Internal Stability

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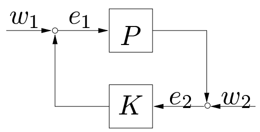
References

- Essentials of Robust Control, by Zhou and Doyle
- A Course in Robust Control Theory, by Dullerud and Paganini
- A Course in $\mathbf{H}_\infty$ Control Theory, by Francis

Outline

- Well-Posedness and Input-Output Stability (Transfer Function Approach)
- Internal Stability (State Space Approach)
- Robust Control Set-up

Well-Posedness and Input-Output Stability



$$\begin{aligned} e_1 &= w_1 + Ke_2 \\ e_2 &= w_2 + Pe_1 \end{aligned} \tag{1}$$

Two questions arise

- Can we solve for the internal signals e_1, e_2 for all w_1, w_2 (well-posedness)?
- Are all transfer functions $w_k \mapsto e_l$ $k, l = 1, 2$ stable (input-output stability)?

Example 1. If $P = K = 1$ and $w_2 = 0$ then we get

$$e_1 = w_1 + PKe_1 = w_1 + e_1 \quad \Leftrightarrow \quad w_1 = 0$$

which is not a well defined feedback loop.
Example 2. If $P = \frac{s-1}{s+1}$, $K = 1$ and $w_2 = 0$ then

$$e_1 = \frac{s+1}{2} w_1$$

which is not a strictly proper transfer function.

These two example systems are not well-posed. The problem is that there are ill-defined algebraic loops in these system.

System (1) can equivalently be written

$$\begin{bmatrix} I & -K \\ -P & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Definition 1. The system (1) is

$$\begin{matrix} \text{well-posed if} \\ \text{input-output stable if} \end{matrix} \quad \begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} \quad \begin{matrix} \text{is defined and proper} \\ \text{in addition is stable} \end{matrix}$$

Recall that a transfer function $G(s)$ is proper if $G(\infty)$ is finite.

It is called strictly proper if $G(\infty) = 0$

It is called stable if all poles are in the open left half plane.

Well-Posedness

Let

$$\Phi(s) = \begin{bmatrix} I & -K(s) \\ -P(s) & I \end{bmatrix} = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right]$$

Claim: $\Phi(s)^{-1}$ is well defined and proper $\Longleftrightarrow \Phi(\infty) = D_{cl}$ is invertible.
Proof: We have

$$\Phi(s)^{-1} = \left[\begin{array}{c|c} A_{cl} - B_{cl}D_{cl}^{-1}C_{cl} & B_{cl}D_{cl}^{-1} \\ \hline -D_{cl}^{-1}C_{cl} & D_{cl}^{-1} \end{array}\right]$$

which is a proper and well defined transfer function iff $\Phi(\infty) = D_{cl}$ is invertible.

Proposition 1. *System (1) is well-posed iff either of the following equivalent conditions hold*

(*i*) $\begin{bmatrix} I & -K(\infty) \\ -P(\infty) & I \end{bmatrix}$ *is invertible*

(*ii*) $I - K(\infty)P(\infty)$ *is invertible*

(*ii*) $I - P(\infty)K(\infty)$ *is invertible*

Proof: (*i*) *follows from the above claim. The other equivalences are a consequence of the identity*

$$det \begin{bmatrix} I & -K(\infty) \\ -P(\infty) & I \end{bmatrix} = det(I - K(\infty)P(\infty)) = det(I - P(\infty)K(\infty))$$

Input-Output Stability

Proposition 2. *From the definition we have that (1) is internally stable iff*

$$\begin{bmatrix} I & -K(s) \\ -P(s) & I \end{bmatrix}^{-1} = \begin{bmatrix} (I-KP)^{-1} & K(I-PK)^{-1} \\ P(I-KP)^{-1} & (I-PK)^{-1} \end{bmatrix}(s)$$

is stable.

Comment: Since

$$\det \begin{bmatrix} I & -K(s) \\ -P(s) & I \end{bmatrix} = \det(I - K(s)P(s)) = \det(I - P(s)K(s))$$

one would like to believe that it is enough check stability of $I - P(s)K(s))^{-1}$. The next example shows that this may lead to wrong conclusions.

Example 3 (From Zhou). If $P(s) = \dfrac{s-1}{s+1}$ and $K(s) = -\dfrac{1}{s-1}$ then

$$\begin{bmatrix} I & -K(s) \\ -P(s) & I \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{s+1}{s+2} & -\dfrac{s+1}{(s-1)(s+2)} \\ \dfrac{s-1}{s+2} & \dfrac{s+1}{s+2} \end{bmatrix}$$

The problem is that there is an unstable pole-zero cancelation $P(s)K(s) = \dfrac{1}{s+1}$.

Theorem 1. *The system (1) is input-output stable iff it is well-posed and*

(i) $\#RHP(P(s)K(s)) = \#RHP(P(s)) + \#RHP(K(s))$

(ii) $(I-PK)^{-1}$ *is stable*

Proof: *A proof can be found in Zhou*

Several simpler results can be obtained easily. For example using the identities

- $(I-PK)^{-1} = I + (I-PK)^{-1}PK$
- $P(I-KP)^{-1} = (I-PK)^{-1}P$
- $\det(I-PK) = \det(I-KP)$

it follows that (1) is input-output stable iff it is well-posed and K and $P(I-KP)^{-1}$ are stable.

Another case is when P and K are stable. Then the system is input-output stable iff $(I-PK)^{-1}$ is stable, i.e. iff $\det(I - P(s)K(s)) \neq 0$ for all $\mathrm{Re}\, s \geq 0$.

