The Structured Singular Value

Ulf Jönsson

Optimization and Systems Theory

Royal Institute of Technology

Sweden

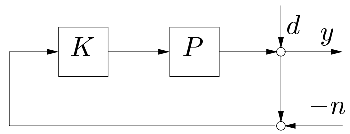
References

- Books
  1. Essentials of Robust Control, by Zhou and Doyle
  2. A course in Robust Control Theory, by Dullerud and Paganini
- Software: μ-Analysis and Synthesis Toolbox
- Journal papers
  1. Complex Structured Singular Value, by Packard, Doyle
  2. Analysis of Feedback Systems with Structure Uncertainty, by Doyle
  3. Stability Margins of Diagonally Perturbed Multi-variable Feedback Systems, by Safonov
- LIDS MIT. Linear Multi-variable Systems by A. Megretski

Outline

- The following concepts are introduced via an example
  - Performance Analysis
  - Robust Stability Analysis
  - Robust Performance Analysis
- Linear Fractional Transformation (LFT)
- A Zero Exclusion Principle
- The Structured Singular Value (μ)
- The Main Loop Theorem
- Bounds on μ and the S-procedure

Performance Analysis



Suppose the system is internally stable. We have

$$y = (I - PK)^{-1}(d - PKn) = Sd - Tn$$

where

$S = (I - PK)^{-1}$	Sensitivity Function
$T = (I - PK)^{-1}PK$	Complementary Sensitivity Function

- S small $\Rightarrow$ insensitive to load disturbance d
- T small $\Rightarrow$ insensitive to measurement noise n

Both cannot be small simultaneously since $S - T = I$. In practice we often have

- d is a low frequency signal $\Rightarrow d = W_1 \tilde{d}$, where W_1 is a low pass filter e.g. $W_1(s) = \dfrac{1}{s+1} I$, and $\tilde{d} \in \mathbf{L}_2[0, \infty)$ is arbitrary.
- n is a high frequency signal $\Rightarrow -n = W_2 \tilde{n}$, where W_2 is a high pass filter e.g. $W_2(s) = \dfrac{s}{s+1} I$, and $\tilde{n} \in \mathbf{L}_2[0, \infty)$ is arbitrary.

A good S must have high pass character in this case.

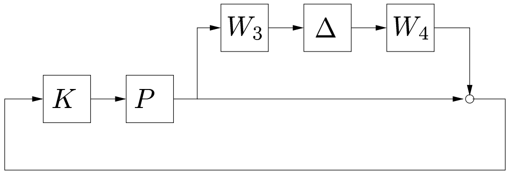
We say that the system has performance level γ if

$$\|y\|_2^2 \leq \gamma^2 (\|\tilde{d}\|_2^2 + \|\tilde{n}\|_2^2), \quad \forall \tilde{d}, \tilde{n} \in \mathbf{L}_2[0, \infty)$$

or equivalently

$$\| \begin{bmatrix} (I - PK)^{-1} W_1 & (I - PK)^{-1} PKW_2 \end{bmatrix} \|_{\mathbf{H}_\infty} \leq \gamma$$

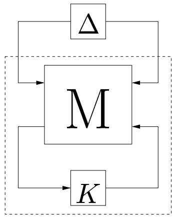
Robust Stability Analysis



System with multiplicative uncertainty

- $W_3, W_4 \in \mathbf{RH}_\infty$ weighting functions
- $\Delta \in \mathbf{RH}_\infty$ norm bounded uncertain dynamics $\|\Delta\|_{\mathbf{H}_\infty} \leq 1$
- The system has robust stability if the closed loop is internally (input output) stable for all $\|\Delta\|_{\mathbf{H}_\infty} \leq 1$

Transform to standard form (G-Δ, LFT form)



where $M = \begin{bmatrix} 0 & W_3 P \\ W_4 & P \end{bmatrix}$. If K internally stabilizes M then $G \in \mathbf{RH}_\infty$.

