infty) \geq 0$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$

One such controller is

$$K(s) = \left[\begin{array}{c|c} A_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$A_\infty = A + \gamma^{-2}B_1B_1^TX_\infty + B_2F_\infty + Z_\infty L_\infty C_2$$

$$F_\infty = -B_2^TX_\infty,\ L_\infty = -Y_\infty C_2^T,\ Z_\infty = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}$$

The set of all such controllers is parametrized as $K(s) = \mathcal{S}_l(M_\infty, Q)$, *where* $Q \in \boldsymbol{RH}_\infty$ *and*

$$M_\infty(s) = \left[\begin{array}{c|cc} A_\infty & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$