Linear Multivariable Systems

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References

- Essentials of Robust Control, by Zhou and Doyle
- A course in Robust Control Theory, by Dullerud and Paganini
- 5B1822 Matematisk systemteori, fortsättningskurs
- Linear Systems by T. Kailath

Outline

- Controllability and Stabilizability
- Observability and Detectability
- Duality
- Observer-Based Controllers
- Operations on Transfer Functions
- Poles and Zeros of Multivariable Systems

Controllability

Definition 1. The linear system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$

is controllable if for every $x_0, x_1 \in \mathbf{R}^n$, $t_1 > 0$ there exists a integrable control $u : [0, t_1] \to \mathbf{R}^m$ such that

$$x_1 = e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1 - s)}Bu(s)ds \tag{1}$$

We say that the pair (A, B) is controllable if the condition in this definition holds.

Theorem 1. *The following are equivalent*

(i) *The pair (A, B) is controllable*

(ii) *The controllability matrix*

$$\mathcal{C} = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

has full (row) rank

(iii) *The controllability Gramian*

$$W_t := W_C(0,t) = \int_0^t e^{As}BB^Te^{A^Ts}ds$$

is positive definite for all $t > 0$

(iv) *The matrix $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ has full row rank for all $\lambda \in \boldsymbol{C}$*

(v) *If $x^*A = \lambda x^*$ then $x^*B \neq 0$*

(vi) *The eigenvalues of $A + BF$ can be assigned arbitrarily*

(symmetric with respect to the real axis if (A, B) are real-valued) by suitable choice of F

Sketch of Proof: $(i) \Rightarrow (ii)$: *By Cayley-Hamilton it follows that*

$$e^{At} = \sum_{k=0}^{n-1} \phi_k(t)A^k$$

for some suitable functions $\phi_k(t)$. Hence

$$x_1 - e^{At}x_0 = \int_0^t e^{A(t-s)}Bu(s)ds = \mathcal{C}\begin{bmatrix} \int_0^t \phi_0(t-s)u(s)ds \\ \vdots \\ \int_0^t \phi_{n-1}(t-s)u(s)ds \end{bmatrix}$$

which shows that $\mathcal{C}$ must be of full row rank to ensure that any $x_1 - e^{At}x_0$ is in the image.

$(ii) \Rightarrow (iii)$: *Assume by contradiction that there exists $v \in \boldsymbol{R}^n$ such*

that $v^*W_tv = 0$. This implies that $\phi(t) := v^Te^{At}B \equiv 0$, which in turn implies

$$\begin{bmatrix} \phi(0) & \dot{\phi}(0) & \dots & \phi^{(n-1)}(0) \end{bmatrix} = v^T\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = 0$$

This contradicts (ii) so (iii) must hold.

$(iii) \Rightarrow (i)$: *Let $u(s) = B^Te^{A^T(t-s)}v$ then*

$$\int_0^t e^{A(t-s)}Bu(s)ds = \int_0^t e^{A(t-s)}BB^Te^{A^T(t-s)}dsv$$
$$= \int_0^t e^{As}BB^Te^{A^Ts}dsv = W_tv$$

We obtain a solution to (1) when $v = W_t^{-1}(x_1 - e^{At}x_0)$.

$(ii) \Rightarrow (iv), (v)$: *Assume that*

$$\operatorname{rank}\begin{bmatrix} A - \lambda I & B \end{bmatrix} < n$$

It is no restriction to assume that λ is an eigenvalue of A. Hence, there exists v such that $v^*A = \lambda v^*$ and $v^*B = 0$. Then $v^*A^kB = \lambda^kv^*B = 0$ and thus $v^*\mathcal{C} = 0$, which contradicts (ii).