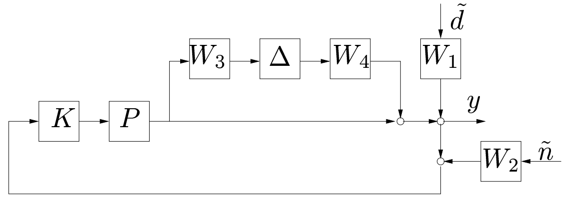
Proposition 1. *Suppose K internally stabilizes P. The system has robust stability iff $\|G\|_{\mathbf{H}_\infty} < 1$.*

Proof. $(I - G\Delta)^{-1} \in \mathbf{RH}_\infty$ is a necessary and sufficient condition since $G, \Delta \in \mathbf{RH}_\infty$. This condition follows from the small gain theorem since

$$\|G\Delta\|_{\mathbf{H}_\infty} \le \|G\|_{\mathbf{H}_\infty}\|\Delta\|_{\mathbf{H}_\infty} \le \|G\|_{\mathbf{H}_\infty} < 1$$

The necessity follows by construction of a perturbation that makes the loop singular. $\square$

Robust Performance Analysis



Let $P_\Delta = (W_4\Delta W_3 + I)P$. For robust performance we need to show

1. The system has robust stability
2. $\left\| \begin{bmatrix} (I - P_\Delta K)^{-1}W_1 & (I - P_\Delta K)^{-1}P_\Delta K W_2 \end{bmatrix} \right\|_{\mathbf{H}_\infty} \le \gamma$

The following idea will be used to derive conditions for robust performance

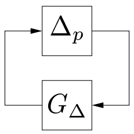
1. Suppose we already proved robust stability. Then (not rectangular)

$$G_\Delta = \gamma^{-1}\begin{bmatrix} (I - P_\Delta K)^{-1}W_1 & (I - P_\Delta K)^{-1}P_\Delta K W_2 \end{bmatrix} \in \mathbf{RH}_\infty$$

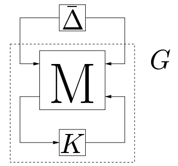
2. By the small gain theorem

$$\|G_\Delta\|_{\mathbf{H}_\infty} < 1 \quad \Longleftrightarrow \quad (I - G_\Delta\Delta_p)^{-1} \in \mathbf{RH}_\infty, \ \forall \|\Delta_p\|_{\mathbf{H}_\infty} \le 1$$

which is equivalent to stability of system



3. Pull out the uncertainty from G_Δ. This leads to the system



$$\text{where} \quad M = \begin{bmatrix} 0 & 0 & 0 & W_3P \\ W_4 & W_1 & 0 & P \\ W_4 & W_1 & W_2 & P \end{bmatrix}, \qquad \bar{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_p \end{bmatrix}$$

The following holds

(a) If K internally stabilizes M then $G \in \mathbf{RH}_\infty$

(b) From the main loop theorem presented later it follows[a]

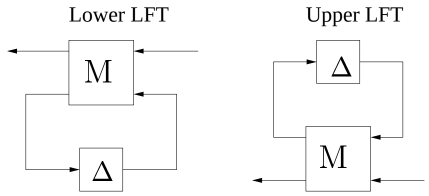
$$\left.\begin{matrix} i) & G_\Delta \text{ stable} \\ ii) & \|G_\Delta\|_{\mathbf{H}_\infty} < 1 \end{matrix}\right. \quad \Longleftrightarrow \quad (I - G\bar{\Delta})^{-1} \text{ stable}$$

[a] if the perturabtions are extended to complex rational transfer functions.

Systematic tools for robustness analysis

- Linear fractional transformations (LFT): By pulling out the uncertainty we obtain feedback interconnections of known nominal part and a structured uncertainty such as the $G-\bar{\Delta}$ interconnection above.
- The zero exclusion principle will allow us to treat uncertain systems in $\mathbf{RH}_\infty$ on a frequency by frequency basis, i.e. everything reduces to a problem of robustness of matrices to structured perturbations.
- The structured singular value, μ, provides a theory for dealing with structured perturbations of matrices.
- The main loop theorem shows that robust performance analysis is equivalent to robust stability analysis
- The S-procedure is used in investigating computable bounds on μ

Linear Fractional Transformations



Let $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$

The Lower LFT $z_1 = \mathcal{S}_l(M,\Delta)w_1$, where

$$\mathcal{S}_l(M,\Delta) = M_{11} + M_{12}\Delta(I-M_{22}\Delta)^{-1}M_{21}$$

is well defined if $I - M_{22}\Delta$ is invertible and corresponds to system

$$\begin{bmatrix} z_1 \\ y_1 \end{bmatrix} = M \begin{bmatrix} w_1 \\ u_1 \end{bmatrix}$$
$$u_1 = \Delta y_1$$