State Space Approach

If $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ then we get

$$\begin{bmatrix} \dot{x} \\ \dot{x}_K \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} x \\ x_K \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & C_K \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_K \end{bmatrix} + \begin{bmatrix} 0 & D_K \\ D & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

For well-posedness we need that $\begin{bmatrix} I & -D_K \\ -D & I \end{bmatrix}$ is invertible since then we can solve for $e = \begin{bmatrix} e_1^T & e_2^T \end{bmatrix}^T$

ıen the closed loop system matrix becomes

$$A_{cl} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C & 0 \end{bmatrix}$$

Definition 2. The closed loop system in (1) is well-posed iff $\begin{bmatrix} I & -D_K \\ -D & I \end{bmatrix}$ is invertible. System (1) is called internally stable if in addition A_{cl} is Hurwitz. This implies input-output stability.

Note that stability of A_{cl} only is a sufficient condition for input-output stability. Indeed, there may be unstable pole zero cancelations that hide the unstable modes of A_{cl} from the input-output relation of the closed loop transfer functions. This would in practice be very undesirable and we will next provide stronger conditions.

Let

$$\Psi(s) = \begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right]$$

Claim: If (A_{cl}, B_{cl}) is stabilizable and (C_{cl}, A_{cl}) detectable then $\Psi(s)$ stable $\Longleftrightarrow A_{cl}$ is Hurwitz.
Proof: $\Leftarrow$ is obvious. For $\Rightarrow$ we assume on the contrary that A_{cl} has an unstable eigenvalue at s_0. Since $\Psi(s)$ is stable there must be an unstable pole-zero cancelation at s_0. However, this is impossible because

$$\begin{bmatrix} A_{cl} - s_0 I & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}$$

has full rank for Re $s \geq 0$ by stabilizability, detectability, and the invertibility of D_{cl}. Do you see why?

We have

$$\Psi(s) = \begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right]$$

where

$$A_{cl} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C & 0 \end{bmatrix}$$

$$B_{cl} = \begin{bmatrix} B & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D & I \end{bmatrix}^{-1}, \quad C_{cl} = \begin{bmatrix} I & -D_K \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C & 0 \end{bmatrix}$$

$$D_{cl} = \begin{bmatrix} I & -D_K \\ -D & I \end{bmatrix}^{-1}$$

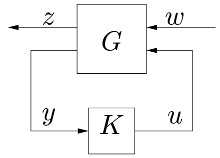
This is a stabilizable and detectable realization because

- (A, B) stabilizable $\iff (A+BF, BM)$ is stabilizable for every state feedback F and any invertible input transformation M.
- (C, A) detectable $\iff (MC, A+LC)$ is detectable for every output feedback L and any invertible output transformation M.

We obtain the following strengthening of Proposition (2)

Theorem 2. *The closed loop system in (1) is well-posed iff D_{cl} is invertible. Moreover, if in addition (A_{cl}, B_{cl}) is stabilizable and (C_{cl}, A_{cl}) is detectable then system (1) is input-output stable iff A_{cl} is Hurwitz.*

The Robust Control Set-up



- z regulated output
- y sensor output
- w exogenous input (disturbances)
- u control input

t

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \qquad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

This implies that

$$G_{cl} = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right]$$

We provide two results

1. An internal stability result which simply states conditions on when the closed loop system matrix is Hurwitz.
2. An input-output result that states a condition when internal stability is necessary for stability of G_{cl}.

Proposition 3. *The closed loop system in the robust control set-up is well-posed when* $\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}$ *is invertible. It is called internally stable if in addition*

$$A_{cl} = \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix}$$

is Hurwitz.

We next consider the system under an assumption used in the basic formulations for $\mathbf{H}_2$ and $\mathbf{H}_\infty$-optimal control:

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

Theorem 3. *Suppose*

$$\begin{bmatrix} A - \lambda I & B_2 \\ C_1 & D_{12} \end{bmatrix}, \qquad \begin{bmatrix} A - \lambda I & B_1 \\ C_2 & D_{21} \end{bmatrix}$$

has full column rank for Re $\lambda \geq 0$ and full row rank for all Re $\lambda \geq 0$, respectively. Assume also that the controller realization is minimal.

Then internal stability, i.e.

$$A_{cl} = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix}$$

Hurwitz, is equivalent to stability of G_{cl}, i.e. G_{cl} has no poles in the closed RHP.

Proof: The following lemma is used

Lemma 1. *If the realization of G_{cl} is stabilizable and detectable then*

$$G_{cl}(s) \text{ stable} \quad \Longleftrightarrow \quad A \text{ is Hurwitz}$$

Using this lemma we need to show that the rank conditions implies stabilizability and detectability of the closed loop system. Closed

ɔp detectability follows if

$$\begin{bmatrix} A_{cl} - \lambda I \\ C_{cl} \end{bmatrix} = \begin{bmatrix} A + B_2 D_K C_2 - \lambda I & B_2 C_K \\ B_K C_2 & A_K - \lambda I \\ C_1 + D_{12} D_K C_2 & D_{12} C_K \end{bmatrix} \tag{2}$$

has full column rank for all Re $s \geq 0$.

This matrix can be factorized as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A - \lambda I & B_2 & 0 \\ C_1 & D_{12} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ D_K C_2 & C_K \\ B_K C_2 & A_K - \lambda I \end{bmatrix}$$

The first factor is invertible and the second has by assumtion full column rank for Re $\lambda \geq 0$. Hence, to find an element in the kernel of (2) we need to find a vector in the kernel of the last factor. It is

clear that this vector must have the form $x^T = \begin{bmatrix} 0 & x_K^T \end{bmatrix}^T$ and satisfy

$$\begin{bmatrix} A_K - \lambda I \\ C_K \end{bmatrix} x_K = 0 \quad \forall \text{Re } \lambda \geq 0$$

This is impossible since K has a detectable realization (it is even minimal). Stabilizability follows similarly.