$(iv) \Leftrightarrow (v)$: *Obvious.*

$(iv) \Rightarrow (ii)$: *Suppose $\operatorname{rank}\mathcal{C} = r < n$. Then there exists a change of coordinates such that*

$$T\,Im\mathcal{C} = Im\begin{bmatrix} I_r \\ 0 \end{bmatrix}$$

Next we use that $Im\mathcal{C}$ is an A invariant subspace, i.e. $AIm\mathcal{C} \subset Im\mathcal{C}$, which follows since

$$Im\,A\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \subset Im\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

Cayley Hamilton. In the new coordinate system this implies

$$TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}, \quad TB = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix} \tag{2}$$

Consider (iv) in the new coordinates

$$T \begin{bmatrix} \lambda I - A & B \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \lambda I - \bar{A} & \bar{B} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda I - \bar{A}_{11} & \bar{A}_{12} & \bar{B}_1 \\ 0 & \lambda I - \bar{A}_{22} & 0 \end{bmatrix}$$

This shows that the matrix in (iv) looses rank when $\lambda \in \mathrm{eig}(\bar{A}_{22})$, which proves that (iv) $\Rightarrow$ (ii).

(ii) $\Leftrightarrow$ (vi): We only do the SISO case. If

$$\mathcal{C} = \begin{bmatrix} B & AB & \dots A^{n-1}B \end{bmatrix}$$

is stable then there exists a coordinate transform T s.t.

$$\bar{A} = TAT^{-1} = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix}, \qquad \bar{B} = TB = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

where the a_k are coefficients of the characteristic polynomial $\det(sI - A) = s^n + a_{n-1}s^{n-1} + \dots + a_0$ (see e.g. Kailat's book). If we use the state feedback $u = Fx$ with $F = \begin{bmatrix} a_0 - \bar{a}_0 & \dots a_{n-1} - \bar{a}_{n-1} \end{bmatrix}$ then we get the new characteristic polynomial

$$\det(sI - \bar{A} - \bar{B}F) = s^n + \bar{a}_{n-1}s^{n-1} + \dots + \bar{a}_0$$

Since the $\bar{a}_k$ are arbitrary it follows that the closed loop poles can be placed anywhere we want.

It is easy to see that the (3) is a controllable pair.

(2) we showed that if the pair (A, B) is not controllable then there exists a change of coordinates such that

$$TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}, \quad TB = \begin{bmatrix} \bar{B}_{11} \\ 0 \end{bmatrix}$$

The pair $(\bar{A}_{11}, \bar{B}_{11})$ is controllable while $\bar{A}_{22}$ corresponds to states that are not controllable.

Definition 2. The pair (A, B) is stabilizable if $\bar{A}_{22}$ is Hurwitz.

Remark 1. Note that a controllable pair (A, B) also is stabilizable.

Remark 2. $\bar{A}_{22}$ corresponds to the uncontrollable states.

Remark 3. If the system is stabilizable then the uncontrollable modes converge to zero.

The next result is a direct consequence of the previous theorem.

Theorem 2. *The following are equivalent*

(*i*) *The pair* (A, B) *is stabilizable*

(*ii*) *The matrix* $\begin{bmatrix} A - \lambda I & B \end{bmatrix}$ *has full row rank for all Re* $\lambda \geq 0$

(*v*) *If* $x^*A = \lambda x^*$ *and Re* $\lambda \geq 0$ *then* $x^*B \neq 0$

(*vi*) *There exists a matrix* F *such that* $A + BF$ *is Hurwitz*

Observability

Definition 3. The system

$$\dot{x} = Ax, \quad x(0) = x_0$$
$$y = Cx$$

is observable if every initial condition x_0 can be uniquely determined by observing the output $y(t)$ over an arbitrary nonzero time inteval $[0, T]$.

We will next see that a number of equivalent conditions for observability can be obtained from directly from the corresponding equivalences for controllability. Define $\Psi : \mathbf{R}^n \to C^\infty(\mathbf{R}, \mathbf{R}^m)$ as $(\Psi x_0)(t) = Ce^{At}x_0$ for $t \in [0, T]$. In order to uniquely determine x_0 from $y(t)$ it is necessary and sufficient that $\operatorname{Ker}\Psi = 0$.