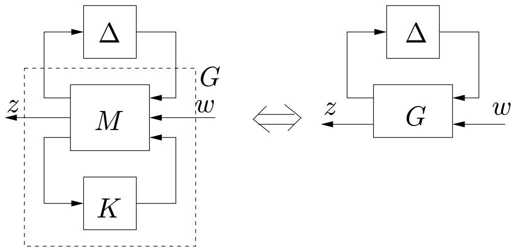
The Upper LFT $z_2 = \mathcal{S}_u(M,\Delta)w_2$, where

$$\mathcal{S}_u(M,\Delta) = M_{22} + M_{21}\Delta(I-M_{11}\Delta)^{-1}M_{12}$$

is well defined if $I - M_{11}\Delta$ is invertible and corresponds to the system

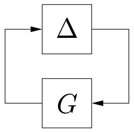
$$\begin{bmatrix} y_2 \\ z_2 \end{bmatrix} = M \begin{bmatrix} u_2 \\ w_2 \end{bmatrix}$$
$$u_2 = \Delta y_2$$

ɔplication to uncertain systems



- Conditions for when K internally stabilizes M have been given
- Many problems in robust control reduces to robust stability analysis of G-Δ structures. If $G,\Delta \in \mathbf{RH}_\infty$ then $(I-G_{11}\Delta)^{-1} \in \mathbf{RH}_\infty$ is sufficient for internal stability
- The zero exclusion principle shows that the above analysis problem is equivalent to μ-analysis.

Zero Exclusion Principle



Suppose $G \in \mathbf{RH}_\infty$ and that the uncertainty is structured such that

$$\Delta(s) = \text{diag}(\delta_1 I_1, \ldots, \delta_S I_S, \Delta_1, \ldots, \Delta_F)$$

where $\delta_s \in \mathbf{RH}_\infty$ and $\Delta_f \in \mathbf{RH}_\infty^{m_f \times m_f}$ and norm bounded such that $\|\Delta\|_{\mathbf{H}_\infty} \leq 1$. If we let

$$\mathbf{\Delta} = \left\{\Delta = \text{diag}(\delta_1 I_1, \ldots, \delta_S I_S, \Delta_1, \ldots, \Delta_F),\ \delta_s \in \mathbf{C},\ \Delta_f \in \mathbf{C}^{m_f \times m_f}\right\}$$

then the above uncertainty can be described as $\Delta \in \mathbf{RH}_\infty$ with $\Delta(s) \in \mathbf{\Delta}$ for Re $s \geq 0$ and $\|\Delta\|_{\mathbf{H}_\infty} \leq 1$

Theorem 1. *Under the above assumption the following are equivalent*

(i) $(I - G\Delta)^{-1} \in \boldsymbol{RH}_\infty$ *for all possible* $\Delta(s)$ *defined above*

(ii) $det(I - G(j\omega)\hat{\Delta}) \neq 0$, $\forall \omega \in [0, \infty]$, $\forall \hat{\Delta} \in \mathbf{\Delta}$ *with* $|\hat{\Delta}| \leq 1$

Idea behind proof: *((ii) ⇒ (i)) We have* $(I - G\Delta)^{-1} \in \boldsymbol{RH}_\infty$ *iff* $det(I - G(s)\Delta(s)) \neq 0$ *for all Re* $s \geq 0$. *Since* $\rho_\Delta(s) = det(I - G(s)\Delta(s))$ *is analytic in the RHP it follows that* $\min_{Re\ s \geq 0} \rho_\Delta(s) = \min_\omega \rho_\Delta(j\omega)$. *Hence, since* $\rho_0(s) = 1$ *when* $\Delta \equiv 0$ *and since (ii) implies that* $\rho_\Delta(j\omega) \neq 0$ *it follows that* $\rho_\Delta(s) > 0$ *for Re* $s \geq 0$.
((i) ⇒ (ii)) Construct a counter example.

The Structured Singular Value (μ)

Consider the uncertainty structure $\mathbf{\Delta}$ defined above. If the identity matrices have size m_s then $m = \sum_{s=1}^S m_s + \sum_{f=1}^F m_f$ is the size of Δ.
Definition 1. For $M \in \mathbf{C}^{m \times m}$ the structured singular value is defined as

$$\mu_{\mathbf{\Delta}}(M) = \frac{1}{\inf\{|\Delta| : \det(I - M\Delta) = 0,\ \Delta \in \mathbf{\Delta}\}}$$

- $\mu_{\mathbf{\Delta}}(M)$ is a robustness margin for M. The smaller μ is the larger structured perturbation from $\mathbf{\Delta}$ is needed to make $I - M\Delta$ singular.
- From the zero exclusion principle we conclude that $(I - G\Delta)^{-1} \in \mathbf{RH}_\infty$ if $\sup_\omega \mu_{\mathbf{\Delta}}(G(j\omega)) < 1$ (this is also necessary if Δ can be complex rational transfer function)

The Main Loop Theorem

Let $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ and consider the three uncertainty structures $\mathbf{\Delta}_1$, $\mathbf{\Delta}_2$ (where the size of Δ_k is compatible with M_{kk}, $k = 1, 2$) and

$$\mathbf{\Delta} = \left\{ \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} : \Delta_1 \in \mathbf{\Delta}_1,\ \Delta_2 \in \mathbf{\Delta_2} \right\}$$

Theorem 2.

$$\mu_{\mathbf{\Delta}}(M) < 1 \quad \Leftrightarrow \quad \begin{cases} \mu_{\mathbf{\Delta_1}}(M_{11}) < 1 \\ \sup_{\substack{\Delta_1 \in \mathbf{\Delta_1} \\ |\Delta_1| \leq 1}} \mu_{\Delta_2}(\mathcal{S}_u(M, \Delta_1)) < 1 \end{cases}$$

Proof: *(⇐) follows from*

$$
\begin{aligned}
& det(I - M\Delta) \\
&= det(I - M_{11}\Delta_1)det(I - M_{22}\Delta_2 - M_{21}\Delta_1(I - M_{11}\Delta_1)^{-1}M_{12}\Delta_2) \\
&= det(I - M_{11}\Delta_1)det(I - \mathcal{S}_u(M, \Delta_1)\Delta_2) \quad (1)
\end{aligned}
$$

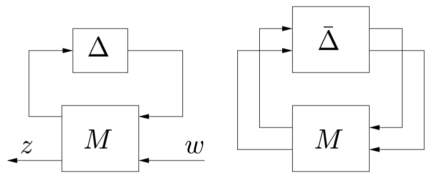
Both factors on the RHS are nonzero by the assumptions.
(⇒) The inequality

$$\mu_{\boldsymbol{\Delta}}(M) \geq \max(\mu_{\boldsymbol{\Delta_1}}(M_{11}), \mu_{\boldsymbol{\Delta_2}}(M_{22}))$$

shows that $\mu_{\boldsymbol{\Delta_1}}(M_{11}) < 1$. *Then from (1) we get* $det(I - \mathcal{S}_u(M, \Delta_1)\Delta_2) \neq 0$ *for all uncertainties, which shows*

$$\sup_{\substack{\Delta_1 \in \boldsymbol{\Delta_1} \\ |\Delta_1| \leq 1}} \mu_{\boldsymbol{\Delta_2}}(\mathcal{S}_u(M, \Delta_1)) < 1$$

Application to Robust Performance



$$\left.\begin{array}{l} \sup_\omega \mu_{\boldsymbol{\Delta}}(M_{11}(j\omega)) < 1 \\ \sup_\omega \sup_{\Delta \in \boldsymbol{\Delta}} \mathcal{S}_u(M(j\omega), \Delta) < 1 \end{array}\right. \iff \sup_\omega \mu_{\overline{\boldsymbol{\Delta}}}(M(j\omega)) < 1$$

where

$$\overline{\boldsymbol{\Delta}} = \begin{bmatrix} \boldsymbol{\Delta} & 0 \\ 0 & \Delta_p \end{bmatrix}, \qquad \Delta_p \in C^{m \times m} \text{ with } \|\Delta_p\|_{\mathbf{H}_\infty} \leq 1$$

Bounds on μ

Elementary bounds

Proposition 2.