Lemma 1. *We have* $Ker\,\Psi = Ker\,\mathcal{O}$ *where the observability matrix is defined as*

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Proof: *If* $x_0 \in Ker\,\Psi$ *then* $\dfrac{d^k}{dt^k}Ce^{At}|_{t=0}x_0 = 0$ *for all* k*. Using Cayley Hamilton this leads to* $x_0 \in \mathcal{O}$*, which proves* $Ker\,\Psi \subset Ker\,\mathcal{O}$*.*

For the other direction, notice that

$$Ce^{At}x_0 = \sum_{k=0}^{n-1} \phi_k(t)CA^kx_0$$

for some suitable functions ϕ_k*. This shows* $Ker\,\mathcal{O} \subset Ker\,\Psi$*.*

The lemma implies that (C, A) is observable iff (A^T, C^T) is controllable because

$$\begin{aligned} &\operatorname{Ker}\mathcal{O} = 0 \\ &\iff \\ &\operatorname{rank}\mathcal{O} = n \\ &\iff \\ &\operatorname{rank}\begin{bmatrix} C^T & A^TC^T & \dots & (A^T)^{n-1}C^T \end{bmatrix} = n \end{aligned}$$

The next result follows as an immediate consequence

Theorem 3. *The following are equivalent*

(i) *The pair (C, A) is observable*

(ii) *The observability matrix $\mathcal{O}$ has full rank*

(iii) *The observability Grammian*

$$W_O(0,t) = \int_0^t e^{A^T t} C^T C e^{At} dt$$

is positive definite for all $t > 0$

(iv) *The matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank for all $\lambda \in \boldsymbol{C}$*

(v) *If $Ax = \lambda x$ then $Cx \neq 0$*

(vi) *The eigenvalues of $A + LC$ can be assigned arbitrarily*

The unobservable subspace in $\operatorname{Ker}\mathcal{O}$ is A-invariant, i.e. $A\operatorname{Ker}\mathcal{O} \subset \operatorname{Ker}\mathcal{O}$. If we introduce a change of coordinates such that

$$T\operatorname{Ker}\mathcal{O} = \operatorname{Im}\begin{bmatrix} 0 \\ I \end{bmatrix}$$

the we obtain a realization on the form

$$CT^{-1} = \begin{bmatrix} \bar{C}_{11} & 0 \end{bmatrix} \qquad TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{12} & \bar{A}_{22} \end{bmatrix}$$

The pair $(\bar{C}_{11}, \bar{A}_{11})$ is observable while $\bar{A}_{22}$ corresponds to unobservable states.

Definition 4. The pair (C, A) is detectable if $\bar{A}_{22}$ is Hurwitz.

Remark 4. Note that an observable pair (C, A) also is detectable.

Remark 5. Detectability ensure that the unobservable states are not very important since they decay to zero.

The next result is a direct consequence of the previous theorem.

Theorem 4. *The following are equivalent*

(i) *The pair (C, A) is detectable*

(ii) *The matrix $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank for all Re $\lambda \geq 0$*

(iii) *If $Ax = \lambda x$ and Re $\lambda \geq 0$ then $Cx \neq 0$*

(iv) *There exists a matrix L such that $A + LC$ is Hurwitz.*

Observer

The standard full order observer for the system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad x(0) = x_0 \\ y &= Cx + Du \end{aligned}$$

has the form

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} + Du - y)$$

The observer error $e = x - \hat{x}$ satisfies

$$\dot{e} = (A + LC)e$$

which can be made stable if (C, A) is detectable.

Observer-Based Controllers

The idea is to

1. Design a full state observer to estimate the state vector
2. Design a state feedback controller to stabilize the system
3. Use the observer state in the state feedback controller

The resulting controller is

$$\begin{aligned}\dot{x}_K &= Ax_K + Bu + L(Cx_K + Du - y)\\ u &= Fx_K\end{aligned}$$

The closed loop system becomes

$$\begin{aligned}\dot{x} &= Ax + BFx_K\\ \dot{x}_K &= Ax_K + BFx_K + L(Cx_K - Cx)\end{aligned}$$

If we change coordinates such that $(e = x - x_K, x_K)$ is the new state then

$$\begin{bmatrix}\dot{e}\\ \dot{x}_K\end{bmatrix} = \begin{bmatrix}A+LC & 0\\ -LC & A+BF\end{bmatrix}\begin{bmatrix}e\\ x_K\end{bmatrix}$$

This system is stable if $A + BF$ and $A + LC$ are stable which is possible to achieve if (A, B) is stabilizable and (C, A) is detectable.