$$\rho(M) \leq \mu_{\boldsymbol{\Delta}}(M) \leq \overline{\sigma}(M)$$

Proof: *If* $\boldsymbol{\Delta} = \{\delta \mathbf{I} : \delta \in \boldsymbol{C}\}$ *then* $det(I - \delta M) = 0 \iff \frac{1}{\delta} \in \text{eig}(M)$. *It follows that* $\mu_{\boldsymbol{\Delta}}(M) = \rho(M)$, *the spectral radius of* M. *Since this is the smallest (most restricted) uncertainty structure this gives a lower bound.*

Let $\boldsymbol{\Delta} = \boldsymbol{C}^{\mathbf{m} \times \mathbf{m}}$, *the largest (least restricted) uncertainty structure. If* $M = \sum_{i=1}^m \sigma_i u_i v_i^*$ *is a SVD with* $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m$, *we get* $det(I - M\Delta) = 0$ *if* $\Delta = \frac{1}{\sigma_1} v_1 u_1^*$. *We have* $|\Delta| = 1/\sigma_1 = 1/\overline{\sigma}(M)$, *which gives* $\mu_{\boldsymbol{\Delta}}(M) = \bar{\sigma}(M)$.

Matrix Transformations

Let

$$
\begin{aligned}
&\mathcal{U} = \{U \in \boldsymbol{\Delta} : UU^* = I\} \\
&\mathcal{D} = \{\text{diag}(D_1, \ldots, D_S, d_1 I_{m_1}, \ldots, d_{F-1} I_{m_{F-1}}, I_{m_F}) : \\
&\qquad D_s \in C^{m_s \times m_s}; D_s \text{ nonsingular; } d_f > 0\}
\end{aligned}
$$

Proposition 3. *For all* $U \in \mathcal{U}$ *and* $D \in \mathcal{D}$

$$\mu_{\boldsymbol{\Delta}}(MU) = \mu_{\boldsymbol{\Delta}}(UM) = \mu_{\boldsymbol{\Delta}}(M) = \mu_{\boldsymbol{\Delta}}(DMD^{-1})$$

Proof: *Since* $D\Delta = \Delta D$ *we get* $\Delta = D^{-1}\Delta D$ *and*

$$det(I - M\Delta) = det(I - MD^{-1}\Delta D) = det(I - DMD^{-1}\Delta)$$

where the last equality follows since $det(I - AB) = det(I - BA)$. *This proves* $\mu_{\boldsymbol{\Delta}}(M) = \mu_{\boldsymbol{\Delta}}(DMD^{-1})$.

For the other equivalences use $det(I - M\Delta) = det(I - MUU^*\Delta)$ and that $U^*\Delta \in \mathbf{U}$ satisfies $\bar{\sigma}(U^*\Delta) = \bar{\sigma}(\Delta)$. This gives $\mu_{\mathbf{\Delta}}(MU) = \mu_{\mathbf{\Delta}}(M)$.

The following bounds are evident from the last two propositions

$$\max_{U \in \mathcal{U}} \rho(MU) \leq \mu_{\mathbf{\Delta}}(M) \leq \inf_{D \in \mathcal{D}} \bar{\sigma}(D^{-1}MD)$$

Remark 1. The lower bound can be shown to be an equality, see survey by Packard and Doyle. However, the optimization is non-convex.

Remark 2. The upper bound is quasi convex optimization problem. Indeed, $\bar{\sigma}(D^{-1}MD) < \gamma$ is equivalent to $D^*M^*(D^{-1})^*D^{-1}MD < \gamma^2$. The last can be written as

$$M^*XM - \gamma^2 X < 0$$

where $X = (D^{-1})^*D^{-1}$ which is a positive definite matrix in $\mathcal{D}$.

Hence if we let

$$\mathcal{X} = \{X \in \mathbf{C}^{m \times m} : X = X^* > 0;\ X \in \mathcal{D}\}$$

then the upper bound can be obtained as

$$\min \gamma \quad \text{s.t} \quad M^*XM - \gamma^2 X < 0;\ X \in \mathcal{X}$$

We will discuss such problems in further detail later.

The S-Procedure

- The upper bound can be derived using the S-procedure
- Conditions for exactness of the upper bound can be obtained using the S-procedure

F= / S=	0	1	2	3	4
0		yes	yes	yes	no
1	yes	yes	no	no	no
2	no	no	no	no	no