- The state feedback controller and the observer can be designed separately by placing the poles of $A + BF$ and $A + LC$ suitably.
- This separation between state feedback control design and observer design also takes place in
  - LQG and $\mathbf{H}_2$ optimal control.
  - classical $\mathbf{H}_\infty$ control
  - many methods for LMI synthesis of controllers (but it is then more implicit)

Duality

A transfer function realization is in robust control often denoted

$$G(s) = C(sI - A)^{-1}B + D =: \left[\begin{array}{c|c} A & B\\ \hline C & D\end{array}\right]$$

The dual system is

$$G(s)^T = \left[\begin{array}{c|c} A^T & C^T\\ \hline B^T & D^T\end{array}\right]$$

An operation on a system often has a corresponding operation on the dual.

Some dual relations

Operation	Operation on dual system
Controllability	Observability
Stabilizability	Detectability
State feedback design	Observer design
LQ control	Kalman filter design
$\mathbf{H}_\infty$ state feedback	$\mathbf{H}_\infty$ observer

Operations on Transfer Functions

If

$$G_1(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right], \qquad G_2(s) = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

then the series connection becomes

$$G_1(s)G_2(s) = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]$$

Proof:

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + B_1(C_2x_2 + D_2u) = A_1x_1 + B_1C_2x_2 + B_1D_2u \\ \dot{x}_2 &= A_2x_2 + B_2u \\ y &= C_1x_1 + D_1(C_2x_2 + D_2u) = C_1x_1 + D_1C_2x_2 + D_1D_2u \end{aligned}$$

Another interpretation

$$G_1G_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & I & 0 \\ \hline C_1 & 0 & D_1 \end{array}\right] \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & A_2 & B_2 \\ \hline 0 & C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right]$$

Parallel connection (summation)

$$G_1 + G_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right]$$

Concatenation (parallel connection, summation)

$$\begin{bmatrix} G_1 & G_2 \end{bmatrix} = \left[\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ \hline C_1 & C_1 & D_1 & D_2 \end{array}\right]$$

Inversion

$$G_1^{-1} = \left[\begin{array}{c|c} A_1 - B_1D_1^{-1}C_1 & -B_1D_1^{-1} \\ \hline D_1^{-1}C_1 & D_1^{-1} \end{array}\right]$$

- These operations are useful when implementing the robust control methods in software packages
  - μ-Analysis and Synthesis Toolbox
  - IQC Toolbox
  - LMI toolbox
- The operations may not result in a minimal realization (controllable and observable). To reduce the order
  - make a Kalman decomposition and remove states
  - use algorithms for model order reduction

Multivariable System Poles and Zeros

The normal rank of $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ is the maximally possible rank considered for all $s \in \mathbf{C}$. The normal column and normal row rank are defined similarly.

Definition 5. If $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ has full normal column rank then s_0 is an invariant zero of $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ if there exists (x, u) such that

$$\begin{bmatrix} A - s_0 I & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0$$

Similarly, if $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ has full normal row rank then s_0 is an invariant zero of $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ if there exists (x, u) such that

$$\begin{bmatrix} x^* & u^* \end{bmatrix} \begin{bmatrix} A - s_0 I & B \\ C & D \end{bmatrix} = 0$$

The interpretation is that $G(s_0)u = 0$ in the first case and $u^* G(s_0) = 0$ in the second case. This is easy to prove

Lemma 2.

(i) *for every state feedback F and any invertible input transformation M we have*

$$(A, B) \text{ stabilizable} \iff (A + BF, BM) \text{ is stabilizable}$$

(ii) *for every output feedback L and any invertible output transformation M we have*

$$(C, A) \text{ detectable} \iff (MC, A + LC) \text{ is detectable}$$

Proof. This is a homework problem. □